Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

January 2016 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages.
Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

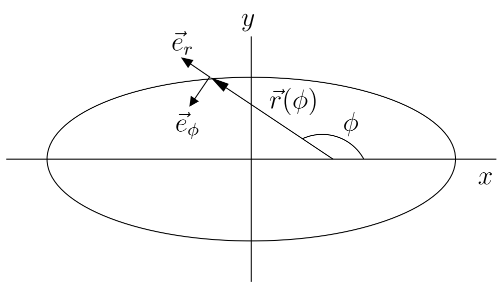
Orbits of planets

As you know, the orbits $\vec{r}(\phi)$ of planets are ellipses, but, as you will see, the orbits $\vec{v}(\phi)$ in velocity space are circles! If one adds a perturbation $\frac{\beta}{r^2}$ to the $\frac{1}{r}$ potential, the ellipses start to precess, and the circles become a kind of epicycles. In this problem we prove these statements, and construct exact solutions for the potential

$$V(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2} \qquad (\alpha > 0,\ \beta \geq 0)\,. \tag{1}$$

Consider a point particle with mass m and negative energy $E = -|E|$ in this potential.

a) (2 points) First prove that $\vec{e}_r = -\frac{d\vec{e}_\phi}{d\phi}$, where $\vec{e}_r$ is the unit vector along the radius and $\vec{e}_\phi$ the unit vector orthogonal to $\vec{e}_r$ in the direction of increasing ϕ (see the figure)



Set $\beta = 0$. Then prove that

$$\frac{d\vec{v}}{d\phi} = \gamma \frac{d\vec{e}_\phi}{d\phi}\,. \tag{2}$$

What is the constant γ?

b) (4 points) It follows from (2) that $\vec{v}(t) = \vec{w} + \gamma\vec{e}_\phi(t)$ with constant $\vec{w}$. By taking the scalar product of this equation with $\vec{e}_\phi$, show that one obtains elliptical orbits given by $r(\phi)$ (see definition below*). *Hint:* Express $v_\phi \equiv \vec{v}(t)\cdot\vec{e}_\phi(t)$ in terms of polar coordinates, and choose a coordinate system such that $\vec{w}$ lies along the positive y-axis.

*An ellipse $(x/a)^2 + (y/b)^2 = 1$ can be parametrized in polar coordinates relative to the focus by $r(\phi)$

$$r(\phi) = \frac{b^2}{a + c\cos\phi}\,,$$

where $c = \sqrt{a^2 - b^2}$ is the distance between the (x, y) origin and the focus. The angle ϕ and radius $r(\phi)$ are indicated in the figure.

c) (4 points) On the other hand, show that $\vec{v}(\phi)$ describes circles. Draw pictures of these elliptical and circular orbits and locate in these pictures the angle ϕ.

d) (2 points) Now consider the case that $\beta > 0$. Derive the relation

$$\tfrac{1}{2}m\dot{r}^2 + \left(\tfrac{A}{r} - B\right)^2 = \boldsymbol{E}^2\,. \tag{3}$$

What are A, B and $\boldsymbol{E}$? Set $\sqrt{\frac{m}{2}}\,\dot{r} = \boldsymbol{E}\sin f(t)\,;\ \left(\frac{A}{r} - B\right) = \boldsymbol{E}\cos f(t)$. Is this always possible?

e) (4 points) Show that $\frac{\dot{f}(t)}{\dot{\phi}} = \frac{df}{d\phi} = \omega = \text{constant}$. Show that the orbits $r(\phi)$ are now ellipses with precession.

f) (4 points) Find the equation which generalizes (2) to the case when β is nonvanishing. How would you solve this equation?

Solutions

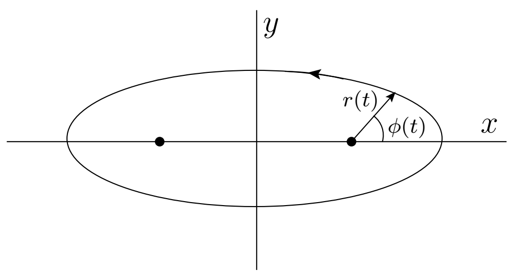
a) $m\dot{\vec{v}} = -\frac{\alpha}{r^2}\vec{e}_r$ where $\vec{e}_r$ is the unit vector along the radius. With $mr^2\dot{\phi} = l =$ angular momentum, we get

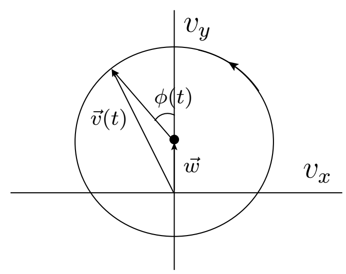
$$\frac{\dot{\vec{v}}}{\dot{\phi}} = -\frac{\alpha}{l}\vec{e}_r = \frac{\alpha}{l}\frac{d\vec{e}_\phi}{d\phi}$$

because $\vec{e}_r = -\frac{d\vec{e}_\phi}{d\phi}$. (Proof: $\vec{e}_r = (\cos\phi, \sin\phi, 0)$ and $\vec{e}_\phi = (-\sin\phi, \cos\phi, 0)$, so $\frac{d\vec{e}_\phi}{d\phi} = -\vec{e}_r$.) Clearly $\gamma = \frac{\alpha}{l}$.

b) We evaluate $v_\phi = r\dot{\phi} = \vec{w}.\vec{e}_\phi + \frac{\alpha}{l}$. Hence $\frac{l}{mr(t)} = w\cos\phi(t) + \frac{\alpha}{l}$ if we choose a coordinate system such that $\vec{w}$ lies along the positive y-axis. This is an ellipse. (The equation for an ellipse in general is given by $\frac{1}{r} = \frac{a + c\cos\phi}{b^2}$ and in our case $\frac{a}{b^2} = \frac{m\alpha}{l^2}$ and $\frac{c}{b^2} = \frac{mw}{l}$.)

c) Squaring $\vec{v}(\phi) - \vec{w} = \frac{\alpha}{l}\vec{e}_\phi$ with constant $\vec{w}$, we obtain $(\vec{v} - \vec{w})^2 = \left(\frac{\alpha}{l}\right)^2$. These are circles, with the origin at $\vec{w}$ and radius $\frac{\alpha}{l}$.





If $\phi = 0$, the velocity is maximal, so the point $(a, 0)$ on the ellipse corresponds to the north pole of the circle.

d) The energy is given by $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{\alpha}{r} + \frac{\beta}{r^2} = E$, hence

$$\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}\tfrac{l^2}{mr^2} + \frac{\beta}{r^2} - \frac{\alpha}{r} = E\,, \text{ or}$$

$$\tfrac{1}{2}m\dot{r}^2 + \left(\tfrac{A}{r} - B\right)^2 = \boldsymbol{E}^2\,,$$

with $A^2 = \beta + \frac{l^2}{2m}$, $B = \frac{\alpha}{2A}$, and $\boldsymbol{E}^2 = E + \frac{\alpha^2}{4A^2}$. If $a^2 + b^2 = \boldsymbol{E}^2$ one can always write $a = \boldsymbol{E}\sin f(t)$ and $b = \boldsymbol{E}\cos f(t)$.

e) Taking the time derivative of the relation with $\cos f(t)$, gives

$$-\frac{A}{r^2}\frac{\boldsymbol{E}}{\sqrt{m/2}}\sin f(t) = -\boldsymbol{E}\dot{f}(t)\sin f(t)\,, \text{ so}$$
$$\frac{\dot{f}}{\dot{\phi}} = \frac{df}{d\phi} = \frac{A}{l}\sqrt{2m} \equiv \omega = \sqrt{1+\frac{2m\beta}{l^2}}.$$

Then $f = \omega\phi$ and $\left(\frac{A}{r} - B\right) = \boldsymbol{E}\cos\omega\phi$. If $\omega \neq 1$ ($\beta \neq 0$), this is an ellipse with precession: if ϕ runs from 0 to 2π, the cosine is no longer periodic.

f) The relation for $\frac{\dot{\vec{v}}}{\dot{\phi}} = \frac{d\vec{v}}{d\phi}$ is no longer a total derivative. Instead, one finds

$$m\dot{\vec{v}} = \left(-\frac{\alpha}{r^2} + \frac{2\beta}{r^3}\right)\vec{e}_r\,, \qquad \frac{d\vec{v}}{d\phi} = \frac{\alpha}{l}\frac{d\vec{e}_\phi}{d\phi} - \frac{2\beta}{lr}\frac{d\vec{e}_\phi}{d\phi}\,.$$

Since the term with β is not a total derivative, this is not a circle.

If one substitutes the equation for $\frac{1}{r}$ one obtains an equation that is easy to solve

$$\begin{pmatrix}\frac{dv_x}{d\phi}\\ \frac{dv_y}{d\phi}\end{pmatrix} = \left[\frac{\alpha}{l} - \frac{2\beta}{l}\left(\frac{\boldsymbol{E}\cos\omega\phi + B}{A}\right)\right]\begin{pmatrix}-\cos\phi\\ -\sin\phi\end{pmatrix},$$

or more explicitly

$$\begin{aligned}\frac{dv_x}{d\phi} &= \left(-\frac{\alpha}{l} + \frac{2\beta B}{lA}\right)\cos\phi + \frac{\beta\boldsymbol{E}}{lA}\big(\cos(\omega+1)\phi + \cos(\omega-1)\phi\big)\\ \frac{dv_y}{d\phi} &= \left(-\frac{\alpha}{l} + \frac{2\beta B}{lA}\right)\sin\phi + \frac{\beta\boldsymbol{E}}{lA}\big(\sin(\omega+1)\phi - \sin(\omega-1)\phi\big).\end{aligned}$$

The first terms on the right-hand side give again a circle, but the second terms give two counter-precessing ellipses. Their sum constitutes a complicated kind of epicycle motion.

Comment 1: For these precessing ellipses it is still true that the total energy E is proportional to a, and for fixed E, the maximal value of l is obtained for circular motion. However for given angular momentum, there is an upper bound on the eccentricity which excludes linear motion. To prove these statements we recall the equation of motion and its solution:

$$\frac{d^2u}{d\phi^2} + \omega^2 u = \frac{\alpha m}{l^2} \qquad \text{with } \omega^2 = 1 + \frac{2m\beta}{l^2} \text{ and } u = \frac{1}{r}$$
$$\Rightarrow u = A + B\cos(\omega\phi)\,; \qquad \omega^2 A = \frac{\alpha m}{l^2}\,.$$

The energy is given by

$$
\begin{aligned}
E &= \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 - \tfrac{\alpha}{r} + \tfrac{\beta}{r^2} \\
&= \frac{l^2}{2m}\left[\left(\frac{du}{d\phi}\right)^2 + \omega^2 u^2\right] - \alpha u \qquad \left(\text{where } u = \frac{1}{r}\right) \\
&= \tfrac{l^2}{2m}\left[\omega^2 A^2 + \omega^2 B^2\right] - \alpha A = \tfrac{l^2}{2m}\omega^2\left[-A^2 + B^2\right].
\end{aligned}
$$

On the other hand, the semi-major axis follows from $\frac{a}{b^2} = A$, $B = \frac{c}{b^2}$, so $\frac{1}{b^2} = A^2 - B^2$, and $a = \frac{A}{A^2 - B^2}$. Then E depends indeed only on a (and α, but not on b or β)

$$
E = -\frac{l^2}{2m}\omega^2 \cdot \frac{A}{a} = -\frac{\alpha}{2a}\,.
$$

The angular momentum is most easily evaluated when $\cos(\omega\phi) = 1$, because then $\frac{1}{2}m\dot{r}^2 = 0$, and the velocity has only a ϕ-component

$$
l = mv_\phi r \;(\text{at } \phi = 0) = \frac{m}{A+B}\sqrt{\frac{2}{m}\left(E + \frac{\alpha}{r} - \frac{\beta}{r^2}\right)}
$$

Using $\frac{1}{r} = A + B = \frac{a+c}{b^2} = \frac{1}{a-c}$, we get

$$
\begin{aligned}
l &= \sqrt{2m}\,(a-c)\sqrt{E + \frac{\alpha}{a-c} - \frac{\beta}{(a-c)^2}} \\
&= \sqrt{2m}\,\sqrt{E(a-c)^2 + \alpha(a-c) - \beta} = \sqrt{2m}\,\sqrt{\frac{1}{2}\alpha a - \beta - \frac{\alpha c^2}{2a}}\,.
\end{aligned}
$$

Clearly, for fixed a (for fixed energy), l is maximal if $c = 0$, so for circular orbits. The value of c is bounded by

$$
c^2 \le a^2 - \tfrac{2a\beta}{\alpha}\,.
$$

Apparently, the repulsive potential $\frac{\beta}{r^2}$ keeps the point particle away from the point $r = 0$ so that c can not become too large (not equal to a). Thus linear motion (motion along the x-axis) is excluded.

