Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

August 2016 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages.
Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

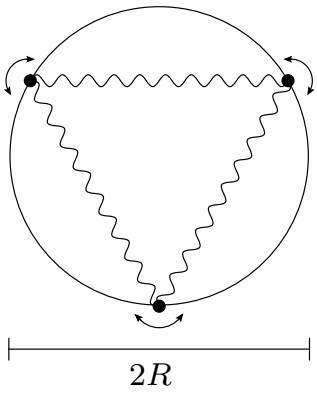
Classical Mechanics 1

Three beads

Consider three beads of mass m, connected by three identical springs with spring constant k and unstretched equilibrium length l_0. In equilibrium this system has the shape of an equilateral triangle whose sides have length l_0. We are going to study the small oscillations of this system.

1. (3 points) First consider the motion of this system in a horizontal plane. All three beads can freely move in the plane, only restricted by the springs attached to them. The motion is assumed to be without friction. (i) How many normal modes does this system have? (ii) How many zero modes are there (by zero mode we mean a normal mode with frequency $\omega = 0$)? (iii) How many nonzero modes are there, and is there degeneracy (by degeneracy we mean that two or more normal modes have the same frequency)? (iv) Sketch the motion of the normal modes with nonvanishing frequencies. Do *not* calculate the values of these frquencies.

2. (2 points) Next consider the case that the beads can only move along a fixed ring of radius R in the plane. To fit the triangle on the ring, one may need to stretch or compress the springs, depending on the values of l_0 and R. Consider an arbitrary value of l_0. Again the motion of the beads along the ring is assumed to be frictionless. We



   ask again the same questions: (i) How many modes are there? (ii) How many zero modes are there? (iii) How many nonzero modes are there, and is there degeneracy?

3. (1 points) Depending on the value of $\frac{l_0}{R}$, the triangle on the ring may be stable or unstable. Can you guess a case when it is stable?

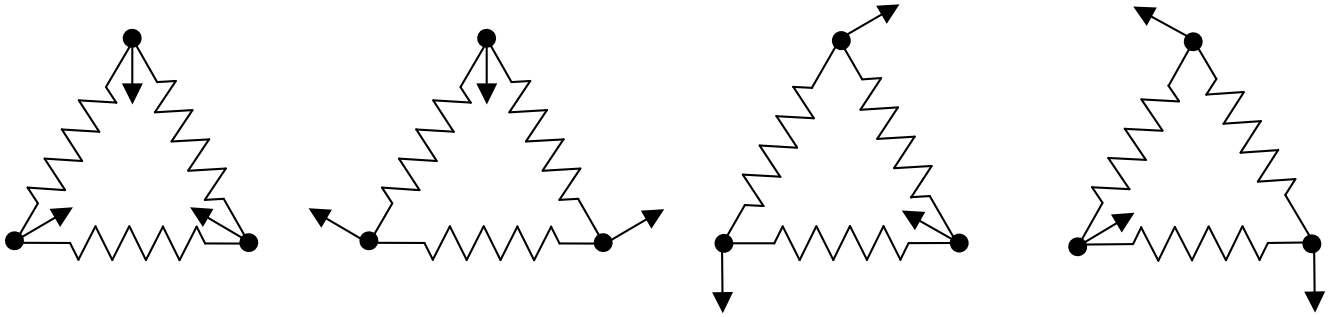
4. (6 points) Determine the value of $\frac{l_0}{R}$ when the system changes from stable to unstable.

5. (3+5 points) Expand the potential to second order in small deviations. Compute the frequencies of small oscillations in the stable regime.

Solution

1. Each of the beads, moving in a plane, has 2 degrees of freedom, so that the system of 3 of them have 3×2 = 6 degrees of freedom. According to the general theory of linearly coupled particles, the system has the same number (6) of normal modes, but 3 of them have zero frequencies: free translational motions of the system as the whole along 2 mutually perpendicular directions, and the free rotation of the system about its center of mass.

So, there are 3 modes with non-zero frequency, all conserving the linear and angular momenta of the system. One of them is evidently the radial oscillation of the beads toward the center of mass and back (with their angular positions conserved) – see the left panel in Fig. below. The remaining 3 panels of that Fig. show 3 possible choices for the remaining 2 modes. All these 3 modes differ only by $2\pi/3$ rotations. Due to the similarity of the beads, the oscillation frequency has to be invariant with respect to such rotations, so that the frequencies of all these modes are equal. Moreover, each of these modes may be represented as a weighed sum of the other two, so there are only 2 linearly-independent, doubly-degenerate modes of this type.



2. The ring constraint reduces the number of degrees of freedom to 3 - characterized, say, by the angles φ_j (with j = 1, 2, 3) at which the j^{th} bead is visible from the center of the ring. This reduction obviously kills both translational modes and the radial oscillation mode, leaving the system with 3 modes of motion: 1 free rotational mode (of zero frequency) and 2 angular modes with the same frequency $\omega \neq 0$.

3. We may expect the symmetric, equilibrium position of the system (with the magnitudes of all 3 angles $\varphi_{jj'} \equiv \varphi_j - \varphi_{j'}$ between the beads being equal to $\varphi_0 = 2\pi/3$) to be stable if the springs are pre-compressed, i.e. if l_0 is larger that the equilibrium linear distance between the beads,

$$l_e = 2R\sin\frac{\varphi_0}{2} = 2R\sin\frac{\pi}{3} = \sqrt{3}R,$$

because the springs "try" to push the beads apart from each other.[1] Moreover, if $l_0 = l$, the system is evidently also stable. Hence we may expect that even if the springs are slightly pre-stretched, i.e. if

$$\Delta l_0 \equiv l_e - l_0$$

[1] The only requirement on the pre-compression is to avoid spring "buckling" (which would break our implicit assumption that the springs are straight).

is positive but not very large, the symmetric equilibrium position is stable as well. However, a large pre-stretch ($\Delta l_0 >> R$) means that the beads are strongly attracted to each other, so that we may expect the symmetric equilibrium position to be unstable with respect to an avalanche process leading to all beads eventually collecting into a tight group with all linear distances of the order of l_0.

4. The potential energy of the spring connecting beads number *j* and number *j'* is

$$U_{jj'} = \frac{k}{2}\left(l_{jj'} - l_0\right)^2,$$

where $l_{jj'}$ is spring's length:

$$l_{jj'} = 2R\sin\frac{\varphi_{jj'}}{2}.$$

Combining these two relations, we get

$$U_{jj'} = \frac{k}{2}\left(2R\sin\frac{\varphi_{jj'}}{2} - l_0\right)^2.$$

In the symmetric stationary state, all three angles are equal: $\varphi_{jj'} = \varphi_0$. Expanding the sine function in a Taylor series in small deviations $\tilde{\varphi}_{jj'} \equiv \varphi_{jj'} - \varphi_0$ from this equilibrium, and dropping all terms higher than $O\left(\tilde{\varphi}_{jj'}^2\right)$, we get

$$U_{jj'} \approx \frac{k}{2}\left[2R\left(\sin\frac{\varphi_0}{2} + \frac{1}{2}\cos\frac{\varphi_0}{2}\,\tilde{\varphi}_{jj'} - \frac{1}{4}\sin\frac{\varphi_0}{2}\,\frac{\tilde{\varphi}_{jj'}^2}{2}\right) - l_0\right]^2 = \frac{k}{2}\left(\Delta l_0 + R\cos\frac{\varphi_0}{2}\,\tilde{\varphi}_{jj'} - \frac{R}{4}\sin\frac{\varphi_0}{2}\,\tilde{\varphi}_{jj'}^2\right)^2$$

$$\approx \frac{k}{2}\left(\Delta l_0\right)^2 + \frac{k}{2}2\left(\Delta l_0\right)\left(R\cos\frac{\varphi_0}{2}\right)\tilde{\varphi}_{jj'} + \frac{k}{2}\left[\left(R\cos\frac{\varphi_0}{2}\right)^2 - 2\left(\Delta l_0\right)\left(\frac{R}{4}\sin\frac{\varphi_0}{2}\right)\right]\tilde{\varphi}_{jj'}^2.$$