Comment 2: Conservation laws correspond to symmetries and vice-versa (the Noether theorem). Because Kepler orbits are 1-dimensional, but the orbits for $\beta \neq 0$ are 2-dimensional (in the sense that the precessing ellipses fill up a disk in the xy plane), there should be precisely one extra conserved quantity for $\beta = 0$. The vector $\vec{w} = \vec{v} - \gamma\vec{e}_\phi$ is conserved but that seems to yield 3 extra conserved quantities instead of one. Why is there only one extra conserved quantity? The vector $\vec{w}$ lies in the xy plane if $\vec{J}$ is along the z-axis and also is constant. Thus there is, after all, only one more extra conserved quantity. We cannot take $C = \vec{a} \cdot (\vec{v} - \gamma \vec{e}_\phi)$ as generator of symmetries because we should encode in C the knowledge that the motion is in a plane (for simplicity the xy plane). So consider instead

$$C = \vec{a} \cdot \left(\vec{J} \times \vec{w} \right) = \vec{a} \cdot \left(\vec{J} \times \vec{v} - \vec{J} \times \gamma \vec{e}_\phi \right) \text{ with } \vec{J} = J \vec{e}_z \,.$$

One can compute the infinitesimal transformation law δx^i which is generated by C and then one finds that the Lagrangian transforms into a total derivative: $\delta \mathcal{L} = \frac{dF}{dt}$. $\delta x^i = \{C, x^i\}$ where $\{\, , \,\}$ are Poisson brackets with $\{x^i, p_j\} = \delta^i_j$. Using $\vec{e}_z \times \vec{e}_\phi \equiv -\vec{e}_r$, we get

$$\begin{aligned} C &= a^i \epsilon^{ijk} \left(\epsilon^{jmn} x^m p^n \right) \frac{p^k}{m} + \alpha \frac{\vec{a} \cdot \vec{x}}{r} \\ &= a \cdot p \, x \cdot \frac{p}{m} - a \cdot x \, \frac{p^2}{m} + \alpha \frac{a \cdot x}{r} \,. \end{aligned}$$

So

$$\delta x^i = \{C, x^i\} = -a^i x \cdot \frac{p}{m} - a \cdot p \, \frac{x^i}{m} + 2a \cdot x \, \frac{p^i}{m} \,.$$

Now $\delta L = \frac{\partial L}{\partial \vec{x}} \cdot \delta \vec{x} + \frac{\partial L}{\partial \vec{v}} \cdot \delta \vec{v} = \frac{dF}{dt}$ and $\delta \vec{v} = \frac{d}{dt} \delta \vec{x}$. So

$$\delta L = \left(\frac{\partial L}{\partial x^k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^k} \right) \delta x^k + \frac{d}{dt} \left(\frac{\partial L}{\partial v^k} \delta x^k \right) .$$

The quantity $C = \frac{\partial L}{\partial v^k} \delta x^k - F = p_k \delta x^k - F$ is on-shell conserved: $\frac{dC}{dt} = 0$ if the Euler-Lagrange equations $\frac{\partial L}{\partial x^k} - \dot{p}_k = 0$ hold. So

$$\begin{aligned} F = p_k \delta x^k - C &= -2a \cdot p \, \frac{x \cdot p}{m} + 2a \cdot x \, \frac{p^2}{m} - a \cdot p \, x \cdot \frac{p}{m} + a \cdot x \, \frac{p^2}{m} - \alpha \frac{a \cdot x}{r} \\ &= -3a \cdot p \, \frac{x \cdot p}{m} + 3a \cdot x \, \frac{p^2}{m} - \alpha \frac{a \cdot x}{r} \,. \end{aligned}$$

Direct evaluation of F would be very tedious.

Comment 3: Richard Feynman tried to reproduce Newton's pre-calculus (!) proof that the velocity orbits of planets are circles in a lecture for freshman and sophomores at Caltech in 1964. (The lecture audiotape was lost and then later found and published.) Unknown to him, Maxwell had already proven this in his little book "Matter and Motion" in 1877. But Maxwell refers to Hamilton, who coined the term hodograph for velocity diagrams. Before Hamilton already Laplace, and before him Bernoulli, had studied aspects of this problem.

Classical Mechanics 2

A bead on a hoop

A bead of mass m is constrained to move (without friction) on a hoop of radius R. The hoop rotates with constant angular velocity ω around the vertical axis. The bead is subjected to the force of gravity at the surface of the Earth.

a) Write down the Lagrangian for the system and the Lagrangian equations of motion. [4pts]

b) Find any constants of motion that may exist. Construct the Hamiltonian. Is it equal to the energy in the fixed (i.e. non-rotating) frame? Is the fixed-frame energy conserved? [2pts]

c) Find the critical angular velocity Ω below which the bottom of the hoop is a position of *stable* equilibrium. Find the stable equilibrium positions for both $\omega < \Omega$ and $\omega > \Omega$. [7pts]

d) Calculate the frequencies of small oscillations around the positions of stable equilibrium. [7pts]

Solution

a) The motion of the bead is one-dimensional. We can use as generalized coordinate its angular position θ along the hoop, measured from the center of the hoop, with $\theta = 0$ at the bottom and $\theta = \pi$ at the top. In terms of cylindrical coordinates ρ, φ, z in the inertial frame of the Earth (with origin at the center of the hoop) the position of the hoop is given by (choosing the time origin $t = 0$ when $\varphi = 0$):

$$z = -R\cos\theta\,, \quad \rho = R\sin\theta\,, \quad \varphi = \omega t\,. \tag{1}$$

The Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - V = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - mgz \\ &= \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin\theta^2) + mgR\cos\theta\,.\end{aligned} \tag{2}$$

The Lagrangian equation of motion is

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= 0 \\ R^2\ddot{\theta} - R\omega^2\sin\theta\cos\theta + g\sin\theta &= 0\,.\end{aligned} \tag{3}$$

We have chosen to evaluate the Lagrangian in the inertial system of the Earth. Alternatively, one may use the non-inertial system attached to the hoop. In that case, the velocity has only the component along the hoop, so the kinetic energy is

$$T_{non-in} = \frac{1}{2}mR^2\dot{\theta}^2\,, \tag{4}$$

but one must introduce an additional term in the potential to account for the fictitious centrifugal force,

$$V_{non-in} = -mgR\cos\theta - mR^2\omega^2\sin\theta^2\,. \tag{5}$$

The result for $\mathcal{L} = T_{non-in} - V_{non-in}$ is of course the same as (2)

b) Since $\partial\mathcal{L}/\partial t = 0$, $\dot{\theta}\partial\mathcal{L}/\partial\dot{\theta} - \mathcal{L}$ is conserved,

$$\dot{\theta}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \mathcal{L} = \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}mR^2\omega^2\sin\theta^2 - mgR\cos\theta = \text{constant}\,. \tag{6}$$

This conserved quantity can be identified with $T_{non-in} + V_{non-in}$, but it is not the same as $T + V$.

The momentum conjugate to θ is

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\dot{\theta}\,, \tag{7}$$

and the Hamiltonian

$$\mathcal{H} = p_\theta \dot{\theta} - \mathcal{L} = \frac{p_\theta^2}{2mR^2} - \frac{1}{2} m R^2 \omega^2 \sin\theta^2 - mgR\cos\theta\,, \tag{8}$$

which coincides of course with the conserved quantity (6). The fixed-frame energy has a different sign in front of the term with $\sin\theta^2$ and is thus not conserved.

c) At equilibrium, $\dot{\theta} = \ddot{\theta} = 0$ so

$$\sin\theta(\omega^2\cos\theta - \Omega^2) = 0\,, \quad \text{where } \Omega^2 \equiv \frac{g}{R}\,. \tag{9}$$

If $\omega < \Omega$, the only solutions come from $\sin\theta = 0$, and they are $\theta = 0$ and $\theta = \pi$. If $\omega > \Omega$, in addition to $\theta = 0, \pi$ we also have an equilibrium position as

$$\cos\theta = \frac{\Omega^2}{\omega^2} = \frac{g}{R\omega^2}\,. \tag{10}$$

Let θ_0 be an equilibrium position, and take $\theta = \theta_0 + \alpha$ where α is small. To linear order in α, the equation of motion is

$$\ddot{\alpha} + (\Omega^2\cos\theta_0 - \omega^2\cos 2\theta_0)\alpha = 0\,. \tag{11}$$

Using $\cos 2\theta_0 = 2\cos^2\theta_0 - 1$, this can also be written as

$$\ddot{\alpha} + \left[\cos\theta_0(\Omega^2 - 2\omega^2\cos\theta_0) + \omega^2\right]\alpha = 0\,. \tag{12}$$

For $\omega < \Omega$, the coefficient of α is positive for the equilibrium at $\theta_0 = 0$ and negative for the equilibrium at $\theta_0 = \pi$. This shows that for $\omega < \Omega$ the bottom of the hoop ($\theta = 0$) is a stable equilibrium position, while the top of the hoop ($\theta = \pi$) is unstable.

For $\omega > \Omega$ both $\theta = 0$ and $\theta = \pi$ are unstable. On the other hand, for the additional equilibrium position at $\cos\theta_0 = \Omega^2/\omega^2$, the coefficient of α is

$$\frac{1}{\omega^2}\left(\omega^4 - \Omega^4\right) > 0\,, \tag{13}$$

so the equilibrium is stable.

To summarize, the stable equilibrium positions are $\theta = 0$ for $\omega < \Omega$ and $\theta_0 = \arccos\Omega^2/\omega^2$ for $\omega > \Omega$.

d) The angular frequency ω' of small oscillations around a point of stable equilibrium θ_0 is found by looking at the coefficient of α in the linearized equation (11),

$$\omega' = \sqrt{\Omega^2\cos\theta_0 - \omega^2\cos 2\theta_0}\,. \tag{14}$$

For $\theta_0 = 0$ (stable equilibrium for $\omega < \Omega$) this gives

$$\omega' = \sqrt{\Omega^2 - \omega^2}\,. \tag{15}$$

For $\theta_0 = \arccos\Omega^2/\omega^2$ (stable equilibrium for $\omega > \Omega$) this gives

$$\omega' = \sqrt{\omega^2 - \Omega^4/\omega^2}\,. \tag{16}$$

Classical Mechanics 3

Magnetic mirrors

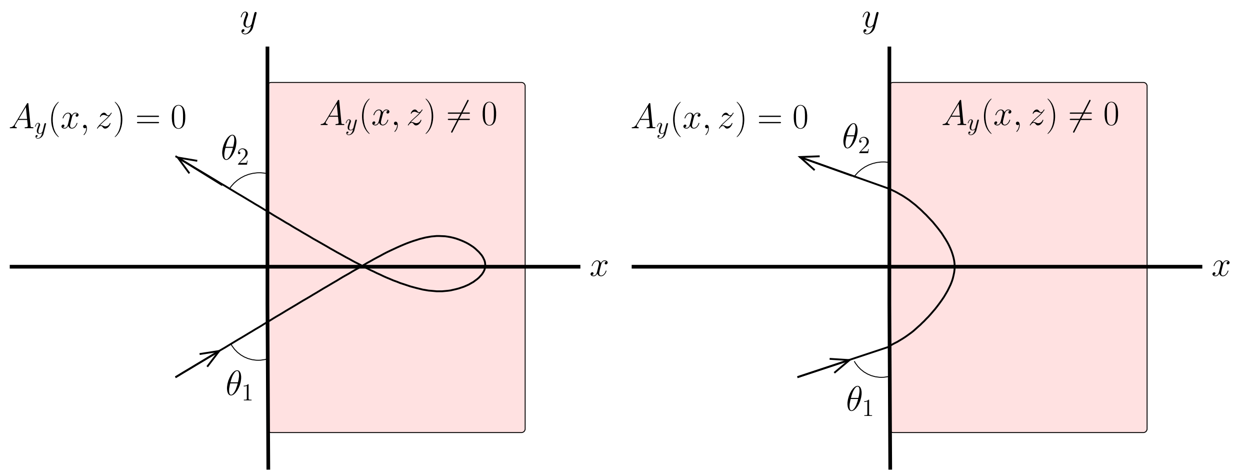


Fig. 1: Planar view of the system. The z-axis is perpendicular to the plane with $\hat{e}_z = \hat{e}_x \times \hat{e}_y$.

A relativistic electron (charge $e = -|e|$, rest mass m) with mechanical momentum

$$\vec{p} = p_0 \left(\hat{e}_x \cdot \sin\theta_1 + \hat{e}_y \cdot \cos\theta_1 \right) \tag{1}$$

propagates from free space with zero magnetic field (and zero vector potential) at $x < 0$ into a time-independent magnetic field (see Fig. 1). The magnetic field has no y-component and is due to a vector potential which has only a y-component:

$$\vec{A} = \hat{e}_y A_y(x,z)\,; \quad \vec{B} = \vec{\nabla} \times \vec{A}\,;$$
$$\vec{B} = \hat{e}_x B_x(x,z) + \hat{e}_z B_z(x,z)\,.$$

At $z = 0$ the magnetic field is perpendicular to the $x - y$ plane.

a) (2 points) Show that a trajectory of an electron located at $z = 0$ with its momentum in $x - y$ plane (as in equation (1)) will stay in $x - y$ plane.

b) (5 points) Construct the relativistic Hamiltonian of the system and the canonical momentum of the particle. Is the Lagrangian, or the action, or the Hamiltonian Lorentz invariant? Exlain. (Hint: Recall that the Lagrangian of a point particle in an external electromagnetic potential is

$$L = -mc^2\sqrt{1 - \vec{v}^2/c^2} - e\,\varphi(\vec{x}(t)) + \frac{e}{c}\,\vec{v} \cdot \vec{A}(\vec{x}(t)) \tag{2}$$

where $\vec{v}$ is the particle velocity, φ is the electrostatic potential, and $\vec{A}$ is the vector potential.)

c) (4 points) Obtain two integrals of the motion for the problem described above. Using these two integrals of the motion, derive an effective 1D Hamiltonian for motion in the x-direction of the following form:

$$H^* = \frac{p_x^2}{2m^*} + U(x)\,.$$

Find an expression for m^* and for $U(x)$ in terms of the integrals of the motion and $A_y(x)$.

Hints: (1) One of these invariants is generic for any motion in a magnetic field, while the other is specific to this system's translation symmetry. (2) Write the equations of motion for x-components using the full Hamiltonian and substitute the two invariants into these equations. Compare these equations with those from the effective Hamiltonian to define m^* and $U(x)$.

d) (5 points) Using the two integrals of the motion show that this system is indeed a "mirror" for trajectories in the $x - y$ plane, namely, an electron with initial momentum (1) is reflected such that angle of the incoming and outgoing electron in the figure are equal:

$$\theta_2 = \theta_1.$$

e) (5 points) Again, for a trajectory in $x - y$ plane, find an equation for the depth of penetration for the cases illustrated in Fig. 1(a) and Fig. 1(b). Solve this equation for the field of a quadrupole with field gradient G:

$$\vec{B} = G\left(\hat{e}_x z - \hat{e}_z x\right); \quad x > 0\,.$$

Which signs of G correspond to the trajectories in Fig. 1(a) and Fig. 1(b)?
(Denote the charge of the electron by e, where $e = -|e|$.)