The first term in the last expression for $U_{jj'}$ is a constant, and the sum of the second terms for all springs vanishes because

$$\tilde{\varphi}_{12} + \tilde{\varphi}_{23} + \tilde{\varphi}_{31} \equiv \left(\tilde{\varphi}_2 - \tilde{\varphi}_1\right) + \left(\tilde{\varphi}_3 - \tilde{\varphi}_2\right) + \left(\tilde{\varphi}_1 - \tilde{\varphi}_3\right) = 0.$$

The last, quadratic, term may be rewritten as

$$\tilde{U}_{jj'} = \frac{k'}{2}R^2\tilde{\varphi}_{jj'}^2,$$

where *k'* is the effective spring constant:

$$k' \equiv k\left[\left(R\cos\frac{\varphi_0}{2}\right)^2 - 2\left(\Delta l_0\right)\left(\frac{R}{4}\sin\frac{\varphi_0}{2}\right)\right]/R^2 \equiv k\left(\cos^2\frac{\varphi_0}{2} - \frac{\Delta l_0}{2R}\sin\frac{\varphi_0}{2}\right).$$

The constant is positive, and hence the potential energy of the spring grows with the deviation from equilibrium, if

$$\Delta l_0 < 2R\frac{\cos^2(\varphi_0/2)}{\sin(\varphi_0/2)}, \quad \text{i.e. if } \; l_0 > 2R\left[\sin(\varphi_0/2) - \frac{\cos^2(\varphi_0/2)}{\sin(\varphi_0/2)}\right].$$

In our particular case of 3 beads[2]

$$\varphi_0 \equiv \frac{2\pi}{3}, \qquad \text{so that} \quad \sin\frac{\varphi_0}{2} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\varphi_0}{2} = \frac{1}{2},$$

and the above conditions take the form

$$\Delta l_0 < \frac{R}{\sqrt{3}} \approx 0.577\ R, \quad \text{i.e. if } \; l_0 > \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)R \approx 1.155\ R.$$

This result confirms our preliminary hand-waving arguments: the condition is always fulfilled if $\Delta l_0 \le 0$, i.e. if the springs are either pre-compressed or not stretched in the equilibrium state.

5. If the system is stable, i.e. if $k' > 0$, we may analyze small oscillations around the equilibrium position by constructing system's Lagrangian

$$\begin{aligned} L = T - U &= (T_1 + T_2 + T_3) - (U_{12} + U_{23} + U_{31}) = \frac{m}{2}(v_1^2 + v_2^2 + v_3^2) - (\tilde{U}_{12} + \tilde{U}_{23} + \tilde{U}_{31}) \\ &= \frac{m}{2}R^2(\dot{\tilde{\varphi}}_1^2 + \dot{\tilde{\varphi}}_2^2 + \dot{\tilde{\varphi}}_3^2) - \frac{k'}{2}R^2(\tilde{\varphi}_{12}^2 + \tilde{\varphi}_{23}^2 + \tilde{\varphi}_{31}^2) \\ &= \frac{m}{2}R^2(\dot{\tilde{\varphi}}_1^2 + \dot{\tilde{\varphi}}_2^2 + \dot{\tilde{\varphi}}_3^2) - \frac{k'}{2}R^2\left[(\tilde{\varphi}_2 - \tilde{\varphi}_1)^2 + (\tilde{\varphi}_3 - \tilde{\varphi}_2)^2 + (\tilde{\varphi}_3 - \tilde{\varphi}_1)^2\right] \end{aligned}$$

with

$$k' = k\left(\cos^2\frac{\varphi_0}{2} - \frac{\Delta l_0}{2R}\sin\frac{\varphi_0}{2}\right) = \frac{k}{4}\left(1 - \sqrt{3}\,\frac{\Delta l_0}{R}\right).$$

With the angle deviations $\tilde{\varphi}_j$ taken as the generalized coordinates q_j, the Lagrange equations of motion read:

$$m\ddot{\tilde{\varphi}}_j + k'(2\tilde{\varphi}_j - \tilde{\varphi}_{j+1} - \tilde{\varphi}_{j-1}) = 0.$$

Looking for a solution of the usual form $\tilde{\varphi}_j = c_j e^{-i\omega t}$, we reduce this system to that of three linear algebraic equations:

$$-m\omega^2 c_j + k'(2c_j - c_{j+1} - c_{j-1}) = 0. \qquad (*)$$

Now there are two ways to proceed to find the eigenvalues ω^2, with the same result. The general way is to write the condition of self-consistency of system (*), equating its determinant to zero, and solving the resulting characteristic equation to find 3 roots for ω^2. A more elegant way is to use system's symmetry, which expresses itself in the similarity of all 3 equations (*), and to look for a solution of the form, $c_j = ae^{+i\alpha j}$. This substitution immediately yields the characteristic equation

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[2] Please note that until this point, our quantitative analysis is valid for an arbitrary number ($N > 1$) of beads, when $\varphi_0 = 2\pi/N$.

$$-m\omega^2 + k'\left(2 - e^{i\alpha} - e^{-i\alpha}\right) = 0,$$

giving

$$\omega = \pm 2\left(\frac{k'}{m}\right)^{1/2} \sin\frac{\alpha}{2}.$$

In an infinite 1D system of particles, the phase shift α may take any (real) values. In our current case, we have just 3 particles, on a ring, so that the wave assumption, $c_j = ae^{+i\alpha j}$, is only valid if we impose the additional periodicity condition $c_{j+3} = c_j$. This condition yields $3\alpha = 2\pi n$, with integer n, so that there are only 3 physically different (i.e. located on a single 2π-segment, say $-\pi \le \alpha < \pi$) values of the constant α:[3]

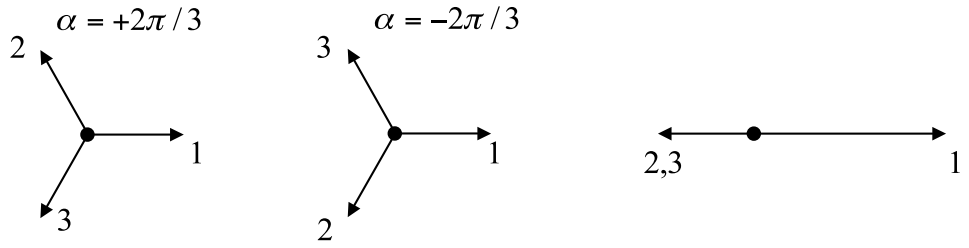
$$\alpha_1 = 0, \qquad \alpha_{2,3} = \pm\frac{2\pi}{3},$$

corresponding to just 2 physically different (say, non-negative) eigenfrequencies:

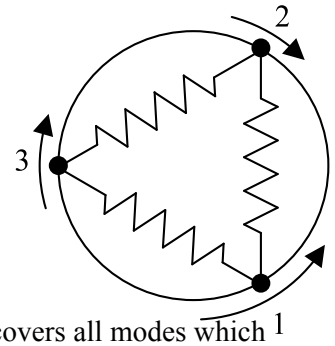
$$\omega_1 = 0, \qquad \omega_{2,3} = 2\left(\frac{k'}{m}\right)^{1/2} \sin\frac{\pi}{3} = \left(\frac{3k'}{m}\right)^{1/2}.$$

The first solution is the formal description of the evident fact that an arbitrary displacement of the 3 beads from their equilibrium state by an arbitrary (constant) shift $\Delta\varphi$ does not disturb the mutual force balance, and hence does not result in oscillations. The second and third solutions describe oscillations with the same frequency, but are physically different, by the sign of the phase shift α between the oscillations of any fixed pair of beads, thus confirming the qualitative argumentation above.

Two left panels of Fig. below show the "phasor diagrams", i.e. the sets of complex amplitudes of oscillations of each bead, of these two modes.



Note that due to the linearity of the equation of motion (for small oscillations only!), any linear superposition of these two basic phasor diagrams, with arbitrary complex amplitudes a, also represents a possible motion of the system. In particular, if the initial conditions are such that the amplitudes a of these two modes are equal, we should sum up the vectors of both basic diagrams, getting the new diagram shown on the right panel of Fig. above. It corresponds to a rather curious mode of oscillations in which beads



[3] The fact that the wave assumption gives 3 different oscillation modes means that it covers all modes which might be calculated using the more general method mentioned above.

2 and 3 oscillate together in antiphase with bead 1 (with a twice smaller amplitude), so that the length of one spring, l_{23}, does not change – see Fig. on the right. (Evidently, there are three such modes, which differ only by bead numbers, and may be obtained by the superposition of the two basic modes with equal $|a|$, but specific phase shifts, $\arg a$. Note, however, that only two of them are linearly independent.)

Classical Mechanics 2

The rocking of a half-cylinder

Consider a uniform half-cylinder of mass m and radius a rocking without slipping on a horizontal plane.

1. (2 points) Determine the position of the center of mass of the half-cylinder and the moment of inertia I_{cm} around the center of mass.

2. (4 points) Derive the Lagrangian L in terms of the angle θ between the flat surface of the half-cylinder and the horizontal.

3. (4 points) Write down the Lagrange equation of motion and find the frequency ω of small oscillations around the equilibrium.

4. (5 points) Determine the force $\vec{F}(\theta)$ produced by the plane on the half-cylinder at the line of contact as a function of the angle θ of the half-cylinder in the regime of small oscillations.

5. (5 points) Now consider large oscillations. Give physical arguments to predict whether the normal force exerted by the plane is larger or smaller than the gravitational force in the following two cases: when the angle θ is maximal, and when $\theta = 0$.

Solution

1. The center of mass is located on the symmetry plane of the half-cylinder a distance d away from the cylinder axis. The standard definition of the center of mass gives for the distance d:

$$d = \frac{1}{\pi a^2/2}\int_0^a dy 2y(a^2-y^2)^{1/2} = \frac{2a}{\pi}\int_0^1 d\xi(1-\xi)^{1/2} = \frac{4a}{3\pi}\,.$$

The moment of inertia I_{cm} for the rotation around the center of mass is most directly obtained with the help of the parallel-axis theorem:

$$I_{cm} = \frac{1}{2}ma^2 - md^2 = \frac{1}{2}ma^2\left(1-\frac{32}{9\pi^2}\right).$$

2. The potential energy $V(\theta)$ of the rocking half-cylinder comes from the gravitational force mg applied to the center of mass, and is given by the elevation $d(1-\cos\theta)$ of the center of mass relative to its lowest position at $\theta = 0$:

$$V(\theta) = mgd(1-\cos\theta)\,.$$

The kinetic energy K is a sum of the translational energy, and the rotational energy relative to the center of mass:

$$K = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_{cm}\dot\theta^2\,,$$

where v is the velocity of the center of mass. As a vector, this velocity has a horizontal and vertical components (v_h, v_v):

$$v_h = \frac{d}{dt}\Big[d\sin\theta - a\theta\Big] = \big[d\cos\theta - a\big]\dot\theta\,, \quad v_v = \frac{d}{dt}\Big[d(1-\cos\theta)\Big] = d\sin\theta\dot\theta\,,$$

and

$$v^2 = v_v^2 + v_h^2 = \big[a^2 + d^2 - 2da\cos\theta\big]\dot\theta^2\,.$$

This result can also be obtained by viewing the motion of the center of mass as rotation around the contact line with the plane.

The two energies determine the Lagrangian:

$$L = \tfrac{1}{2}I(\theta)\dot\theta^2 - mgd(1-\cos\theta)\,, \quad I(\theta) \equiv I_{cm} + m(a^2+d^2-2da\cos\theta) = \tfrac{3}{2}ma^2 - 2mda\cos\theta.$$

3. The Lagrange equation of motion reads

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = I(\theta)\ddot\theta + \frac{dI(\theta)}{d\theta}\dot\theta^2, \quad \frac{\partial L}{\partial\theta} = -mgd\sin\theta + \frac{1}{2}\frac{dI(\theta)}{d\theta}\dot\theta^2,$$

and finally,

$$I(\theta)\ddot\theta + \frac{1}{2}\frac{dI(\theta)}{d\theta}\dot\theta^2 = -mgd\sin\theta\,.$$

For small angles θ this equation can be linearized as

$$I(0)\ddot{\theta} = -mgd\theta\,,$$

and gives the frequency of small oscillations:

$$\omega = \left[\frac{mgd}{I(0)}\right]^{1/2} = \left(\frac{g}{a}\right)^{1/2}\left[\frac{8}{9\pi-16}\right]^{1/2}.$$

4. The most direct way of finding the vector of the contact force $\vec{F}(\theta)$ is from the acceleration of the center of mass, $m\vec{a} = \vec{F} - mg\hat{y}$, where $\hat{y}$ is the unit vector in the vertical direction. From the expression above for the velocity v, we have:

$$\vec{a} = \tfrac{d}{dt}\{v_h,\, v_v\} = \{d\cos\theta - a,\, d\sin\theta\}\ddot{\theta} + d\{-\sin\theta,\, \cos\theta\}\dot{\theta}^2.$$

For small oscillations, this reduces to $\vec{a} = \{(d-a)\ddot{\theta},\, 0\} = \left\{\frac{mgd(a-d)\theta}{I(0)},\, 0\right\}$, and