Solution

a) The charged particle initially in the xy-plane (left), will stay in the same plane (right) if $(e = -|e|)$

$$F_z = \frac{e}{c}\left(\vec{v}\times\vec{B}\right)_z = \frac{e}{c}\left(v_x B_y - v_y B_x\right) = 0 \tag{3}$$

which is the case if $B_{x,y} = 0$.

b) The relativistic Lagrangian is

$$\mathcal{L} = -mc^2\left(1-\frac{\dot{\vec{x}}^2}{c^2}\right)^{\frac{1}{2}} + \frac{e}{c}\vec{A}\cdot\dot{\vec{x}} \equiv -\frac{mc^2}{\gamma} + \frac{e}{c}\vec{A}\cdot\dot{\vec{x}} \tag{4}$$

with $\vec{A} = A_y(x,z)\hat{y}$. The canonical momentum is

$$\vec{\Pi} = \gamma m\dot{\vec{x}} + \frac{e}{c}\vec{A} \equiv \vec{p} + \frac{e}{c}\vec{A} \tag{5}$$

The Hamiltonian follows canonically as

$$H = \vec{\Pi}\cdot\dot{\vec{x}} - \mathcal{L} = \gamma mc^2 = \left(m^2c^4 + \vec{p}^2c^2\right)^{\frac{1}{2}} \tag{6}$$

Time translational invariance of the external vector potential implies energy conservation $H = \text{constant} \equiv E$ (first integral of motion), while y-translational invariance of the external vector potential implies $\Pi_y = \text{constant} \equiv p_0\cos\theta_1$ (second integral of motion). With this in mind, we may re-write (6) as

$$H \to \frac{H^2}{2E} \equiv \frac{p_x^2}{2m^*} + U(x) \tag{7}$$

with

$$\begin{aligned} m^* &= \frac{E}{c^2} = m\left(1+\frac{p_0^2}{m^2c^2}\right)^{\frac{1}{2}} \\ U(x) &= \frac{1}{2E}\left(m^2c^4 + \left(p_0\cos\theta_1 - \frac{e}{c}A_y\right)^2 c^2\right) \end{aligned} \tag{8}$$

c) The (squared) energy and momentum integrals of motion can be re-written respectively in the form

$$\begin{aligned} &p_x^2 + p_y^2 = p_0^2 \to p_x = \pm\left(p_0^2 - p_y^2\right)^{\frac{1}{2}} = \pm p_0\sin\theta_1 \\ &p_y = p_0\cos\theta_1 \end{aligned} \tag{9}$$

The $+\theta_1$ solution corresponds to the entering charge, and the $-\theta_1$ solution corresponds to the exiting charge. Hence the mirror solution $\theta_1 = \theta_2$ upon deflection.

d) The charge will stop propagating in the x-direction at $x = xmin$ when $p_x = \sqrt{p_0^2 - p_y^2} = 0$. This occurs when

$$p_y = \pm p_0 = p_0 \cos\theta_1 - \frac{e}{c} A_y(xmin, z = 0) = 0 \tag{10}$$

or

$$A_y(xmin, z = 0) = \frac{cp_0}{e} (\mp 1 + \cos\theta_1) \tag{11}$$

For the quadrupole magnetic field $\vec{B} = G(\hat{x}z - \hat{z}x)$ we have $\vec{A}(x, z) = -\frac{G}{2}(x^2 + z^2)\hat{y}$. Upon insertion in (11) we obtain

$$xmin^2 = \frac{2cp_0}{eG} (\pm 1 - \cos\theta_1) \tag{12}$$

Thus the two solutions

$$\begin{aligned} G > 0 \qquad xmin &= \left(\frac{2cp_0}{|e||G|} (1 + \cos\theta_1) \right)^{\frac{1}{2}} \qquad \text{Fig.1a} \\ G < 0 \qquad xmin &= \left(\frac{2cp_0}{|e||G|} (1 - \cos\theta_1) \right)^{\frac{1}{2}} \qquad \text{Fig.1b} \end{aligned} \tag{13}$$

Electromagnetism 1

Radiation from a relativistic electron

Consider a relativistic electron (of charge e) traveling with an initial speed v_o along the z-axis. At time $t = 0$ it slows down to a stop over a time τ while moving along the z-axis

$$v(t) = v_o \left(1 - \frac{t}{\tau}\right), \qquad 0 \le t \le \tau \,. \tag{1}$$

Recall that the electric field in the far field radiated from a point charge following a trajectory with position $\boldsymbol{x}(t)$, and velocity $\boldsymbol{v}(T) = \boldsymbol{x}'(t)$ is

$$\boldsymbol{E}_{\rm rad}(t, \boldsymbol{r}) = \frac{e}{4\pi c^2} \left[\frac{\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a}}{R\,(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} \right]_{\rm ret} , \tag{2}$$

where all quantities in square brackets are evaluated at the *retarded time*, $T(t, \boldsymbol{r})$ (which you will define below). The other symbols are defined as $\boldsymbol{n} \equiv (\boldsymbol{r} - \boldsymbol{x}(T))/|\boldsymbol{r} - \boldsymbol{x}(T)|$, $R \equiv |\boldsymbol{r} - \boldsymbol{x}(T)|$, and $\boldsymbol{\beta} = \boldsymbol{v}/c$.

(a) (3 points) Define the retarded time and compute the derivatives $\partial T/\partial t$ and $\partial T/\partial \boldsymbol{r}^i$

(b) (3 points) The radiation field $\boldsymbol{E}_{\rm rad}$ is derived from the *Liénard-Wiechert* potentials

$$\varphi(t, \boldsymbol{r}) = \frac{e}{4\pi} \left[\frac{1}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret} , \tag{3}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e}{4\pi c} \left[\frac{\boldsymbol{v}}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret} . \tag{4}$$

Using far field approximations, show that the Lorenz gauge condition is satisfied by these potentials.

(c) (6 points) For the decelerating electron described above, compute:

(i) the energy radiated per solid angle per *retarded time.*

(ii) the energy radiated per solid angle *per time.*

Describe in what physical situations you would be interested in (i) and (ii) respectively. Use no more than two sentences to describe each case.

(d) (4 points) Now consider a relativistic electron with initial energy of 1 GeV.

Examining your results of part (c), you should find that at $t = 0$ the radiation is initially emitted (predominantly) at a characteristic angle. Give an order of magnitude estimate for this angle. Explain your estimate by pointing to specific terms in your formulas from part (c).

(e) (4 points) Determine the total energy per solid angle emitted as the electron decelerates to a stop.

Solution

(a) The retarded time is the time that light was emitted at the source such that it arrives at space-time observation point $(t, \boldsymbol{r})$. It satisfies the implicit equation

$$t - T = |\boldsymbol{r} - \boldsymbol{x}(T)|/c \,. \tag{5}$$

Differentiating

$$1 - \frac{\partial T}{\partial t} = - \frac{(\boldsymbol{r} - \boldsymbol{x}(T))^\ell}{|\boldsymbol{r} - \boldsymbol{x}(T)|} v_\ell(T)/c \frac{\partial T}{\partial t} \,, \tag{6}$$

$$1 - \frac{\partial T}{\partial t} = - \boldsymbol{n} \cdot \beta(T) \frac{\partial T}{\partial t} \,. \tag{7}$$

Thus

$$\frac{\partial T}{\partial t} = \frac{1}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}(T)} \,. \tag{8}$$

Similarly,

$$- \frac{\partial T}{\partial r^k} = \frac{(\boldsymbol{r} - \boldsymbol{x}(T))^\ell}{|\boldsymbol{r} - \boldsymbol{x}(T)|} \left(\delta_{\ell k} - \frac{v_o(T)_\ell}{c} \frac{\partial T}{\partial r^k} \right) \,. \tag{9}$$

Thus

$$\frac{\partial T}{\partial r^k} = \frac{-n_k}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta}(T))} \,. \tag{10}$$

(b) The Lorenz gauge condition reads

$$\frac{1}{c} \partial_t \varphi + \partial_i A^i = 0 \,. \tag{11}$$

In the far field we neglect differentiating $1/R$ and $\boldsymbol{n}$ which lead to subleading terms in $1/R$. Then in the far field we differentiate

$$\frac{1}{c} \partial_t \varphi = \frac{e}{4\pi R c^2} \frac{\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} \frac{\partial T}{\partial t} \,, \tag{12}$$

$$= \frac{e}{4\pi R c^2} \frac{\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} \,. \tag{13}$$

Similarly,

$$\partial_i \boldsymbol{A}^i = \frac{e}{4\pi R c^2} \left[\frac{a^i}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \frac{\partial T}{\partial r^i} + \frac{\beta^i}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} (\boldsymbol{n} \cdot \boldsymbol{a}) \frac{\partial T}{\partial r^i} \right] \,, \tag{14}$$

$$= \frac{e}{4\pi R c^2} \left[\frac{-\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} + \frac{-\boldsymbol{n} \cdot \beta}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} (\boldsymbol{n} \cdot \boldsymbol{a}) \right] \,, \tag{15}$$

$$= \frac{e}{4\pi R c^2} \left[\frac{-\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} \right] \,. \tag{16}$$

So we verify that

$$\frac{1}{c} \partial_t \varphi + \partial_i \boldsymbol{A}^i = 0 \,. \tag{17}$$

(c) In this case $\boldsymbol{\beta} \times \boldsymbol{a} = 0$, $|\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{a}| = a\sin(\theta)$, and thus the magnitude of $\boldsymbol{E}$ is

$$E = \frac{e}{4\pi R c^2} \frac{a \sin\theta}{(1 - \beta(T)\cos\theta)^3} \tag{18}$$

So the energy per time per solid angle

$$\frac{dW}{dt d\Omega} = \lim_{r\to\infty} c|r\boldsymbol{E}|^2 \tag{19}$$

$$= \frac{e}{(4\pi)^2 c^3} \frac{a^2 \sin^2\theta}{(1 - \beta(T)\cos\theta)^6} \tag{20}$$

where $a = v_o/\tau$, and $\beta(T) = \beta_o(1 - T/\tau)$. The energy *per retarded time* per solid angle is

$$\frac{dW}{dT d\Omega} = \frac{dW}{dt d\Omega}\frac{dt}{dT} \tag{21}$$

$$= \frac{e^2}{(4\pi)^2 c^3} \frac{a^2 \sin^2\theta}{(1 - \beta(T)\cos\theta)^5} \tag{22}$$

The energy per time is useful if you want to know whether a remote detector will burn up. The energy per retarded time is useful if you want to calculate how much energy is lost to radiation over a given element of a particles trajectory, $d\boldsymbol{x} = \boldsymbol{v}(T)dT$.

(d) We see that the denominator function, $1 - \beta_o \cos\theta$, is approaching zero at small angle since $\beta_o \simeq 1$. Expanding $\beta_o \simeq 1 - \frac{1}{2\gamma^2}$ and $\cos\theta \simeq 1 - \frac{\theta^2}{2}$,

$$\frac{1}{1 - \boldsymbol{n}\cdot\boldsymbol{\beta}} \simeq \frac{1}{\frac{1}{2\gamma_o^2} + \frac{\theta^2}{2}} = \frac{2\gamma_o^2}{1 + (\gamma_o\theta)^2}\,. \tag{23}$$

So the characteristic angle is $\theta \sim 1/\gamma_o$. For a 1 GeV electron, $\gamma \simeq E/m_e c^2 \sim 2000$. So $\theta \sim 1/2000$.

(e) The total energy is

$$\frac{dW}{d\Omega} = \int_0^\tau dT \frac{dW}{dT d\Omega}\,. \tag{24}$$

So with the result of Eq. 21 we have

$$\frac{dW}{d\Omega} = \frac{e^2}{(4\pi)^2 c^3}(a^2 \sin^2\theta) \int_0^\tau dT \frac{1}{(1 - \beta_o(1 - \frac{T}{\tau})\cos\theta)^5}\,, \tag{25}$$

$$= \frac{e^2}{(4\pi)^2 c^3} \frac{\tau(a^2 \sin^2\theta)}{4\beta_o \cos\theta} \left[\frac{-1}{(1 - \beta_o(1 - \frac{T}{\tau})\cos\theta)^4}\right]_0^\tau, \tag{26}$$

$$= \frac{e^2}{(4\pi)^2 c^3} \frac{\tau(a^2 \sin^2\theta)}{4\beta_o \cos\theta} \left[\frac{1}{(1 - \beta_o\cos\theta)^4} - 1\right]. \tag{27}$$

In the ultra relativistic limit we have

$$\frac{1}{1-\beta_o \cos\theta} \simeq \frac{1}{\frac{1}{2\gamma_o^2}+\frac{\theta^2}{2}} = \frac{2\gamma_o^2}{1+(\gamma_o\theta)^2}\,, \tag{28}$$

and thus

$$\frac{dW}{d\Omega} \simeq \frac{e^2 a^2 \tau}{(4\pi)^2 c^3}\,4\gamma_o^2 \left[\frac{(\gamma_o\theta)^2}{(1+(\gamma_o\theta)^2)^4}\right]\,. \tag{29}$$

Electromagnetism 2

Induction and the energy in static magnetic fields

Consider a closed circuit of wire formed into a circular coil of n turns with radius a, resistance R, and self-inductance L. The coil rotates around the z-axis in a uniform magnetic field H directed along the x-axis (see below).

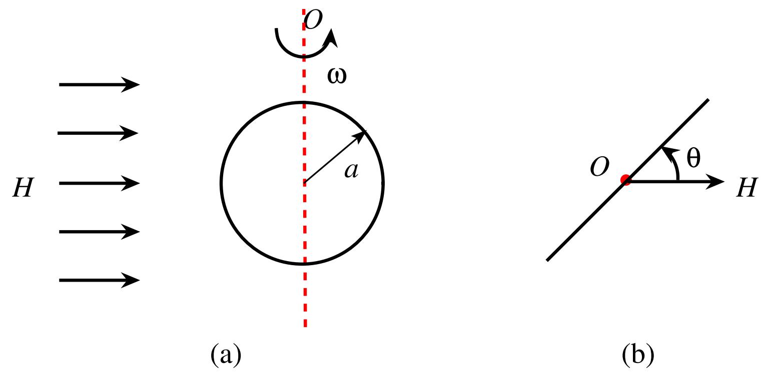


Figure 1: (a) side view; (b) top view.

a) (6 points) Find the current in the coil as a function of θ for rotation at a constant angular velocity ω. Here $\theta(t) = \omega t$ is the angle between the plane of the coil and H (the x-axis).

b) (4 points) Find the externally applied torque that is needed to maintain the coil's uniform rotation.

c) Because of the time-dependent currents induced in the coil, electromagnetic waves are radiated. Briefly answer the following questions:

(i) (2 points) What is the frequency of the radiation? Explain.

(ii) (2 points) What is the the polarization of the radiated waves propagating along the positive z-axis? Explain.

d) (6 points) Compute the total power radiated by the rotating coil of wire.

Note: in all parts you should assume that all transient effects have died away.

Solution:

a) Let I be the current in the coil, we have

$$\mathcal{E} = IR = -L\frac{dI}{dt} - \frac{1}{c}\frac{\partial \Phi_H}{\partial t}\,, \tag{1}$$

where the flux is given by $\Phi_H = \pi a^2\, n\, H \sin\theta(t)$ with $\theta(t) = \omega t$. With these phase conventions, the area vector of the loop points in the negative $\hat{\boldsymbol{y}}$ direction at $t = 0$ and in the $\hat{\boldsymbol{x}}$ direction at $\omega t = \pi/2$. Thus the circulation of a positive current at $t = 0$ is specified with the right hand rule with the thumb pointing in the negative $\hat{\boldsymbol{y}}$ direction.