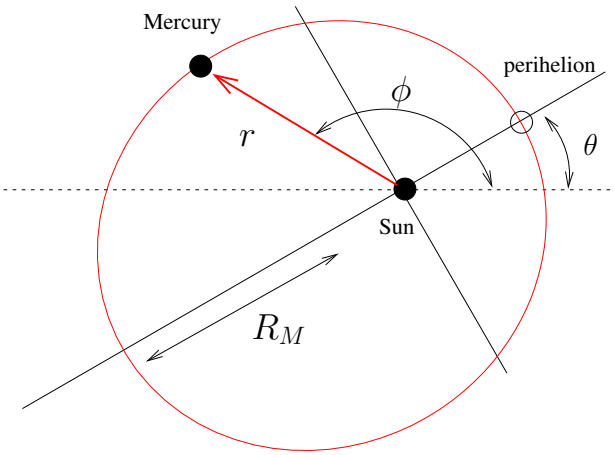
$$\vec{F}(\theta) = mg\left\{\frac{4}{3\pi}\cdot\frac{6\pi-8}{9\pi-16}\theta\,,1\right\}.$$

5. When the angle is maximal, the velocity vanishes at that moment, and the center of mass starts falling. At that moment the force exerted by the plane is less than the gravitational force. On the other hand, at the moment when the angle of the rocking cylinder vanishes, the center of mass reaches the bottom of its trajectory and then the centrifugal force adds to the gravitational force, so the force exerted by the plane is larger than the gravitational force.

Classical Mechanics 3

The precession of Mercury due to Jupiter

Recall that the trajectory of Mercury $r(\phi)$ is an ellipse with the sun at one focus as shown below. The perihelion (defined as the distance of closest approach – see below) is rotated relative to the x-axis by an angle θ. The semi-major axis is denoted R_M. The eccentricity of Mercury is small, $\epsilon = 0.2$, although it is the most eccentric of the Sun's planets.



Due to perturbations from the other planets, the angle of the perihelion θ changes (it precesses) as function of time. The precession rate due to the plantes is small. The contribution of Jupiter to the precession rate is of order 150 arcsec/century, or (since the orbital period of Mercury is 88 days) approximately 1.78×10^{-6} rad/turn.

The goal of this problem is to estimate Jupiter's contribution to the precession rate[1]. Specifically, we will model Jupiter as a ring of mass M_J at the orbital radius of Jupiter R_J (not shown), and compute how the ring perturbs Mercury's orbit and causes the perihelion of Mercury to precess. Jupiter's orbital radius is significantly larger than Mercury's, $R_J \simeq 10\, R_M$, and its mass is significantly smaller than the sun's, $M_J \simeq 0.95 \times 10^{-3}\, M_\odot$.

1. (4 points) Show that for $R_J \gg R_M$ the Lagrangian of Mercury interacting with the sun of mass $M_\odot$, and a ring of mass M_J and radius R_J is approximately

$$L \simeq \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + \frac{GmM_\odot}{r} + \alpha r^2 \,, \tag{1}$$

and determine the coefficient α.

[1] Famously, general relativity also perturbs the classical orbit and contributes 43 arcsecs/century to the total precession rate. This "anomalous" precession of Mercury was measured in the nineteenth century by Le Verrier and finally explained by Einstein in 1915. The total precession rate due to all the planets is approximately 550 arcsec/century.

2. (2 points) By introducing a dimensionless radius $\underline{r} = r/R_M$ (the radius in units of Mercury's semi-major axis) and other dimensionless variables show that a dimensionless Lagrangian for the system is

$$\underline{L} \simeq \frac{1}{2}\left(\frac{d\underline{r}}{d\underline{t}}\right)^2 + \frac{1}{2}\underline{r}^2\left(\frac{d\phi}{d\underline{t}}\right)^2 + \frac{1}{\underline{r}} + \underline{\alpha}\,\underline{r}^2\,, \tag{2}$$

where the dimensionless constant $\underline{\alpha}$ is of order

$$\frac{M_J}{M_\odot}\left(\frac{R_M}{R_J}\right)^3 \simeq 0.4 \times 10^{-6}\,. \tag{3}$$

To lighten the notation, stop underlining the variables in what follows.

3. (2 points) Determine the Hamiltonian of the dimensionless system, and use the Hamiltonian to determine the equations of motion.

4. (6 points) For $\alpha = 0$ determine the trajectory of Mercury $r(\phi)$, in terms of the energy and the angular momentum.

   For convenience, we note the elementary integral

$$\int^x \frac{dy}{\sqrt{1-y^2}} = \sin^{-1}(x)\,. \tag{4}$$

5. (6 points) Determine the change in the orbital period and precession of the perihelion of Mercury to first order in α. You may treat the eccentricity of Mercury as small so that the orbit is approximately circular. Evaluate the precession rate numerically in radians per turn and compare to the experimental result of 1.78×10^{-6}rad/turn.

Solution:

1. The potential from a ring of mass of density $\rho(\boldsymbol{r}_o)$ is found from the law of Newton

$$\Phi(\boldsymbol{r}) = -\int_{\text{ring}} d^3\boldsymbol{r}_o \, \frac{G\rho(\boldsymbol{r}_o)}{|\boldsymbol{r}-\boldsymbol{r}_o|}\,, \tag{5}$$

where $|\boldsymbol{r}_o^2| = R_J^2$ is the ring radius. Expanding the denominator

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_o|} = \left(R_J^2 + r^2 - 2rR_J\cos\phi_o\right)^{-1/2}\,, \tag{6}$$

$$\simeq \frac{1}{R_J}\left[1 + \frac{r}{R_J}\cos\phi_o + \frac{r^2}{R_J^2}(\tfrac{3}{2}\cos^2\phi_o - \tfrac{1}{2})\right]\,, \tag{7}$$

we obtain the integral expression for the potential

$$\Phi(\boldsymbol{r}) = -\lambda G\int d\phi_o \left[1 + \frac{r}{R_J}\cos\phi_o + \left(\frac{r}{R_J}\right)^2\left(\tfrac{3}{2}\cos^2\phi_o - \tfrac{1}{2}\right)\right]\,, \tag{8}$$

where $\lambda = M_J/(2\pi R_J)$ is the linear mass density of the ring. After integrating over the angle ϕ_o, we find

$$\Phi(\boldsymbol{r}) = -\frac{GM_J}{R_J} - \frac{GM_J}{4R_J^3}r^2\,. \tag{9}$$

Thus, the Lagrangian with the potential $V = m\Phi(\boldsymbol{r})$ is

$$L \simeq \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GmM_\odot}{r} + \alpha r^2\,, \tag{10}$$

where we have dropped an irrelevant constant, GmM_J/R_J, and

$$\alpha = \frac{GmM_J}{4R_J^3}\,. \tag{11}$$

2. Then we rescale the radius

$$\underline{r} \equiv \frac{r}{R_M}\,, \tag{12}$$

and choose a rescaling for time

$$\underline{t} \equiv \frac{t}{T_M}\,, \tag{13}$$

where T_M is at this point arbitrary. With these rescalings, the kinetic and potential energies are respectively

$$\frac{1}{2}\frac{mR_M^2}{T_M^2}\left[\left(\frac{d\underline{r}}{d\underline{t}}\right)^2 + \frac{1}{2}\underline{r}^2\left(\frac{d\phi}{d\underline{t}}\right)^2\right], \qquad \text{and} \qquad \frac{GmM_\odot}{R_M}\left[\frac{1}{\underline{r}}\right]. \tag{14}$$

This leads us to define

$$T_M^2 \equiv \frac{R_M^3}{GM_\odot}\,, \tag{15}$$

and the dimensionless Lagrangian is

$$\underline{L} = \frac{L}{E_o} = \frac{1}{2}\left(\frac{d\underline{r}}{d\underline{t}}\right)^2 + \frac{1}{2}\underline{r}^2\left(\frac{d\phi}{d\underline{t}}\right)^2 + \frac{1}{\underline{r}} + \underline{\alpha}\,\underline{r}^2\,, \tag{16}$$

where

$$E_o = \frac{GmM_\odot}{R_M} = m\frac{R_M^2}{T_M^2}\,, \tag{17}$$

and

$$\underline{\alpha} = \frac{\alpha R_M^2}{E_o} = \frac{1}{4}\left(\frac{M_J}{M_\odot}\right)\left(\frac{R_M}{R_J}\right)^3\,. \tag{18}$$

3. The Hamiltonian is

$$H = \frac{1}{2}p_r^2 + \frac{p_\phi^2}{2r^2} - \frac{1}{r} - \alpha r^2\,. \tag{19}$$

The Hamilton equations of motion are

$$\dot{r} = p_r\,, \tag{20}$$

$$\dot{\phi} = \frac{p_\phi}{r^2}\,, \tag{21}$$

$$\dot{p}_r = \frac{p_\phi^2}{r^3} - \frac{1}{r^2} + 2\alpha r\,, \tag{22}$$

$$\dot{p}_\phi = 0\,. \tag{23}$$

4. We have two constants of motion: the angular momentum $p_\phi = \ell$ and the energy $H = E$. Thus

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{r^2 p_r}{p_\phi}\,, \tag{24}$$

where

$$p_r = \left(2E - \frac{\ell^2}{r^2} + \frac{2}{r}\right)^{1/2}\,, \tag{25}$$

Therefore a integral relation determining $r(\phi)$ is

$$\int^r \frac{dr}{r^2}\frac{\ell}{(2E - \frac{\ell^2}{r^2} + \frac{2}{r})^{1/2}} = \int d\phi\,. \tag{26}$$

Substituting

$$u \equiv \frac{1}{r} \tag{27}$$

completing the square

$$2E - \ell^2 u^2 + 2u = \left(2E + \frac{1}{\ell^2}\right) - \left(\ell u - \frac{1}{\ell}\right)^2\,, \tag{28}$$

and using the elementary integral

$$\int^u \frac{\ell du}{\left[(2E + \frac{1}{\ell^2}) - (\ell u - \frac{1}{\ell})^2\right]^{1/2}} = \arcsin\left[\frac{(\ell u - \frac{1}{\ell})}{(2E + \frac{1}{\ell^2})^{1/2}}\right]\,, \tag{29}$$

we find

$$\frac{\ell}{r} - \frac{1}{\ell} = (2E + \frac{1}{\ell^2})^{1/2} \sin(\phi - \phi_o)\,. \tag{30}$$

Or,

$$r = \frac{\ell^2}{1 + \ell(2E + \frac{1}{\ell^2})^{1/2} \sin(\phi - \phi_o)}\,. \tag{31}$$

The minimum energy at fixed ℓ (corresponding to a circular orbit) is found by setting $p_r = 0$ in

$$E = \frac{1}{2}p_r^2 + \frac{p_\phi^2}{2r^2} - \frac{1}{r}\,, \tag{32}$$

and thus the circular orbit parameters are[2]

$$r|_{\rm min} \equiv r_o = \ell^2 \qquad E_{\rm min} = -\frac{1}{2\ell^2}\,. \tag{33}$$

Thus, we can express r as

$$r(\phi) = \frac{r_o}{1 + \epsilon\cos(\phi - \theta)}\,, \tag{34}$$

where

$$r_o = \ell^2 \qquad \text{and} \qquad \epsilon = \sqrt{\frac{(E - E_{\rm min})}{|E_{\rm min}|}} = \sqrt{2E\ell^2 + 1}\,. \tag{35}$$

We have chosen units where the semi-major axis of the ellipse is unity. Since the semi-major axis in physical units is related to the eccentricity of the ellipse

$$R_M = \frac{r_o}{1 - \epsilon^2}\,, \tag{36}$$

we have a relationship between ℓ and ϵ

$$\frac{\ell^2}{1 - \epsilon^2} = 1\,. \tag{37}$$

5. The unperturbed orbit is approximately circular. In the first step, we will determine the radius of the circular orbit r_o to first order in α. In the second step, we will analyze radial perturbations of the circular orbit, and determine the temporal period of radial oscillations. During one full cycle of a radial oscillation, the azimuthal angle ϕ will increase by 2π plus a correction $\delta\phi$ proportional to α, $\phi \to \phi_o + 2\pi + \delta\phi$. $\delta\phi$ is precession rate per turn.

In the first step we determine the radius of the circular orbit to first order in α. For general orbits the exact equation of motion for $r(t)$ reads

$$\frac{d^2r}{dt^2} = \frac{p_\phi^2}{r^3} - \frac{1}{r^2} + 2\alpha r\,. \tag{38}$$

[2]These relations are familiar from the Bohr model where the Bohr radii and energies scale as $r_n \propto n^2$ and $E_n \propto -1/n^2$ respectively.

Thus, for circular orbits the radius r_o is constant and determined by

$$\frac{p_\phi^2}{r_o^3} - \frac{1}{r_o^2} + 2\alpha r_o = 0\,. \tag{39}$$

Solving Eq. (39) order by order in α

$$r_o = r_o^{(0)} + r_o^{(1)} + \dots\,, \tag{40}$$

we find

$$r_0^{(0)} = \ell^2, \qquad \text{and} \qquad r_o^{(1)} = 2\alpha\ell^8\,. \tag{41}$$

In the second step we study perturbations of the circular orbit, and determine the period of radial oscillations. Substituting $r(t) = r_o^{(0)} + r_o^{(1)} + \delta r(t)$ into the exact equation of motion, Eq. (38), and expanding to first order in δr and α, we find

$$\frac{d^2\delta r}{dt^2} = -\frac{1}{\ell^6}(1 - 14\alpha\ell^6)\delta r\,. \tag{42}$$

Thus, the period of radial oscillations is

$$\tau_M = 2\pi\ell^3(1 + 7\alpha\ell^6)\,. \tag{43}$$

Finally, in the last step we examine the angular equation in order to determine $\delta\phi$

$$\dot\phi = \frac{\ell}{r^2}\,, \tag{44}$$

$$= \frac{\ell}{(r_o^{(0)} + r_o^{(1)} + \delta r)^2}\,, \tag{45}$$

$$\simeq \frac{1}{\ell^3}\left(1 - 4\alpha\ell^6 - \frac{2}{\ell^2}\delta r\right)\,. \tag{46}$$

Integrating the over a full period of radial oscillations $t = 0 \dots \tau_M$, the term proportional to δr vanishes upon integration, and we find that the azimuthal angle has changed by

$$\Delta\phi \simeq \frac{\tau_M}{\ell^3}(1 - 4\alpha\ell^6) \simeq 2\pi(1 + 3\alpha\ell^6)\,. \tag{47}$$

Thus the angle ϕ deviates from a full rotation by $+6\pi\alpha\ell^6$.