From Eq. (1), we have the differential equation for the current,

$$L\,\frac{dI}{dt} + R\,I = -\frac{\pi a^2}{c} n\, H\, \omega\, \cos(\omega t)\,, \tag{2}$$

and corresponding solution as

$$\begin{aligned} I(t) &= -\frac{\pi a^2\, n\, H\, \omega}{c}\,\frac{1}{2}\left[\frac{e^{i\omega t}}{R + i\omega L} + \frac{e^{-i\omega t}}{R - i\omega L}\right] \\ &= -\frac{\pi a^2\, n\, H}{c}\,\frac{\omega}{\sqrt{R^2 + \omega^2 L^2}}\,\cos(\omega t + \phi)\,, \end{aligned} \tag{3}$$

where the phase $\phi = \tan^{-1}(-\omega L/R)$.

b) The rotating coil has a magnetic dipole moment, $\boldsymbol{\mu}(t) = I(t)\vec{A}(t)/c$. With the conventions of the previous part we have

$$\boldsymbol{\mu}(t) = m_o \cos(\omega t + \phi)\, \left(-\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}}\right)\,. \tag{4}$$

where

$$m_o \equiv \left(\frac{\pi a^2 n}{c}\right)^2 \frac{\omega}{\sqrt{R^2 + \omega^2 L^2}} H\,. \tag{5}$$

The torque on the loop is $\boldsymbol{\mu} \times \mathbf{H}$, and an external torque of $\boldsymbol{\tau}_{\rm ext} = -\boldsymbol{\mu} \times \mathbf{H}$ is needed to keep the coil rotating at a constant angular velocity is (with $\mathbf{H} = H\hat{\boldsymbol{x}}$) :

$$\boldsymbol{\tau}_{\rm ext}(t) = m_o H \cos(\omega t + \phi)\cos(\omega t)\,\hat{\boldsymbol{z}} \tag{6}$$

c)

(i) From the solution in part (b), we see that the current induces a magnetic moment which is oscillating in time. Writing the magnetic moment in complex notation (with the understanding that the physical quantity corresponds to the real part) we see that

$$\boldsymbol{\mu}(t) = \frac{1}{2} m_o e^{-2i\omega t - i\phi}\,(-i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) + \text{const}\,, \tag{7}$$

where we have neglected a constant vector,

$$\text{const} = \frac{m_o}{2} e^{i\phi} \left(-i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} \right) , \tag{8}$$

which does not contribute to the radiation. We see that the frequency of the radiation is 2ω.

(ii) The polarization must be transverse to the $\hat{\boldsymbol{z}}$, *i.e.* $\hat{\boldsymbol{z}} \cdot \boldsymbol{\epsilon}^* = 0$. Given the fact that the rotating magnetic moment does not prefer the x or y axes, the only two possible choices are right or left handed circularly polarized light. The magnetic moment is rotating around the z axis according to the right hand rule and the magnetic field will follow this orientation. Thus the light traveling on the z-axis will be circularly polarized with positive helicity (*i.e.* right handed).

d) A general formula for magnetic dipole radiation for a harmonic dipole moment $\boldsymbol{\mu}(t) = \boldsymbol{m} e^{-i\omega t}$ is

$$P = \frac{1}{4\pi} |\boldsymbol{m}|^2 \frac{\omega^4}{3c^3} . \tag{9}$$

with $\boldsymbol{m}$ a complex vector.

Adapting this formula to the problem at hand we have the replacements

$$\omega \to 2\omega \qquad \boldsymbol{m} \to \frac{m_o}{2} (-i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) , \tag{10}$$

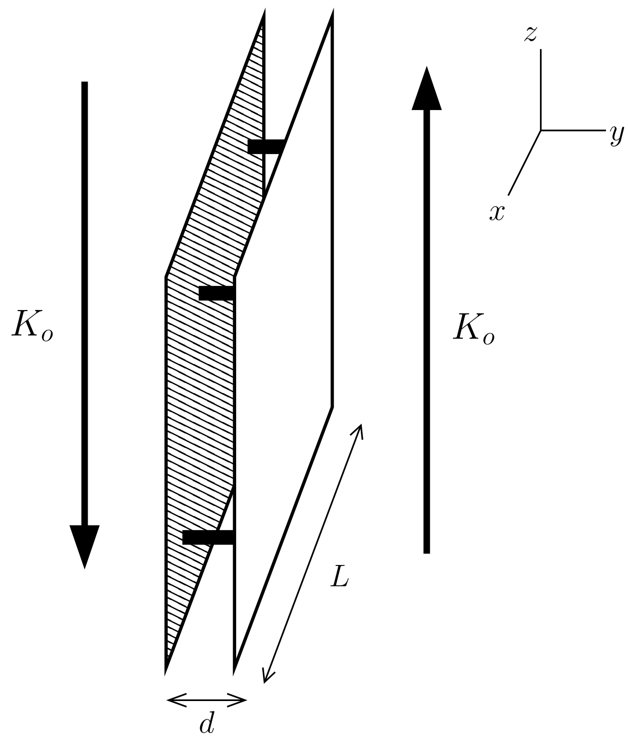
yielding

$$P = \frac{2}{\pi} m_o^2 \frac{\omega^4}{3c^3} \tag{11}$$

Electromagnetism 3

Two current sheets under Lorentz boosts

Consider two large square sheets of conducting material (with sides of length L separated by a distance d, $d \ll L$) each carrying a uniform surface current of magnitude K_o. (The total current in each sheet is $I_o = K_o L$.) The current flows up the right sheet and returns down the left sheet. The mass of the sheets is negligible. The sheets are mechanically supported by four electrically neutral columns of mass M_{col} and cross sectional area A_{col} (three shown). Neglect all fringing fields.



(a) (3 points) Write down the electromagnetic stress tensor $\Theta^{\mu\nu}_{\text{em}}$ covariantly in terms of $F^{\mu\nu}$ and compute all non-vanishing components of $F^{\mu\nu}$ and $\Theta^{\mu\nu}_{\text{em}}$ both in between and outside of the two sheets.

(b) (1 point) Compute the total rest energy of the system (or $M_{\text{tot}}c^2$) including the contribution from the electromagnetic energy.

(c) (3 points) Determine the electromagnetic force per area on the current sheets (magnitude and direction) and the components of the mechanical stress tensor in the columns, $\Theta^{00}_{\text{mech}}$ and $\Theta^{yy}_{\text{mech}}$ (use the coordinates system in the figure). You can assume that the stress is constant across the cross sectional area of the columns.

(d) (6 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/c$ up the z-axis.

(i) Determine the electric and magnetic fields (magnitudes and directions) using a Lorentz transformation. Check that the direction of the Poynting vector measured by this observer is consistent with physical intuition.

(ii) Determine the charge and current densities in the sheets according to this observer. Are your charges and currents consistent with the fields computed in the first part of (d)? Explain.

(e) (7 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/z$ to the *right* along the y-axis (use the coordinate system shown in the figure).

(i) Determine the total mechanical energy in the columns according to this observer.

(ii) Determine the total electromagnetic energy according to this observer.

(iii) Determine the total energy of this configuration. Is your result for the total energy consistent with part (b)? Explain.

Comment: There is of course stress in the sheets. But, since it does not have a yy component the stress in the sheets can be neglected in this problem.

Solution

(a) The stress tensor is

$$\Theta^{\mu\nu}_{\rm em} = F^{\mu\alpha} F^{\nu}_{\ \alpha} + \eta^{\mu\nu}\left(-\tfrac{1}{4}F^2\right) \, . \tag{1}$$

The only nonzero field component is the x component of the magnetic field. Using boundary conditions or Ampère's rule

$$\boldsymbol{n} \times (\boldsymbol{B}_{\rm out} - \boldsymbol{B}_{\rm in}) = \frac{K_o}{c}\,\hat{\boldsymbol{z}} \, , \tag{2}$$

we find

$$B_x = \frac{K_o}{c} \, , \tag{3}$$

in between the sheets and zero outside the sheets. Thus only non-zero component of $F^{\mu\nu}$ is

$$F^{23} = \frac{K_o}{c} \, . \tag{4}$$

The non-zero temporal components of $\Theta^{\mu\nu}_{\rm em}$ are

$$\Theta^{00}_{\rm em} = \tfrac{1}{2}B^2 = \tfrac{1}{2}(K_o/c)^2 \tag{5}$$

The spatial components of $\Theta^{\mu\nu}$ are expressed in terms of the magnetic fields as:

$$\Theta^{ij}_{\rm em} = -B^i B^j + \frac{\delta^{ij}}{2} B^2 \, . \tag{6}$$

So the non-zero spatial components are

$$-\Theta^{xx}_{\rm em} = \Theta^{yy}_{\rm em} = \Theta^{zz}_{\rm em} = \tfrac{1}{2}(K_o/c)^2 \, . \tag{7}$$

(b) The total energy is a sum of the rest energy of the columns and the electromagnetic energy (the energy density in Eq. (5) times the volume)

$$M_{\rm tot}c^2 = 4M_{\rm col}c^2 + \left[L^2 d\, \tfrac{1}{2}(K_o/c)^2\right] \tag{8}$$

(c) The force per area on the sheets is the discontinuity in the stress tensor. For a normal n_i pointing from "in" to "out" the force is

$$\frac{F^j}{A} = -n_i(\Theta^{ij}_{\rm out} - \Theta^{ij}_{\rm in}) \, , \tag{9}$$

and therefore, for the problem at hand, the electromagnetic force per area is

$$\left(\frac{F^y}{A}\right) = \Theta^{yy}_{\rm em} = \tfrac{1}{2}(K_o/c)^2\,. \tag{10}$$

This is the force per area on the right sheet and is directed outward. The force per area on the left sheet is also directed outward

$$\left(\frac{F^y}{A}\right) = -\tfrac{1}{2}(K_o/c)^2\,. \tag{11}$$

Note: the is exactly *half* of what would get for surface current in a uniform magnetic field of K_o/c (the field in between the sheets). Indeed, the force on the currents in the right sheet can be interpreted as arising from the fields generated by the currents in the left sheet. This left-sheet-generated field strength is $\frac{1}{2}K_o/c$.

The net total force on the sheets is zero (otherwise the configuration would not be stable). Thus, the electromagnetic force is balanced by the mechanical forces in the columns. The mechanical force per area in the four columns is therefore

$$\Theta^{yy}_{\rm mech} = -\frac{\frac{1}{2}L^2(K_o/c)^2}{4A_{\rm col}}\,, \tag{12}$$

where the factor of four accounts for the four columns. The mechanical energy density in the columns is

$$\Theta^{00}_{\rm mech} = \frac{M_{\rm col}c^2}{A_{\rm col}d}\,. \tag{13}$$

(d) Now we will boost the configuration. $\boldsymbol{\beta}$ is the velocity of the new observer, $\boldsymbol{\beta} = \beta\hat{\boldsymbol{z}}$.

(i) To determine the boosted fields we note the transformation rules

$$\underline{E}_{\parallel} = E_{\parallel}\,, \tag{14}$$
$$\underline{\boldsymbol{E}}_{\perp} = \gamma\boldsymbol{E}_{\perp} + \gamma\boldsymbol{\beta}\times\boldsymbol{B}_{\perp}\,, \tag{15}$$

and

$$\underline{B}_{\parallel} = B_{\parallel}\,, \tag{16}$$
$$\underline{\boldsymbol{B}}_{\perp} = \gamma\boldsymbol{B}_{\perp} - \gamma\boldsymbol{\beta}\times\boldsymbol{E}_{\perp}\,, \tag{17}$$

and thus in this case we have

$$\underline{E}^y = \gamma\beta(K_o/c)\,, \tag{18}$$
$$\underline{B}^x = \gamma(K_o/c)\,. \tag{19}$$

The direction of $\underline{\boldsymbol{E}}\times\underline{\boldsymbol{B}}$ is in the negative z direction. This makes sense – according to an observer moving the positive z direction the fields have a net momentum in the negative z direction.

(ii) To boost the currents we first record the four components of the current of the right sheet in the original frame

$$J^\mu = (J^0, J^x, J^y, J^z) = (0, 0, 0, K_o/\Delta)\,, \tag{20}$$

where Δ is the infinitesimal width of the sheets. J^0 is proportional to the surface charge density σ:

$$J^0 = \sigma c/\Delta\,, \tag{21}$$

and is zero in the original frame. Under boost we have

$$\underline{J}^\mu = L^\mu_{\ \nu} J^\nu\,. \tag{22}$$

This, together with the entries of the boost matrix

$$L^\mu_{\ \nu} = \begin{pmatrix} \gamma & & & -\gamma\beta \\ & 1 & & \\ & & 1 & \\ -\gamma\beta & & & \gamma \end{pmatrix}\,, \tag{23}$$

yields for the right sheet

$$\underline{\sigma} = -\,\gamma\beta K_o/c\,, \tag{24}$$

$$\underline{K}^z/c = \gamma K_{o/c}\,. \tag{25}$$

The left sheet has $J^z = -K_o/(c\Delta)$ and therefore the boosted charges and currents differ in sign

$$\underline{\sigma} = +\,\gamma\beta K_o/c\,, \tag{26}$$

$$\underline{K}^z/c = -\,\gamma K_{o/c}\,. \tag{27}$$

We can check our result by recognizing that the electric field in the y direction in the boosted frame that of a parallel plate capacitor with surface charges $+\underline{\sigma}$ and $-\underline{\sigma}$ on the left and right sheets:

$$\underline{E}^y = \underline{\sigma} = \gamma\beta K_o/c\,. \tag{28}$$

This agrees with the first part of (d). The magnetic field in the x direction is similarly

$$\underline{B}^x = \underline{K}^z/c = \gamma K_o/c\,, \tag{29}$$

and also agrees with the first part of (d).