Finally, since $\ell^2 = 1 - \epsilon^2 \simeq 1$, the precession is

$$\delta\phi = 6\pi\alpha \qquad \text{rad per turn}\,. \tag{48}$$

With the numerical value of α, we find

$$\delta\phi = \frac{3\pi}{2}\frac{M_J}{M_\odot}\left(\frac{R_M}{R_J}\right)^3 \simeq 1.88 \times 10^{-6} \qquad \text{rad per turn}\,, \tag{49}$$

which should be compared to the nominal value of 1.78×10^{-6} rad/turn. Eq. (49) is 5% larger than the nominal value, and this difference is consistent with order $\epsilon^2 \simeq 4\%$ corrections.

One can also proceed more systematically, using Green functions, and only expand at the very end. That will give the usual result (the ellipse, exact to all orders in ϵ) if $\alpha = 0$, and to order ϵ the terms linear in α are kept. The exact equation of motion for $r(t)$ can be rewritten as an equation for $\sigma(\varphi) \equiv 1/r(\varphi)$ by using $\frac{dr}{dt} = \frac{d\varphi}{dt}\frac{dr}{d\varphi}$, and reads

$$\sigma'' + \sigma - \frac{1}{l^2} = -\frac{2\alpha}{l^2}\frac{1}{\sigma^3} \qquad \text{where } \sigma' = \frac{d\sigma}{d\varphi}.$$

We set $\sigma(\varphi) = \sigma_0(\varphi) + \alpha\chi(\varphi) + O(\alpha^2)$ where $\sigma_0(\varphi) = l^{-2}(1+\epsilon\cos\varphi)$ is the most general solution for $\alpha = 0$ (the ellipse). Then we find from the terms of order α

$$\chi'' + \chi = -\frac{2}{l^2}\left(\frac{l^2}{1+\epsilon\cos\varphi}\right)^3.$$

The general solution for χ is the sum of a particular solution of the inhomogeneous equation which we shall obtain using the Green function and the most general solution of the homogeneous equation,

$$\chi(\varphi) = \int d\varphi'\; G(\varphi-\varphi')\frac{-2l^4}{(1+\epsilon\cos\varphi')^3} + A\cos\varphi + B\sin\varphi$$

with A and B arbitrary coefficients. The Green function satisfies

$$\left(\frac{d^2}{d\varphi^2}+1\right)G(\varphi-\varphi') = \delta(\varphi-\varphi'),$$

and is given by[3]

$$G(\varphi-\varphi') = \frac{1}{2}\epsilon(\varphi-\varphi')\sin(\varphi-\varphi').$$

Because the integrals are complicated, we expand in terms of ϵ and keep only terms at most linear in ϵ. Then we find

$$\chi(\varphi) = -l^4\int_0^{\varphi}\sin(\varphi-\varphi')(1-3\epsilon\cos\varphi')\;d\varphi' + l^4\int_{\varphi}^{2\pi}\sin(\varphi-\varphi')(1-3\epsilon\cos\varphi')\;d\varphi' + A\cos\varphi + B\sin\varphi. \quad \text{(A)}$$

The integrals are now straightforwardly evaluated, but to interpret them later we first do the case with $\epsilon = 0$, which yields

$$\chi(\varphi) = -2l^4(1-\cos\varphi) + A\cos\varphi + B\sin\varphi.$$

Since the equation $\chi''+\chi = -2l^4$ has the obvious solution $\chi = -2l^4$, we set $A\cos\varphi + B\sin\varphi$ equal to $-2l^4\cos\varphi$. Then for $\epsilon = 0$ the solution for χ is $\chi = -2l^4$, and to order ϵ we find, after evaluating the four trigonometric integrals in (A), the following final result for χ,

$$\chi(\varphi) = -2l^4 + \frac{3\epsilon l^4}{2}\left(\sin\varphi(2\varphi - 2\pi + \sin 2\varphi) + \cos\varphi(\cos 2\varphi - 1)\right).$$

[3] This is easy to check if one notices that the first φ derivative must act on $\sin(\varphi-\varphi')$.

A little bit of goniometry reduces this to

$$\chi(\varphi) = -2l^4 + 3\epsilon l^4 \sin\varphi(\varphi - \pi).$$

Thus the complete result for $\sigma(\varphi)$ to linear order in α and $\alpha\epsilon$ (but exact to all orders in ϵ if $\alpha = 0$) reads

$$\begin{aligned}\sigma(\varphi) =& l^{-2}(1 + \epsilon\cos\varphi) + \alpha(-2l^4) + 3\alpha\epsilon l^4 \sin\varphi(\varphi - \pi) \\ =& \frac{1}{\bar{l}^2}\left(1 + \bar{\epsilon}\cos\left[\varphi - 3\alpha l^6(\varphi - \pi)\right]\right),\end{aligned}$$

where $\bar{l}^2 = l^2(1 + 2\alpha l^6)$, $\bar{\epsilon} = \epsilon(1 + 2\alpha l^6)$. In this result one could replace l by $\bar{l}$ to the order of approximation we are using, but we shall not do so.

Since two consecutive perihelia occur when the cosine is maximal, we find

$$\begin{cases}\varphi_{\mathrm{Ph,I}}(1 - 3\alpha l^6) + 3\alpha l^6\pi = 0, \\ \varphi_{\mathrm{Ph,II}}(1 - 3\alpha l^6) + 3\alpha l^6\pi = 2\pi.\end{cases}$$

Thus $\varphi_{\mathrm{Ph,II}} - \varphi_{\mathrm{Ph,I}} = (1 + 3\alpha l^6)2\pi$. Hence, the perihelion precession per turn is

$$\delta\varphi = 6\alpha l^6\pi.$$

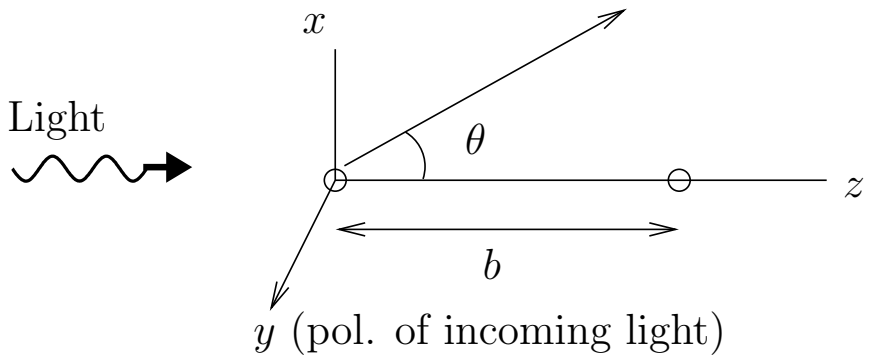
Since $\bar{l}^2 = a^2/b$ for an ellipse, and $b^2 = a^2(1 - \epsilon^2)$, while $a = 1$ in our choice of dimensionless variables, we find $l^2 = 1 + O(\epsilon^2)$. Thus

$$\delta\varphi = 6\alpha\pi = 6\frac{\pi}{4}\frac{M_J}{M_\odot}\left(\frac{R_M}{R_J}\right)^3 = 1.88 \times 10^{-6} \text{ rad per turn.}$$

Electromagnetism 1

Scattering at different scales

Consider the scattering of an electromagnetic plane wave of wavenumber k and frequency ω propagating in the z direction. The incident light is linearly polarized in the y direction, $\boldsymbol{E}(t, \boldsymbol{r}) = \hat{\boldsymbol{y}} E_0 e^{ikz - i\omega t}$ (out of the page in the diagram below). The light is scattered by two small dielectric spheres of radius a separated by a distance b with $b \gg a$. The first sphere is centered at the origin, while the second sphere is located on the z axis with $z = b$. The two spheres have dielectric constant $\epsilon = 1 + \chi$ with $\chi \ll 1$.



(a) (5 points) Consider the scattering of long wavelength light $kb \ll 1$. Determine the total cross section of the two spheres to leading order in kb. *Hint:* in the long wavelength limit the two spheres are polarized by the same (approximately) uniform incoming field.

- How does the cross section of the two spheres compare to the cross section of a single sphere in the long wavelength limit?

(b) (5 points) Remaining in the long wavelength limit $kb \ll 1$, determine the electric field as a function of time at the specific point, $\boldsymbol{r} = (x, y, z) = (2b, 0, 0)$, along the x axis. *Hint:* Is this point in the near or far field? Explain.

(c) (5 points) Now consider the scattering of shorter wavelength light with $kb \sim 1$ but still $ka \ll 1$. Determine the differential cross section $d\sigma/d\Omega$ of the two spheres for light scattered at an angle θ in the z, x plane (see diagram above).

- Sketch the differential cross section $d\sigma/d\Omega$ for scattering at $\theta = \pi/2$ (along the x axis) as a function of k for $kb = 0 \ldots 8\pi$.

(d) (5 points) Now instead of a plane wave of light, consider the scattering of a wave packet with mean wavenumber $\bar{k}$ and bandwidth Δk, with $\Delta k/k \simeq 1/10$. The differential cross section is the energy scattered per solid angle divided by the total energy in the wave packet.

- Qualitatively sketch the differential cross section $d\sigma/d\Omega$ for scattering at $\theta = \pi/2$ as a function of $\bar{k}$, and contrast this sketch with the $\Delta k = 0$ limit drawn in the second part of (c). At large k how does the cross section for the two spheres compare to the cross section for one sphere?

Solution

(a) To leading order in kb the external field induces an identical dipole moment in each sphere of magnitude $\chi V E_0$. The two dipoles then radiate electromagnetic radiation via dipole radiation The total electric dipole moment of the *two* spheres is

$$\boldsymbol{p} = 2\chi V E_0 e^{-i\omega t}\,\hat{\mathrm{y}} \tag{1}$$

The radiated power for dipole radiation

$$P = \frac{\omega^4}{4\pi c^3}\frac{|\boldsymbol{p}|^2}{3} \tag{2}$$

So the cross section is

$$\sigma = \frac{P}{\frac{c}{2}|E_0|^2} = \frac{\omega^4}{4\pi c^2}\frac{2}{3}|p_0|^2 \tag{3}$$

Collecting all factors

$$\sigma = \frac{P}{\frac{c}{2}|E_0|^2} = \frac{\omega^4}{4\pi c^4}\frac{8}{3}\chi^2 V^2 \tag{4}$$

- The total dipole moment is twice as large as a single sphere. The cross goes as the square of the dipole moment and is therefore four times as large

(b) This is in the near field. The electric field is just the electric field of two dipoles, one situated at the origin and one situated at $z = b$. The field from an electric dipole is

$$\boldsymbol{E} = \frac{3(\boldsymbol{p}\cdot\boldsymbol{n})\boldsymbol{n} - \boldsymbol{p}}{4\pi r^3}\,. \tag{5}$$

where $\boldsymbol{n}$ is the vector from the dipole origin to the observation point, and $\boldsymbol{p}$ is the dipole moment. In the current setup, $\boldsymbol{p}$ points in the y direction and $\boldsymbol{n}$ lies in the x, z plane for both dipoles. Thus the sum of the fields from the two dipoles is

$$\boldsymbol{E} = \frac{-\boldsymbol{p}}{4\pi r_1^3} + \frac{-\boldsymbol{p}}{4\pi r_2^3} \tag{6}$$

where r_1 and r_2 are the distances to the two induced dipole moments.

$$r_1 = 2b \tag{7}$$

$$r_2 = \sqrt{(2b)^2 + b^2} = \sqrt{5}b \tag{8}$$

This leads to

$$\boldsymbol{E}(t) = -\hat{\boldsymbol{y}}\frac{\chi V}{4\pi b^3}E_0 e^{-i\omega t}\left[\frac{1}{8} + \frac{1}{5\sqrt{5}}\right] \tag{9}$$

(c) In this case the two dipoles are out of phase

$$\boldsymbol{p}_1 = \hat{\boldsymbol{y}}\chi V E_0 e^{-i\omega t} \tag{10}$$

$$\boldsymbol{p}_2 = \hat{\boldsymbol{y}}\chi V E_0 e^{-i\omega t + ikb} \tag{11}$$

The radiation is

$$\boldsymbol{A}_{\rm rad} = \frac{e^{-i\omega(t-r/c)}}{4\pi rc}\int d^3\boldsymbol{r}_o \boldsymbol{J}(\boldsymbol{r}_o)e^{-ik\boldsymbol{n}\cdot\boldsymbol{r}_o} \tag{12}$$

Thus examining this integral we see that there is an overall phase difference between the two dipoles

$$-k\boldsymbol{n}\cdot\boldsymbol{r}_o = -kb\cos\theta \tag{13}$$

So

$$\boldsymbol{E}_1 = \frac{-\omega^2 e^{-i\omega(t-r/c)}}{4\pi rc^2}\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{p}_1 \tag{14}$$

$$\boldsymbol{E}_2 = \frac{-\omega^2 e^{-i\omega(t-r/c)}}{4\pi rc^2}\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{p}_2 e^{-ikb\cos\theta} \tag{15}$$

For $\boldsymbol{n}$ in the x,z plane and $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ oriented in the y direction we have $\boldsymbol{n}\times\boldsymbol{n}\times\hat{\boldsymbol{y}} = -\hat{\boldsymbol{y}}$

$$\frac{dP}{d\Omega} = \frac{c}{2}|\boldsymbol{E}_1+\boldsymbol{E}_2|^2 \tag{16}$$

$$= \frac{ck^4}{32\pi^2}(\chi V)^2 E_0^2\left|1+e^{ikb(1-\cos\theta)}\right|^2 \tag{17}$$

So the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}(\chi V)^2\left|1+e^{ikb(1-\cos\theta)}\right|^2 . \tag{18}$$