(e) We will now compute the total energy in the boosted frame, $\boldsymbol{\beta} = \hat{\boldsymbol{y}}$. It is important to recognize that the mechanical stress tensor must also be boosted according to the general rule:

$$\underline{\Theta}^{\mu\nu} = L^{\mu}_{\ \rho} L^{\nu}_{\ \sigma} \Theta^{\rho\sigma} \tag{30}$$

(i) The energy density in the columns is

$$\underline{\Theta}^{00}_{\rm mech} = \gamma^2 \Theta^{00}_{\rm mech} + (-\gamma\beta)^2 \Theta^{yy}_{\rm mech} \tag{31}$$

Integrating over the volume of the columns we find the total energy density. In this integration the separation between the sheets is length contracted $d \to d/\gamma$ yielding for the four columns

$$\int_V d^3\underline{\boldsymbol{r}}\, \underline{\Theta}^{00}_{\rm mech} = A_{\rm col} \frac{d}{\gamma} \left[4\gamma^2 \frac{M_{\rm col} c^2}{d A_{\rm col}} - 4\gamma^2\beta^2 \frac{\frac{1}{2}(K_o/c)^2 L^2}{4 A_{\rm col}} \right] \tag{32}$$

$$= 4\gamma M_{col} c^2 - \gamma\beta^2 \left[L^2 d \tfrac{1}{2} (K_o/c)^2 \right] \tag{33}$$

(ii) The electromagnetic stress follows from the transformed fields:

$$\underline{B}^x = \gamma B^x = \gamma \frac{K_o}{c}\,, \tag{34}$$

$$\underline{E}^z = -\gamma\beta B^x = -\gamma\beta \frac{K_o}{c}\,. \tag{35}$$

So the electromagnetic energy density in between the sheets is

$$\underline{\Theta}^{00} = \tfrac{1}{2}(\underline{\boldsymbol{E}}^2 + \underline{\boldsymbol{B}}^2)\,, \tag{36}$$

$$= \tfrac{1}{2}(K_o/c)(\gamma^2\beta^2 + \gamma^2)\,, \tag{37}$$

and the total electromagnetic energy is therefore

$$\int_V d^3\underline{\boldsymbol{r}}\, \underline{\Theta}^{00}_{\rm em} = \left[L^2 d \tfrac{1}{2} (K_o/c)^2 \right] (\gamma + \gamma\beta^2)\,. \tag{38}$$

(iii) Adding the two contributions, the terms proportional to $[L^2 d \frac{1}{2}(K_o/c)^2]\gamma\beta^2$ cancel, and we find

$$\int_V d^3\underline{\boldsymbol{r}}\, \underline{\Theta}^{00}_{\rm tot} = \gamma \left(4 M_{\rm col} c^2 + \left[L^2 d \tfrac{1}{2} (K_o/c)^2 \right] \right)\,. \tag{39}$$

This, as expected, is simply

$$\gamma M_{\rm tot} c^2\,, \tag{40}$$

where $M_{\rm tot} c^2$ was the rest energy computed in part (b).

Quantum Mechanics 1

Interaction of two nucleons

The Schrödinger equation for the interaction of two nucleons can be reduced to the form:

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2}+\left[V(r)+\frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]u(r)=Eu(r) \tag{1}$$

where $u(r)=r\psi(r)$, r is a separation between the proton and the neutron, and the m is the reduced mass of the neutron-proton system. Use: $M_p = 938$ MeV/c^2, $M_n = 939$ MeV/c^2, $\hbar c = 197$ eV$\cdot$ nm, 1 barn $= 10^{-28}\text{m}^2$.

a) (5 points) The deuteron is a bound state of a proton and a neutron which are primarily in an orbital s-wave with total angular momentum $J = 1$ and total spin $S = 1$. The deuteron potential can be approximated as a three-dimensional, spherically symmetric, square-well:

$$V(r)=\begin{cases}-V_0 & \text{for } r<R\\ 0 & \text{for } r\geq R\end{cases} \tag{2}$$

Given the (small!) deuteron binding energy $E \simeq -2\,\text{MeV}$ and the potential range $R \simeq 2$ fm ($\sim$diameter of a deuteron), find an equation for the depth of the potential V_o. By analyzing this equation, show that the depth V_o must be greater than about 25 MeV. (Hint: $\tan\theta$ changes sign at $\pi/2$.)

An exact calculation of the potential depth in part (a) yields $V_0 \simeq 35$ MeV.

b) (10 points) In low energy neutron-proton scattering (incident neutron energy $E_o \leq$ 10 keV), one can use the same potential (Eq. 2) as for the deuteron state in part a). Find the wave function $u(r)$ and determine a formula for the neutron-proton scattering cross section. Which partial wave dominates the cross section? Hint: The wave function at large distances ($r \to \infty$) can be represented as a superposition of an incident wave and a scattered wave:

$$\psi(r,\theta,\phi)\sim e^{ikz}+f(\theta,\phi)\frac{e^{ikr}}{r}, \qquad \text{where } f(\theta,\phi)=\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos\theta),$$

with $k\equiv\sqrt{(2mE/\hbar^2)}$, and the normalization of the Legendre polynomials is:

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta=\frac{2}{2\ell+1}\delta_{\ell\ell'}.$$

c) (5 points) At low energies the experimental value of the unpolarized neutron-proton cross section is $\sigma \simeq 20$ barns (Fig. 1), while an analysis of part b) yields a cross section of 4.5 barns. What is the explanation for the discrepancy between the calculated value of the cross section in part b) and the experimental value? Make a specific prediction for a proton-neutron cross section in different spin channels which can be checked experimentally. Hint: Use the fact that that the deuteron has total spin $S = 1$.

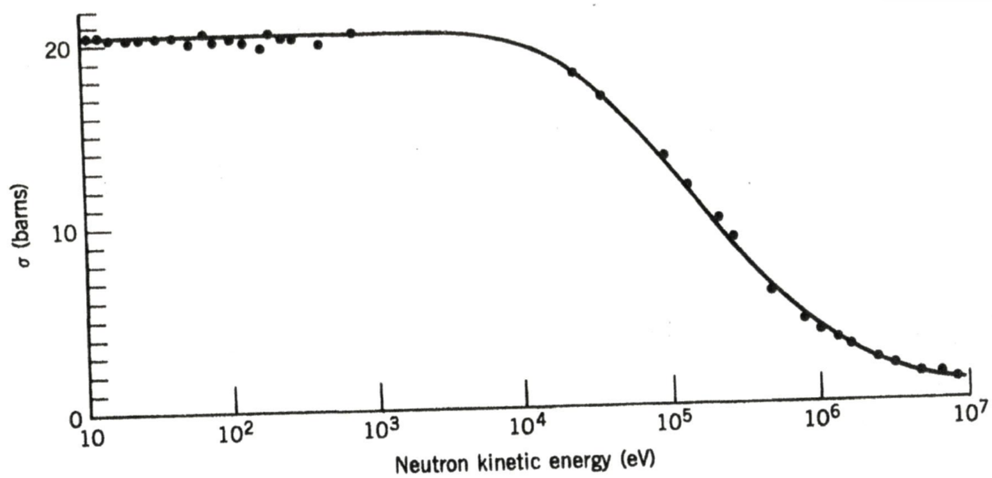


Figure 1: The neutron-proton scattering at low energy. Data taken from R. K. Adair, Rev. Mod. Phys. **22**, 249 (1950) and T. L. Houk, Phys. Rev. C **3**, 1886 (1970).

Solution

a) Assuming $l = 0$ (S-state only) Eq. 1 becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} = [E + V_0]\,u(r) \quad \text{for r} < R$$

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} = Eu(r) \quad \text{for r} \geq R$$

The functions

$$u(r) = \begin{cases} A\sin(k_1 r) + B\cos(k_1 r) & \text{for } r < R, \text{ with } k_1 \equiv \sqrt{2m(E+V_0)/\hbar^2} \\ Ce^{-k_2 r} + De^{k_2 r} & \text{for } r > R, \text{ with } k_2 \equiv \sqrt{-2mE/\hbar^2} \end{cases}, \tag{3}$$

are the solutions of the Schrödinger equation. From the boundary conditions we find: $B = 0, D = 0$ and $k_1 \cot(k_1 R) = -k_2$. Thus we need to solve

$$\frac{k_1}{k_2} = -\tan(k_1 R) \tag{4}$$

with k_2 fixed. This equation determines k_1 and hence V_0. Plotting $-\tan(k_1 R)$ and $k_1 R/k_2 R$ (note $k_2 R$ is small) we see that the first instersection must happen for

$$k_1 R > \frac{\pi}{2}\,. \tag{5}$$

This implies that

$$V_0 > \frac{\hbar^2}{2mR^2}\left(\frac{\pi}{2}\right)^2\,. \tag{6}$$

Substituting numbers we have

$$V_0 > 25.6\,\text{MeV}\,. \tag{7}$$

b) In low energy nucleon-nucleon scattering ($0 < E_o < 10$ keV) we can assume $\ell = 0$. Eq. 1 becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} = [E + V_0]\,u(r) \quad \text{for r} < R\,,$$

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} = Eu(r) \quad \text{for r} \geq R\,.$$

We assume that the wave function inside the well is identical with that in the deuteron problem. This is justified since the total energy inside the potential well is raised by little over 2 MeV corresponding to the binding energy of the deuteron, which is much smaller than the well depth of $V_0 = 35$ MeV. The functions

$$u(r) = \begin{cases} A\sin(k_1 r) + B\cos(k_1 r) & \text{for } r < R, \text{ with } k_1 \equiv \sqrt{2m(E+V_0)/\hbar^2} \\ C\sin(k_2 r + \delta_0) & \text{for } r > R, \text{ with } k_2 \equiv \sqrt{2mE/\hbar^2} \end{cases},$$

are the solutions of the Schroedinger equation. From the boundary conditions we find $B = 0$ and

$$k_2 \cot(k_2 R + \delta_0) = k_1 \cot(k_1 R) \equiv -\alpha \tag{8}$$

Then manipulating Eq. (8)

$$\frac{\cos(k_2 R)\cos(\delta_o) - \sin(k_2 R)\sin(\delta_o)}{\sin(k_2 R)\cos(\delta_o) + \cos(k_2 R)\sin(\delta_o)} = -\frac{\alpha}{k_2} \tag{9}$$

we solve for $\sin(\delta_o)^2$

$$\sin^2 \delta_0 = \frac{(\cos(k_2 R) + (\alpha/k_2)\sin(k_2 R))^2}{1 + (\alpha/k_2)^2}..$$

Standard analysis gives

$$f(\theta, \phi) = \frac{e^{i\delta_0} \sin \delta_0}{k_2}$$

$$\frac{d\sigma(\theta, \phi)}{d\Omega} = |f(\theta, \phi)|^2 \Longrightarrow \sigma = \frac{4\pi \sin^2 \delta_0}{k_2^2}$$

The cross section σ becomes:

$$\sigma = \frac{4\pi}{k_2^2} \times \frac{(\cos(k_2 R) + (\alpha/k_2)\sin(k_2 R))^2}{1 + (\alpha/k_2)^2} \tag{10}$$

$$\simeq \frac{4\pi}{\alpha^2}(1 + \alpha R)^2 \tag{11}$$

In the last approximation we studied Eq. (8) to recognize that $\alpha/k_2 \gg 1$ and $k_2 R \ll 1$. For this problem,

$$k_1 = (2mV_o/\hbar^2)^{1/2} \simeq 0.92\,\text{fm} \tag{12}$$

$$\alpha = -k_1 \cot(k_1 R) \simeq 0.25\,\text{fm}^{-1} \tag{13}$$

$$R \simeq 2.0\,\text{fm} \tag{14}$$

and we find

$$\sigma \simeq 4.5\,\text{barns}\,. \tag{15}$$

c) At low energies, the neutron-proton system can form either a spin triplet or a spin singlet state. Since the potential value $V_0 = 35$ MeV used in b) corresponds to deuteron (spin=1, spin triplet state), the value of the cross section of ~ 4 barn corresponds to the neutron-proton spin-triplet cross section ($\sigma_t \simeq 4$barn). The cross section shown in Fig. 1, is a weighted sum of spin-singlet and spin-triplet state:

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s \simeq 20\,\text{barn (measured)},$$

where 3/4 and 1/4 are probabilities of neutron-proton being in a spin-triplet and spin- singlet states. It follows that

$$\sigma_s = 4\sigma - 3\sigma_t \sim 68\,\text{barn}$$

indicating large differences between the low energy neutron-proton scattering cross sections for the spin-triplet and the spin-singlet states.

Quantum Mechanics 2

An oscillator in an electric field

A particle of mass m and electric charge q moves in 1-dimension under the effects of a a harmonic potential and a homogeneous electrostatic field $\mathcal{E}$. The Hamiltonian for the system is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - q\mathcal{E}x = H_0 - q\mathcal{E}x \tag{1}$$

1. (4 points) Show that H can be written as $H = e^{-A}H_0 e^{A} + B$ by explicitly determining the two operators A, B. Use this to show that the spectrum of H follows from that of H_0 by a shift operator. Use this observation to solve the eigenvalue problem.
2. (4 points) Express A in terms of the creation a and annihilation $a^\dagger$ operator of H_0. Use this to evaluate the probability to find the system in the ground state of H at time t if at time $t = 0$ it is in the ground state of H_0.
3. (4 points) What is the probability for the system to start at $t = 0$ in the ground state of H_0 and remain in this state at time t? For what time this probability is 1?
4. (4 points) Repeat **3** but now for the system to be found in the first excited state of H_0. Comment physically on the similarities and differences between **3** and **4**.
5. (4 points) Express the dipole moment $d = qx$ in terms of $a, a^\dagger$. Use this to calculate the mean value of the dipole moment $d = qx$ at time t, assuming that at $t = 0$ the system is again in the ground state of H_0.

Solution

1. The Hamitonian can be re-written as

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x - \frac{q\mathcal{E}}{m\omega^2}\right) - \frac{q^2\mathcal{E}^2}{2m\omega^2} \equiv e^{-ilp/\hbar}H_0 e^{ilp/\hbar} - \frac{q^2\mathcal{E}^2}{2m\omega^2} \tag{2}$$

with $l = q\mathcal{E}/m\omega^2$. The spectrum of H follows from that of H_0 by a shift. Thus the eigenstates $|\bar{n}\rangle$ are shifted harmonic oscillators $|\bar{n}\rangle = e^{-ilp/\hbar}|n\rangle$ with shifted energies

$$\bar{E}_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{1}{2}m\omega^2 l^2 \tag{3}$$

2. The probability is

$$P_{0\bar{0}}(t) = |\langle 0(0)|\bar{0}(t)\rangle|^2 = \left|\langle 0|e^{-ilp/\hbar}|\bar{0}\rangle\right|^2 \tag{4}$$

Since $p = -i\sqrt{m\hbar\omega/2}(a - a^\dagger)$ and since $[a, a^\dagger] = 1$ we can use the identity

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}e^{-[\mathbf{A},\mathbf{B}]/2} \tag{5}$$

to unwind (4)

$$\langle 0|e^{-ilp/\hbar}|\bar{0}\rangle = \left\langle 0\left|e^{l\sqrt{\frac{m\omega}{2\hbar}}a^\dagger}e^{-l\sqrt{\frac{m\omega}{2\hbar}}a}e^{-m\omega l^2/4\hbar}\right|0\right\rangle \tag{6}$$

Thus

$$P_{0\bar{0}}(t) = e^{-m\omega l^2/2\hbar} \tag{7}$$

3. The exact eigenstate of H_0 at any time t is

$$|\varphi(t)\rangle = e^{-iHt/\hbar}|0\rangle = \sum_{n=0}^{\infty} e^{-i\bar{E}_n t/\hbar}\langle\bar{n}|0\rangle\ |\bar{n}\rangle \tag{8}$$

with

$$\begin{aligned}
\langle 0|\bar{n}\rangle &= \left\langle 0\left|e^{l\sqrt{\frac{m\omega}{2\hbar}}a^\dagger}e^{-l\sqrt{\frac{m\omega}{2\hbar}}a}e^{-m\omega l^2/4\hbar}\right|n\right\rangle \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-l\sqrt{\frac{m\omega}{2\hbar}}\right)^k \langle 0|a^k|n\rangle\, e^{-m\omega l^2/4\hbar} \\
&= \frac{(-1)^n}{\sqrt{n!}}\left(l\sqrt{\frac{m\omega}{2\hbar}}\right)^n e^{-m\omega l^2/4\hbar}
\end{aligned} \tag{9}$$

The probability amplitude to remain in $|0\rangle$ is

$$\langle 0|\varphi(t)\rangle = e^{-i\omega t/2} e^{-m\omega l^2/2\hbar} e^{im\omega^2 l^2 t/2\hbar} e^{m\omega l^2 e^{-i\omega t}/2\hbar} \tag{10}$$

and the probability to remain in the ground state of H_0 is

$$P_{00}(t) = |\langle 0|\varphi(t)\rangle|^2 = e^{-\frac{2}{\hbar} m\omega l^2 \sin^2 \frac{\omega t}{2}} \tag{11}$$

The probability is 1 for $\omega t/2 = \pi \bmod \pi$.

4. The probability is now given by

$$P_{10}(t) = |\langle 1|\varphi(t)\rangle|^2 \tag{12}$$

with

$$\langle 1|\varphi(t)\rangle = \sum_{n=0}^{\infty} e^{-i\bar{E}_n t/\hbar} \langle \bar{n}|0\rangle \langle 1|\bar{n}\rangle \tag{13}$$

using

$$\langle 1|\bar{n}\rangle = \left\langle 0|a e^{-ilp/\hbar}|n\right\rangle = \left\langle 0|e^{-ilp/\hbar} a|n\right\rangle + l\sqrt{\frac{m\omega}{2\hbar}} \left\langle 0|e^{-ilp/\hbar}|n\right\rangle \tag{14}$$

and after a short algebra we have

$$P_{1\bar{0}}(t) = \left(\frac{2}{\hbar} m\omega l^2\right) \left(\sin^2 \frac{\omega t}{2}\right) P_{0\bar{0}}(t) \tag{15}$$

which is seen to vanish when $P_{0\bar{0}}(t)$ is maximum due to the nodal form of the first excited state.

5. The expectation value after some algebra is

$$d(t) = \langle \varphi(t)|qx|\varphi(t)\rangle = 2ql \left(\sin^2 \frac{\omega t}{2}\right) \tag{16}$$

Quantum Mechanics 3

Approximations for a quartic potential

Consider a particle with mass m in a one-dimensional quartic potential $V = \beta x^4$ where β is a positive constant.

(a) (2 points) Use dimensional analysis to determine how the eigenstate energies depend on β. (Hint: write the Schrödinger equation in terms of dimensionless variables

$$\left(-\frac{1}{2}\frac{d^2}{d\bar{x}^2} + \bar{x}^4\right)\psi = \epsilon\psi\,,$$

where $\bar{x}$ is a suitably rescaled coordinate.)

(b) (4 points) Calculate the eigenstate energies E_n with $n = 0, 1, 2, \cdots$ in the WKB approximation. Compare the WKB spectrum of this quartic anharmonic oscillator with the spectrum of the harmonic oscillator and the particle in a box.

(c) (2 points) For which values of n is the WKB method most accurate?

(d) (6 points) Approximate the energy E_0 of the ground state of the βx^4 anharmonic oscillator by applying the variational method with Gaussian wave function $\psi_0 = Ce^{-x^2/\lambda^2}$ where λ is a real variable parameter.

(e) (4 points) Do the results obtained in part (d) satisfy the virial theorem? Explain. Do the variational method and/or the WKB method provide upper and/or lower bounds on the ground state energy?

(f) (2 points) Write down a wave function that can be used for the variational method to obtain an approximate value of the energy E_1 of the first excited state of the quartic anharmonic oscillator.

You may use the following integral:

$$\int_0^1 \mathrm{d}x\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,. \tag{1}$$

Here the gamma function satisfies the recursion relation $\Gamma(z+1) = z\Gamma(z)$, with representative values:

$$\Gamma({}^1\!/\!{}_4) = 3.62561 \tag{2}$$
$$\Gamma({}^1\!/\!{}_2) = \sqrt{\pi} \tag{3}$$
$$\Gamma({}^3\!/\!{}_4) = 1.22542 \tag{4}$$
$$\Gamma(1) = 1 \tag{5}$$

You may also use the following results for Gaussian integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ax^2} x^n = \begin{cases} \sqrt{\frac{\pi}{a}} & \text{for } n = 0 \\ \sqrt{\frac{\pi}{a}}\left(\frac{1}{2a}\right) & \text{for } n = 2 \\ \sqrt{\frac{\pi}{a}}\left(\frac{3}{4a^2}\right) & \text{for } n = 4 \end{cases} \tag{6}$$

Solution

(a) We first define dimensionless variables to simplify the algebra. The Schrödinger equation reads

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \beta x^4\right]\psi = E\psi\,. \tag{7}$$

There is a length scale where the kinetic and potential terms are the same

$$\frac{\hbar^2}{mL^2} = \beta L^4\,, \tag{8}$$

or

$$L \equiv \left(\frac{\hbar^2}{m\beta}\right)^{1/6}\,. \tag{9}$$

Then introducing

$$\underline{x} \equiv x/L\,, \qquad \epsilon = E/(\hbar^2/mL^2)\,, \tag{10}$$

we write the Schrödinger equation in dimensionless form

$$\left[\frac{1}{2}\frac{-d^2}{d\underline{x}^2} + \underline{x}^4\right]\psi = \epsilon\psi\,. \tag{11}$$

We will stop writing the "bar" in $\underline{x}$ to lighten the notation below. From the scalings in Eq. 10 we see that $E \propto \beta^{1/3}$.

(b) For a given energy ϵ, we find the turning points at

$$\epsilon = x^4 \qquad x_{\pm} = \pm\epsilon^{1/4}\,. \tag{12}$$

Then the WKB solution quantization condition is

$$\int_{x_-}^{x+} p(\epsilon, x)dx = \left(n + \tfrac{1}{2}\right)\pi\,, \tag{13}$$

where $p(\epsilon, x)$ is the momentum. In the orginal variables the momentum is

$$p(E, x) = \sqrt{2m(E - V)}\,, \tag{14}$$

while in dimensionless form the momentum reads

$$p(\epsilon, x) = \sqrt{2(\epsilon - x^4)} \tag{15}$$

Thus we find

$$\int_{x_-}^{x^+} dx\,\sqrt{2(\epsilon - x^4)} = (n + \tfrac{1}{2})\pi \tag{16}$$

$$\epsilon^{3/4}\left[\sqrt{2}\int_{-1}^{1} du\,(1 - u^4)^{1/2}\right] = (n + \tfrac{1}{2})\pi \tag{17}$$

where $u = x/\epsilon^{1/4}$. Here the integral can be evaluated numerically if necessary and is of order one. In this case the integral is expressible in terms of the β function

$$C_o \equiv \sqrt{2}\int_{-1}^{1} du\,(1-u^4)^{1/2} \tag{18}$$

$$= \frac{2\sqrt{2}}{4}\int_{0}^{1} dx\,x^{-\frac{3}{4}}(1-x)^{1/2} \tag{19}$$

$$= \frac{1}{\sqrt{2}}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{4}+\frac{3}{2})} \tag{20}$$

$$= 2.4721 \tag{21}$$

So we find

$$\epsilon = \left(\frac{(n+\frac{1}{2})\pi}{C_o}\right)^{4/3} \tag{22}$$

Restoring units

$$E = \beta\left(\frac{\hbar^2}{m\beta}\right)^{2/3}\left(\frac{(n+\frac{1}{2})\pi}{C_o}\right)^{4/3} \tag{23}$$

Thus at large n

$$E \propto n^{4/3} \tag{24}$$

which is steeper than the harmonic oscillator $\epsilon \propto n$, but not as steep as the particle in the box $\epsilon \propto n^2$

(c) The WKB method works best at large n.

(d) Let us take a Gaussian as a variational ansatz

$$\psi = \frac{1}{(2\pi\sigma^2)^{1/4}}e^{-x^2/4\sigma^2}\,. \tag{25}$$

The constants are chosen so that

$$\psi^*\psi = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-x^2/2\sigma^2} \tag{26}$$

is a normalized gaussian.

Let us recall some properties of Gaussian integrals which help to simplify the algebra. First note

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\,\psi^*\psi\,x^2 = \sigma^2 \tag{27}$$

Then recall that a Gaussian is the unique minimum uncertainty wave packet,

$$\sqrt{\langle x^2\rangle\langle p^2\rangle} = \tfrac{1}{2} \tag{28}$$

So, we have without calculation

$$\int_{-\infty}^{\infty} dx\, \psi^* \left(-\frac{d^2\psi}{dx^2}\right) = \frac{1}{4\sigma^2} \tag{29}$$

Finally, let us recall a physically important fact of Gaussians. Namely, all cumulants higher than two vanish. For any probability distribution the fourth cumulant is

$$C_4 \equiv \langle x^4 \rangle - 3 \langle x^2 \rangle^2 \,. \tag{30}$$

Thus

$$\langle x^4 \rangle = 3 \langle x^2 \rangle^2 = 3(\sigma^2)^2 \,, \tag{31}$$

which can be verified by direct integration.

With this information the variational energy is

$$\epsilon(\sigma^2) = \left\langle -\frac{1}{2}\frac{d^2}{dx^2} + x^4 \right\rangle = \frac{1}{8\sigma^2} + 3(\sigma^2)^2 \,. \tag{32}$$

Differentiating with respect to σ^2 to minimize the variational energy,

$$\left.\frac{\partial \epsilon(\sigma^2)}{\partial \sigma^2}\right|_{\sigma^2_{\rm min}} = 0 \,, \tag{33}$$

we find

$$\sigma^2_{\rm min} = \frac{1}{\sqrt[3]{48}} \,. \tag{34}$$

Then the variational energy is

$$\epsilon(\sigma^2_{\rm min}) = \frac{1 + 24(\sigma^2)^3}{8\sigma^2} = \left(\frac{3}{4}\right)^{4/3} \,. \tag{35}$$

(e) The virial theorem says that for any eigenstate eigenstate

$$2 \langle \mathrm{KE} \rangle = \left\langle x \frac{\partial U}{\partial x} \right\rangle \,. \tag{36}$$

The theorem follows by considering the following expectation value in an energy eigenstate

$$\left\langle \frac{d(XP)}{dt} \right\rangle = \frac{-i}{\hbar} \langle [XP, H] \rangle = 0 \,. \tag{37}$$

Evaluating the commutator with $[XP, H] = [X, H]P + X[P, H]$ leads to Eq. 36.

For $U = x^4$ the virial theorem reads

$$2 \langle \mathrm{KE} \rangle = 4 \langle \mathrm{PE} \rangle \,. \tag{38}$$

For the variational eigenstate we see (by comparing the two terms in Eq. 32) that

$$\langle \mathrm{KE} \rangle = \frac{1}{24(\sigma^2)^3} \langle \mathrm{PE} \rangle \ . \tag{39}$$

Or, at minimum where $(\sigma^2)^3 = 1/48$

$$\langle \mathrm{KE} \rangle = 2 \langle \mathrm{PE} \rangle \ . \tag{40}$$

Thus the variational wave function satisfies the virial theorem for this particular case, but not in general.

The variational energy is an upper bound on the ground state energy. The variational energy is

$$\epsilon_{\mathrm{vary}} = 0.68142 \,, \tag{41}$$

while the exact result is found numerically

$$\epsilon_{\mathrm{numerical}} = 0.668 \,. \tag{42}$$

The exact result is below the variational energy by about two percent.

There is no rigorous statement about wether the WKB energy is an upper or lower bound. Comparing the WKB result for the ground state energy, using Eq. 22 with $n = 0$ we find,

$$\epsilon_{\mathrm{WKB}} = 0.546267 \,, \tag{43}$$

which is below the exact energy in this case.

(f) The variational functional form for the first excited state should be orthogonal to the ground state. In this case the symmetry under $x \to -x$ dictates that the ground state is even under interchange. So the first excited variational ansatz should be odd. The function

$$\psi_1 = C x e^{-x^2/\lambda^2} \tag{44}$$

is a natural choice.

Statistical Mechanics 1

Ultra-relativistic electron gas

Consider an ideal 3D gas of N ultra-relativistic electrons with energies $\varepsilon = pc$ (where $\mathbf{p}$ is electron's momentum, and c is the speed of light), confined to volume V.

(a) (3 points) For the gas in equilibrium at zero temperature, calculate its chemical potential μ (i.e. the Fermi energy ε_F) and the total energy E_0, and express E_0 in terms of N and ε_F.

(b) (6 points). Now consider the gas in equilibrium at a low temperature $T << \varepsilon_F/k_B$. In the first nonvanishing approximation in T, calculate the chemical potential, and express your result in terms of ε_F and T.

(c) (4 points) For the same conditions as in Task 2, calculate the specific heat (i.e. the heat capacity per particle) of the gas, and express it in terms of ε_F and T.

(d) (4 points) Obtain general expressions for the grand thermodynamic potential of the gas and its pressure, and express them via the total energy of the gas and its volume. Compare the result with that for an ideal gas of non-relativistic particles.

(e) (3 points) Express the gas pressure at $T = 0$ in terms of N and V.

Hint: You may find the following *Sommerfeld expansion* useful:

$$\int_0^\infty F(\varepsilon) f(\varepsilon) d\varepsilon = \int_0^\mu F(\varepsilon) d\varepsilon + \frac{\pi^2}{6}\left(k_B T^2\right) F'(\mu) + O\left(\frac{k_B T}{\mu}\right)^4,$$

where

$$f(\varepsilon) \equiv \frac{1}{\exp\{(\varepsilon - \mu)/k_B T\} + 1},$$

is the Fermi-Dirac distribution, $F(\varepsilon)$ is any differentiable function, growing slower than $1/f(\varepsilon)$ at $\varepsilon \to \infty$, and $F'(\varepsilon)$ is its derivative.

Solution

(a) Density $g(\varepsilon)$ of quantum states of a single electron maybe found from the standard state counting rule:

$$g(\varepsilon)d\varepsilon = 2V\frac{d^3p}{(2\pi\hbar)^3},$$

where the first factor 2 is due to the spin degeneracy. For an isotropic gas, $d^3p = 4\pi p^2 dp$, so that for an ultra-relativistic gas, with $p = \varepsilon/c$, we get

$$g(\varepsilon) = \frac{V\varepsilon^2}{\pi^2(\hbar c)^3}.$$

Since the electrons are Fermions, and obey the Pauli principle, at $T = 0$ each quantum state with energy ε below the Fermi energy ε_F is occupied with one electron, while all the states with $\varepsilon > \varepsilon_F$ are empty. Hence the total number of electrons may be expressed as

$$N = \int_0^{\varepsilon_F} g(\varepsilon)d\varepsilon = \frac{\varepsilon_F^3 V}{3\pi^2(\hbar c)^3}. \quad (1)$$

If N, rather than ε_F, is given, the Fermi energy may be expressed from this relation as

$$\varepsilon_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}\hbar c. \quad (2)$$

Now we may calculate the total energy of the electrons as

$$E_0 = \int_0^{\varepsilon_F} g(\varepsilon)\varepsilon d\varepsilon = \frac{\varepsilon_F^4 V}{4\pi^2(\hbar c)^3}. \quad (3)$$

Comparing Eqs. (1) and (3), we see that

$$E_0 = \frac{3}{4}N\varepsilon_F.$$

(b) At a nonvanishing but low temperature, we may use the Sommerfeld expression, with $F(\varepsilon)$ - $g(\varepsilon)$, so that

$$F'(\mu) = \frac{dg(\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=\mu} = V\frac{2\mu}{\pi^2(\hbar c)^3},$$

to calculate the generalization of Eq. (1):

$$N = \int_0^\infty g(\varepsilon)f(\varepsilon)d\varepsilon \approx \frac{V}{\pi^2(\hbar c)^3}\left[\int_0^\mu \varepsilon^2 d\varepsilon + \frac{\pi^2}{3}(k_B T)^2\mu\right] = \frac{V}{\pi^2(\hbar c)^3}\left[\frac{\mu^3}{3} + \frac{(\pi k_B T)^2\mu}{3}\right].$$

Plugging N from Eq. (1), we may recast this expression as

$$\frac{\varepsilon_F^3 - \mu^3}{3} \approx \frac{(\pi k_B T)^2\mu}{3}.$$

Since at $T \to 0$, $\mu \to \varepsilon_F$, we approximate the left-hand part of this expression as

$$\frac{\varepsilon_F^3 - \mu^3}{3} \approx \varepsilon_F^2(\varepsilon_F - \mu),$$

and replace μ in the (already small) right-hand part with just ε_F, getting

$$\mu \approx \varepsilon_F \left[1 - \frac{1}{3}\left(\frac{\pi k_B T}{\varepsilon_F} \right)^2 \right]. \qquad (4)$$

(c) The total energy of the gas,

$$E = \int_0^\infty g(\varepsilon)\varepsilon f(\varepsilon) d\varepsilon = \frac{V}{\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^3 f(\varepsilon) d\varepsilon \,, \qquad (5)$$

at low temperature may be calculated using the same Sommerfeld expansion, but now with $F(\varepsilon) = g(\varepsilon)\varepsilon$, so that

$$F'(\mu) = \frac{d[g(\varepsilon)\varepsilon]}{d\varepsilon}\Big|_{\varepsilon=\mu} = \frac{d}{d\varepsilon}\left(\frac{V\varepsilon^3}{\pi^2 (\hbar c)^3} \right)\Big|_{\varepsilon=\mu} = V\frac{3\mu^2}{\pi^2 (\hbar c)^3},$$

giving

$$E \approx \frac{V}{\pi^2 (\hbar c)^3} \left[\int_0^\mu \varepsilon^3 d\varepsilon + \frac{\pi^2}{2} (k_B T)^2 \mu^2 \right] = \frac{V}{\pi^2 (\hbar c)^3} \left[\frac{1}{4}\mu^4 + \frac{1}{2} (\pi k_B T)^2 \mu^2 \right].$$

Plugging into this expression μ from Eq. (4), leaving only two leading terms of the Taylor expansion in small $(\pi k_B T)^2$, and then using Eqs. (1) and (3), we get:

$$E \approx \frac{V}{\pi^2 (\hbar c)^3} \left\{ \frac{\varepsilon_F^4}{4} \left[1 - \frac{4}{3}\left(\frac{\pi k_B T}{\varepsilon_F} \right)^2 \right] + \frac{\varepsilon_F^4}{2} \left(\frac{\pi k_B T}{\varepsilon_F} \right)^2 \right\} = E_0 + \frac{N}{2\varepsilon_F} (\pi k_B T)^2 , \qquad (6)$$

so that the specific heat (the heat capacity per particle) is

$$c_V \equiv \frac{1}{N}\left(\frac{\partial E}{\partial T} \right)_V \approx \pi^2 \frac{k_B^2 T}{\varepsilon_F}.$$

Note that it is much smaller than $c_V \sim k_B$ of classical gases.

(d) The grand thermodynamic potential for each state of a Fermi gas is

$$\Omega(\varepsilon) = -k_B T \ln[1 + \exp\{(\mu - \varepsilon)/k_B T\}],$$

so that the potential of the whole gas may be calculated as

$$\Omega = \int_0^\infty g(\varepsilon)\Omega(\varepsilon) d\varepsilon = -\frac{k_B T V}{\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^2 \ln[1 + \exp\{(\mu - \varepsilon)/k_B T\}] d\varepsilon .$$

Integrating this expression by parts, we get

$$\Omega = -\frac{V}{3\pi^2 (\hbar c)^3} \int_0^\infty \frac{\varepsilon^3}{\exp\{(\varepsilon - \mu)/k_B T\} + 1} d\varepsilon \equiv -\frac{V}{3\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^3 f(\varepsilon) d\varepsilon.$$

But according to Eq. (5), this is just (-1/3)E. So, calculating pressure as

$$P = -\left(\frac{\partial \Omega}{\partial V} \right)_{T,N} = \frac{1}{3\pi^2 (\hbar c)^3} \int_0^\infty \varepsilon^3 f(\varepsilon) d\varepsilon = \frac{E}{3V},$$

we see that for any temperature the following exact relation holds:

$$PV = \frac{E}{3}. \qquad (7)$$

This product is twice smaller that that of the non-relativistic ideal gas (either classical or quantum), for which $PV = Nk_BT = (2/3)E$.

(e) Combining Eqs. (3) and (7), and then using Eq. (2), we get

$$P\Big|_{T=0} = \frac{\left(3\pi^2\right)^{1/3}}{4}\hbar c\left(\frac{N}{V}\right)^{4/3}.$$

This simple expression may be also obtained by counting the average number of particles hitting a container wall during a unit time, and the average momentum transferred to the wall at a single hit.

Statistical Mechanics 2

Magnetic refrigeration

An external magnetic field **B** is applied to a set of N non-interacting spin-½ particles with gyromagnetic ratio γ, and fixed spatial positions. For the thermal equilibrium at temperature T, calculate:

(a) (3 points) the average energy and heat capacity,

(b) (3 points) the average magnetic moment of the system and the variance of its fluctuations, and

(c) (4 points) the entropy per spin.

(d) (5 points). Sketch the temperature dependence of the entropy, for two substantially different field magnitudes, and discuss (qualitatively) what would happen with the entropy and the internal energy of the system if it is first thermally isolated from the environment, and then the applied field is turned off.

(e) (5 points). Suggest a way to use this system as a refrigerator, assuming that its thermal contacts with hot and cold heat baths, and the applied magnetic field, may be controlled at will.

Solution

(a) In the magnetic field, a spin ½ particle may have two energy values,

$$E_\pm = -m_\pm B = -\gamma S_\pm B = \mp \frac{\hbar\gamma B}{2},$$

where $m_\pm = \gamma S_\pm$ are the eigenstates of its magnetic moment's component along applied field's direction. According to the canonical (Gibbs) distribution, in thermal equilibrium the statistical sum (per spin) may be calculated as

$$Z = \sum_\pm \exp\left(-\frac{E_\pm}{T}\right) = \exp\left(+\frac{\hbar\gamma B}{2T}\right) + \exp\left(-\frac{\hbar\gamma B}{2T}\right) \equiv 2\cosh\frac{\hbar\gamma B}{2T},$$

where temperature is in energy units: $T \equiv k_B T_{\text{Kelvin}}$. The respective probabilities of two possible states follow from the Gibbs distribution:

$$P_\pm = \frac{1}{Z}\exp\left(\pm\frac{\hbar\gamma B}{2T}\right),$$

so that the average energy (per spin) may be calculated as

$$\langle E\rangle = \sum_\pm E_\pm P_\pm = \frac{1}{Z}\left[-\frac{\hbar\gamma B}{2}\exp\left(+\frac{\hbar\gamma B}{2T}\right) + \frac{\hbar\gamma B}{2}\exp\left(-\frac{\hbar\gamma B}{2T}\right)\right] \equiv -\frac{\hbar\gamma B}{2}\tanh\frac{\hbar\gamma B}{2T},$$

and the heat capacity as

$$C \equiv \frac{\partial E}{\partial T} = \left(\frac{\hbar\gamma B}{2T}\Big/\cosh\frac{\hbar\gamma B}{2T}\right)^2.$$

These expressions describe the gradual saturation of the state probabilities (at 1 for the lowest-energy state, and 0 for the highest-energy state) at high fields ($\hbar|\gamma B| >> T$), making the heat capacity to have a maximum at $\hbar|\gamma B| \approx 2.3\ T$.

(b) The magnetic moment M of the system of independent spins is just the sum of their moments $m_\pm$, i.e.

$$M = \sum_\pm N_\pm m_\pm = \frac{\hbar\gamma}{2}(N_+ - N_-),$$

where $N_\pm$ (with $N_+ + N_- = N$) are the numbers of particles with spins up and down. Since the statistical averages of $N_\pm$ are, by the probability definition, equal to $NP_\pm$, the average magnetization may be calculated as $N\langle m\rangle$, where

$$\langle m\rangle = \sum_\pm m_\pm P_\pm = \frac{\hbar\gamma}{2}(P_+ - P_-) = \frac{\hbar\gamma}{2}\tanh\frac{\hbar\gamma B}{2T}.$$

Due to the independence of spin fluctuations, the variance of magnetization fluctuation $\tilde{M} \equiv M - \langle M\rangle$ is also just the sum of those of individual spins:

$$\langle \tilde{M}^2 \rangle = N \langle \tilde{m}^2 \rangle, \qquad \text{where } \tilde{m} \equiv m - \langle m \rangle .$$

The latter variance may be calculated from the averages of the moment itself and its square:

$$\langle \tilde{m}^2 \rangle = \langle m^2 \rangle - \langle m \rangle^2,$$

where

$$\langle m^2 \rangle = \sum_{\pm} m_{\pm}^2 P_{\pm} = \left(\frac{\hbar \gamma}{2} \right)^2 (P_+ + P_-) = \left(\frac{\hbar \gamma}{2} \right)^2,$$

so that

$$\langle \tilde{m}^2 \rangle = \left(\frac{\hbar \gamma}{2} \right)^2 - \left(\frac{\hbar \gamma}{2} \tanh \frac{\hbar \gamma B}{2T} \right)^2 \equiv \left(\frac{\hbar \gamma}{2} \Big/ \cosh \frac{\hbar \gamma B}{2T} \right)^2$$

and hence

$$\langle \tilde{M}^2 \rangle = N \left(\frac{\hbar \gamma}{2} \Big/ \cosh \frac{\hbar \gamma B}{2T} \right)^2 .$$

This expression describes a gradual suppression of the fluctuations at high fields, due to the saturation of the magnetization.

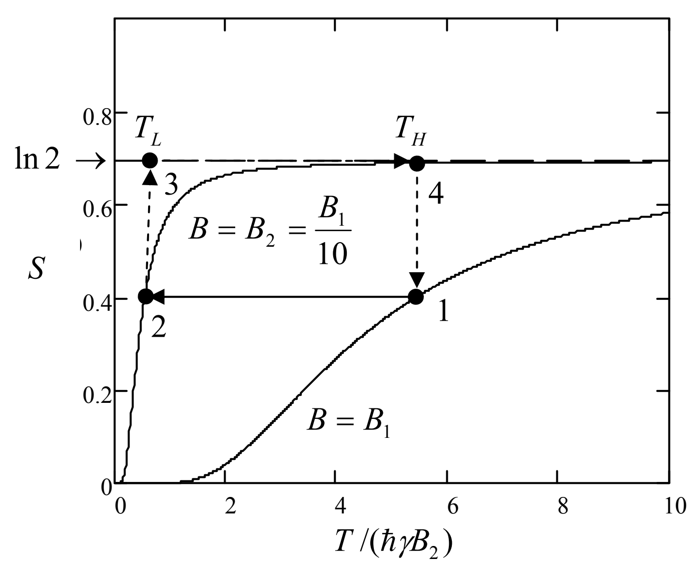
(c) First calculating the free energy (again per spin) from the statistical sum,

$$F = -T \ln Z = -T \ln \left(2 \cosh \frac{\hbar \gamma B}{2T} \right),$$

we may find the entropy as

$$S = \frac{E - F}{T} = -\frac{\hbar \gamma B}{2T} \tanh \frac{\hbar \gamma B}{2T} + \ln \left(2 \cosh \frac{\hbar \gamma B}{2T} \right).$$

According to this expression, the entropy tends to ln2 at low fields (reflecting the two-fold degeneracy of the system at $B = 0$), and vanishes at high fields ($\hbar|\gamma B| >> T$).



(d) Figure on the right shows the entropy as the function of temperature for two magnetic field values that differ by an order of magnitude. Let the system of spins first be at thermal equilibrium with the environment at the higher value B_1 of field, then get thermally isolated, and then the field decreased to the much lower value B_2. Due to its thermal isolation, the spin system's entropy cannot change much, so that it will evolve *approximately* as the solid arrow 1→2 in the Fig. shows, leading to the decrease of the effective temperature of the system. (This transient process is not strictly adiabatic, so that depending on system's details, it may deviate somewhat from the shown horizontal arrow. Nevertheless, it is traditionally called *adiabatic demagnetization*.[1])

(e) In order to use this process for continuous refrigeration, the following Carnot cycle may be used – see dashed lines in Fig. above. Let us start, for example, from point 4, at which the spin system is in contact with "hot bath" of temperature T_H,[2] and kept at vanishing field, so that the entropy per spin is large, $S \approx \ln 2$. Now the field is slowly increased to some value $B_2 \sim T_H/\hbar\gamma$, still keeping the spin system in contact with the hot bath. Since the entropy is being decreased (because almost all spins condense onto the lowest energy level, thus decreasing spin disorder), heat $-Q_H = T_H \Delta S > 0$ is being transferred to the hot bath. Then (at point 1) the refrigerant is being thermally insulated from the hot bath, and then the external field is decreased, leading to a decrease of the spin temperature - as was discussed above. At point 2, when T decreases to temperature T_L of the "cold bath" (the object being cooled), the refrigerant is brought into a thermal contact with that bath, and then field's decrease is continued isothermally until point 3, at which the entropy per spin has its maximum value ln2. The cycle is now completed adiabatically using a slight field increase until the spin system temperature rises to T_H again.

Practical cycles of such "adiabatic magnetic refrigeration" somewhat differ from, and hence have lower $\text{COP}_{\text{cooling}}$ than the Carnot cycle discussed above, mostly because of the difficulties of fast changing the thermal contacts ("heat switches") between the spin system (practically, a solid alloy such as $Gd_5(Si_2Ge_2)$ or $PrNi_5$) and the heat baths – typically letting in and pumping out small potions of gaseous helium.

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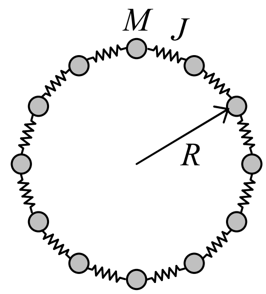
[1] It was suggested independently by P. Debye in 1926 and W. Giauque in 1927, and implemented experimentally by several groups in the early 1930s, enabling them to reach temperatures well below 1 K for the first time.

[2] For a typical application of this technique, with T_H corresponding to ~4 K, the term "hot bath" is pretty awkward, and engineers prefer the term "cooling source" (which is wrong from the point of view of physics :-).

Statistical Mechanics 3

1D vibrational modes



Consider a system of $N >> 1$ similar particles of mass M, equally spaced on a circle of radius R, and constrained to move only around the circle. Nearest neighbor particles are connected by springs with equal spring constants J – see Fig. on the right.

(a) (3 points). Write down the Lagrangian function of the system, and the equation of angular motion of an arbitrary particle, assuming that spring deformations are relatively small.

(b) (2 points). Prove that if the particles are numbered sequentially, the equation of motion of the n^{th} particle is satisfied by the following function:

$$\varphi_n(t) = \text{Re}\sum_{j=0}^{N-1} c_j(t)\exp\left(i\frac{2\pi jn}{N}\right),$$

where $\varphi_n(t)$ is the angular displacement of the particle, and derive the differential equation obeyed by functions $c_j(t)$. What simple physical system obeys the similar differential equation?

(c) (3 points). For a single one-dimensional harmonic oscillator of frequency ω that may be comparable with $k_BT/\hbar$, write down the statistical sum ("partition function"), and calculate its heat capacity $C(T)$ in thermal equilibrium at temperature T. Analyze the low-temperature and high-temperature limits of the function $C(T)$.

(d) (5 points). Returning to the system of N particles on a circle (see Fig. above), calculate its heat capacity in the intermediate temperature range

$$\frac{\hbar\omega_0}{N} << k_BT << \hbar\omega_D, \qquad \text{where } \omega_D \equiv \left(\frac{J}{M}\right)^{1/2}.$$

(e) (4 points). Now suppose that M is so large that there is a broad range of lower temperatures:

$$\frac{\hbar^2 N}{MR^2} << k_BT << \frac{\hbar\omega_D}{N}.$$

What is the heat capacity of the system in this range?

(f) (3 points). Estimate the temperature at which the heat capacity becomes exponentially small, if the particles are distinguishable. How does the answer change if they are indistinguishable bosons?

Hint: You may use the following integral: $\int_0^\infty \frac{xdx}{e^x - 1} = \frac{\pi^2}{6}$.

Solution

(a) The kinetic energy of the n^{th} particle is

$$T_n = \frac{MR^2\dot{\varphi}_n^2}{2},$$

where φ_n is the angular coordinate of the particle, while the potential energy of the spring connecting n^{th} and $(n + 1)^{\text{st}}$ particle, due to its small deformation Δl_n, is

$$U_n = \frac{J\Delta l_n^2}{2} = \frac{J(R\varphi_{n+1} - R\varphi_n)^2}{2}.$$

From here, the Lagrangian function of the system is

$$L = \sum_{n=1}^{N}(T_n - U_n) = R^2\sum_{n=1}^{N}\left(\frac{M\dot{\varphi}_n^2}{2} - \frac{J(\varphi_{n+1} - \varphi_n)^2}{2}\right).$$

From this Lagrangian, the equation of motion of an the n^{th} particle is

$$M\ddot{\varphi}_n + J(2\varphi_n - \varphi_{n-1} - \varphi_{n+1}) = 0.$$

(b) Plugging the suggested solution,

$$\varphi_n(t) = \text{Re}\sum_{j=1}^{N} c_j(t)\exp\left(\frac{2\pi jn}{N}\right), \qquad (1)$$

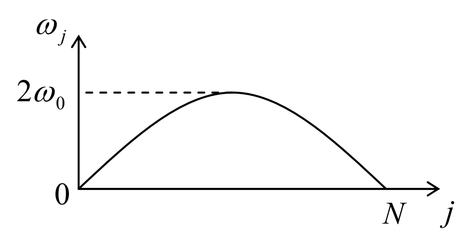
into the above equation of motion, we see that it is indeed satisfied if functions $c_j(t)$ satisfy the usual equations of the usual (one-dimensional) harmonic oscillators,

$$\ddot{c}_j + \omega_j^2 c_j = 0, \qquad (2)$$

with frequencies

$$\omega_j = \left[\frac{J}{M}\left(2 - \exp\left\{\frac{2\pi j}{N}\right\} - \exp\left\{-\frac{2\pi j}{N}\right\}\right)\right]^{1/2} = 2\omega_0 \sin\frac{\pi j}{N}, \qquad \text{with } \omega_0 \equiv \left(\frac{J}{M}\right)^{1/2}.$$

This result is sketched in Fig. on the right; the following features of this "dispersion curve" are important for the next tasks:



- For small j, the frequencies depend on index j (the dimensionless "crystal momentum") linearly, forming the so-called acoustic branch:

$$\omega_j \approx 2\omega_0\frac{\pi j}{N} \propto j, \qquad \text{for } j << N. \qquad (3)$$

- There is a similar acoustic branch, with the negative slope, $\omega_j \propto (N - j)$, for j very close to N.

- The mode with $j = N$, and hence $\omega_j = \omega_N = 0$, is special. Indeed, for this mode, the solution of Eq. (2) is not oscillatory (sinusoidal) as for all other modes, but rather a linear function of time:

$$c_N(t) = \Omega t + \text{const}.$$

This means that this mode describes the rotation of the system of particles around the circle as the whole, without spring deformations, with angular velocity Ω.

- Since this linear system has N degrees of freedom, it can have no other distinguishable modes besides the N modes contributing to Eq. (1).

(c) The quantum energy spectrum of a single harmonic oscillator of frequency ω is

$$E_k = \hbar\omega\left(k + \frac{1}{2}\right),$$

so that its statistical sum is

$$Z \equiv \sum_{k=0}^{\infty} \exp\left\{-\frac{E_k}{k_B T}\right\} = \exp\left\{-\frac{\hbar\omega}{2k_B T}\right\}\sum_{k=0}^{\infty}\exp\left\{-\frac{k\hbar\omega}{T}\right\} = \exp\left\{-\frac{\beta\hbar\omega}{2}\right\}\sum_{k=0}^{\infty}\lambda^k,$$

where

$$\beta \equiv \frac{1}{k_B T}, \qquad \text{and } \lambda \equiv e^{-\beta\hbar\omega} \leq 1.$$

This series is just a geometric progression, equal to $1/(1- \lambda)$, so that

$$Z = \exp\left\{-\frac{\beta\hbar\omega}{2}\right\}\frac{1}{1-\lambda} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}.$$

Now the oscillator's average energy may be calculated as

$$\langle E \rangle = -\frac{\partial}{\partial\beta}\ln Z,$$

giving

$$\langle E \rangle = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \equiv \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/k_B T}-1}, \tag{4}$$

and the heat capacity as

$$C(T) = \frac{\partial\langle E \rangle}{\partial T} = k_B\left(\frac{\hbar\omega}{k_B T}\right)^2 \frac{e^{\hbar\omega/k_B T}}{\left(e^{\hbar\omega/k_B T}-1\right)^2} = k_B\left[\frac{\hbar\omega/2k_B T}{\sinh(\hbar\omega/2k_B T)}\right]^2.$$

This formula shows that at low temperatures ($k_B T << \hbar\omega$) the specific heat is exponentially low, while in the opposite limit of high temperatures it reaches the constant value k_B, which also follows from the equipartition theorem for the classical oscillator (with two quadratic contributions to the Hamiltonian function).

(d) As the above result shows, in the intermediate temperature range

$$\frac{\hbar\omega_0}{N} << k_B T << \hbar\omega_0,$$

we may calculate system's average energy taking into account only the two acoustic branches of its dispersion curve, treating them as the continua. (Due to the left condition, the rotational mode with $j = N$ behaves classically, and its contribution to the heat capacity is negligible in comparison with the sum of comparable contributions from $\sim k_B T/(\hbar\omega_0/N) >> 1$ of classically-behaving oscillatory modes.) As a result, ignoring the first, temperature-independent term of Eq. (4), we may write

$$\langle E\rangle = \sum_{j=1}^{N}\langle E_j\rangle \approx 2\int_0^\infty \langle E_j\rangle dj = 2\int_0^\infty \frac{\hbar\omega_j}{e^{\hbar\omega_j/k_B T}-1}dj,$$

where the front factor of 2 accounts for the second acoustic branch of the dispersion curve. Now using Eq. (3), we may write $dj = (N/2\pi\omega_0)d\omega_j$, bringing the integral to the form

$$\langle E\rangle = 2\int_0^\infty \frac{\hbar\omega_j}{e^{\hbar\omega_j/k_B T}-1}\frac{N}{2\pi\omega_0}d\omega_j = \frac{N\hbar}{\pi\omega_0}\left(\frac{k_B T}{\hbar}\right)^2\int_0^\infty \frac{xdx}{e^x-1}, \qquad \text{with } x \equiv \frac{\hbar\omega_j}{k_B T}.$$

With the dimensionless integral provided in the hint, this expression becomes

$$\langle E\rangle = \frac{N\hbar}{\pi\omega_0}\left(\frac{k_B T}{\hbar}\right)^2\frac{\pi^2}{6} = \frac{\pi N}{6\hbar\omega_0}(k_B T)^2.$$

Now we may readily calculate the heat capacity of the system:

$$C(T) = \frac{d\langle E\rangle}{dT} = \frac{\pi}{3}k_B N\frac{k_B T}{\hbar\omega_0}.$$

This expression confirms that the number of low-frequency modes contributing to the heat capacity ($\sim k_B$ each) in this temperature range is of the order of $N(k_B T/\hbar\omega_0) >> 1$, so that the rotational mode contribution is indeed negligible.

(e) On the contrary, in the lower temperature range,

$$\frac{\hbar^2 N}{MR^2} << k_B T << \frac{\hbar\omega_0}{N}.$$

the contribution of the rotational mode dominates, due to the right inequality. On the other hand, due to the left inequality, its quantum properties (see the solution of Task (f) for their discussion) are negligible, so that the heat capacity may be immediately calculated from the classical equipartition theorem. Since the system, rotating as the whole (with all $\varphi_n(t)$ equal to $\varphi(t) = c_N(t)$), has just one quadratic term in its Hamiltonian function

$$H = T = \sum_0^N \frac{MR^2\dot{\varphi}_n^2}{2} = \frac{NMR^2}{2}\dot{\varphi}^2 = \frac{L_z^2}{2I}, \qquad (5)$$

(where $I = MNR^2$ is the moment of inertia of the system, and $L_z = I\dot{\varphi} = I\Omega$ its angular momentum), in thermal equilibrium its average has to be equal $k_B T/2$, so that

$$C(T) = \frac{d\langle E\rangle}{dT} = \frac{k_B}{2}.$$

(f) If the temperature is further decreased, quantum properties of this aggregate plane rotator become important. The Hamiltonian operator corresponding to the classical Hamiltonian function (5) is

$$\hat{H} = \frac{\hat{L}_z^2}{2I}, \qquad \text{with } \hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi},$$

whose eigenfunctions and eigenvalues are

$$\psi_m(\varphi) = c_m e^{im\varphi}, \qquad E_l = \frac{\hbar^2 m^2}{2I}.$$

If the component particles are distinguishable, then the wavefunctions should be 2π-periodic (because the angular translation by 2π is indistinguishable), so that $\exp\{im(\varphi + 2\pi)\}$ has to equal $\exp\{im\varphi\}$, i.e. $\exp\{i2\pi m\}$ should be equal to 1. This means that index m may take any integer values. Hence the thermal excitations of the system, and its heat capacity becomes exponentially small when the temperature is decreased to

$$k_B T \sim E_1 - E_0 = \frac{\hbar^2}{2I} = \frac{\hbar^2}{2MNR^2}.$$

On the other hand, if the component particles are indistinguishable bosons,[1] an angular translation as small as $2\pi/N$ should already be indistinguishable, so that $\exp\{im(\varphi + 2\pi/N)\} = \exp\{im\varphi\}$, i.e. $\exp\{i2\pi m/N\} = 1$. This means that the quantum number m may only take only values multiple of N. As a result, the heat capacity becomes exponentially small when the temperature is decreased to a higher value:

$$k_B T \sim E_N - E_0 = \frac{\hbar^2 N^2}{2I} = \frac{\hbar^2 N}{2MR^2} >> \frac{\hbar^2}{2MNR^2}.$$

(This result explains the left condition in Task (e), which ensures that the system may be treated as a classical one for any particle statistics.)

[1] In this case the elastic forces between the particles are supposed to be provided by some field, rather than by macroscopic mechanical springs – which are always distinguishable.