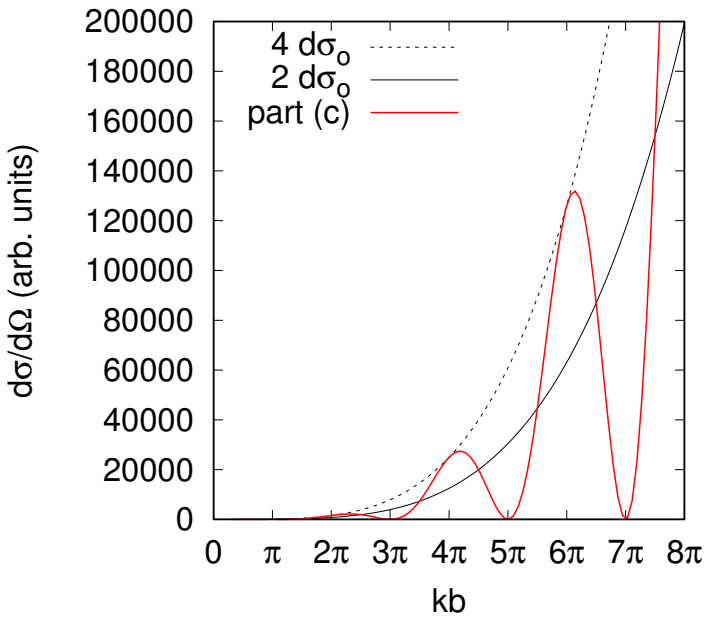
To make a graph we first note that

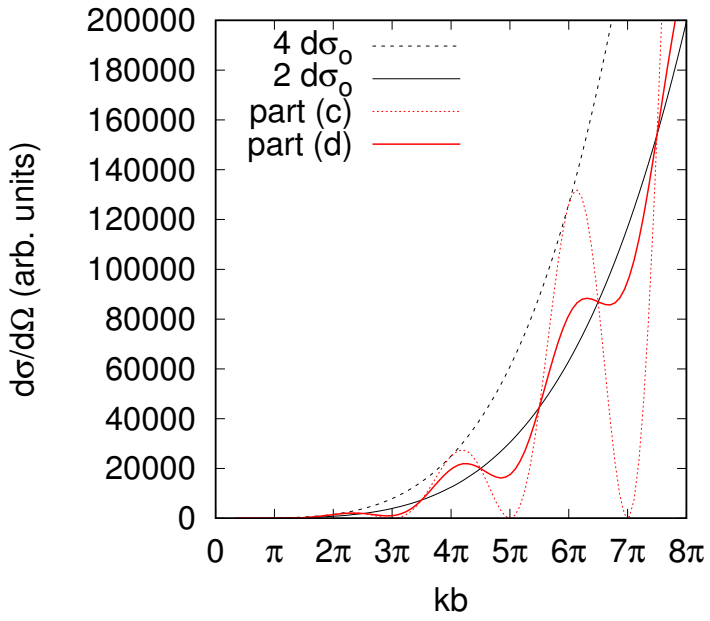
$$\left|1+e^{ikb(1-\cos\theta)}\right|^2 = |2\cos(kb\sin^2(\theta/2))|^2 . \tag{19}$$

Thus at $\theta=\pi/2$ we are plotting

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=\pi/2} \propto (kb)^4\cos^2(kb/2) , \tag{20}$$

We graph this function below





(d) If the wave packet has a finite band width Δk, it has a spatial size of order $\Delta x \sim \frac{1}{\Delta k}$. When this size comes comparable to the spacing b, i.e. $\Delta x \sim b$ or $\Delta k\, b \sim 1$, the interference between the scattering centers will not be complete. Indeed, when the mean Fourier compment of the wave packet $\bar{k}b$ is at an interference maximum, most of the Fourier components in the packet, $k \sim (\bar{k} \pm \Delta k)\, b$, will not be at an interference maximum if $\Delta k\, b \sim 1$. In the limit $\Delta x \ll b$ (or $\Delta k b \gg 1$) the wave packet will scatter off the first sphere and then scatter off the second sphere. The cross section for scattering off of the two spheres is thus twice the cross section for scattering off one of the spheres in this limit.

Since $\Delta k/\bar{k} \sim 10$, when

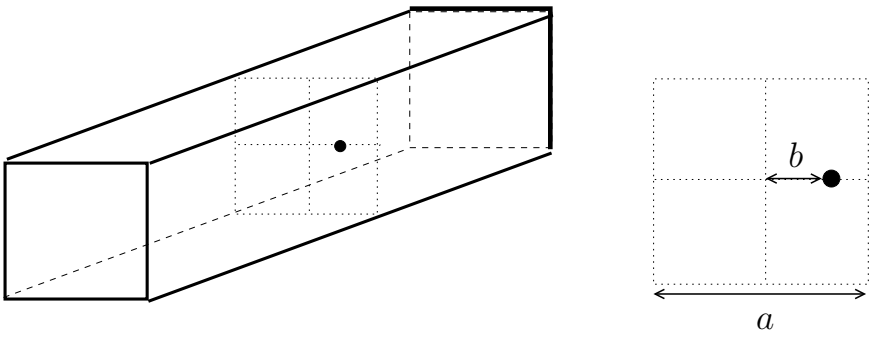
$$\bar{k}\, b \sim 10\,, \tag{21}$$

the interference betweeen the two scattering centers will begin to wash out. A schematic sketch of the cross section in this case is shown above. In the coherent limit (part (c)) the cross section varies between zero and four times the cross section for the scattering off a single sphere correspoinding to destructive and constructive interference respectively. If there is a finite coherence length Δx then the cross section transitions from the coherent limit to the incoherent limit. In the incoherent limit the cross section is twice the cross section of a single sphere.

Electromagnetism 2

A charge in a rectangular tube

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross sectional area of the tube have length a.



1. (2 points) Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

$$\Phi(k_x z)\,\Phi(k_y y)\,e^{\pm\kappa_z z} \qquad \text{where} \qquad \Phi(u) = \left\{ \cos(u) \text{ or } \sin(u) \right\} \tag{1}$$

   for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

2. (4 points) Now consider a point charge displaced from the center of the tube by a distance b in the x direction, i.e. the coordinatess of the charge are $\boldsymbol{r}_o = (x, y, z) = (b, 0, 0)$. Use the method of images to determine the potential.

3. (7 points) As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (1) to determine the potential from the point charge described in part (2).

4. (7 points) Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$.

Solution

1. The Laplace equation is

$$-\nabla^2\varphi = 0 \tag{2}$$

Separating variables with $\varphi = X(x)Y(y)Z(z)$ we must have

$$-\frac{d^2X}{dx^2} = k_x^2 X \tag{3a}$$

$$-\frac{d^2Y}{dy^2} = k_y^2 Y \tag{3b}$$

$$-\frac{d^2Z}{dz^2} = k_z^2 Z \tag{3c}$$

In order to satisfy Eq. (2), the separation constants satisfy

$$k_x^2 + k_y^2 + k_z^2 = 0 \tag{4}$$

and thus

$$\frac{d^2Z}{dz^2} = \kappa^2 Z \qquad \text{with} \qquad \kappa = \sqrt{k_x^2 + k_y^2} \tag{5}$$

The solutions to Eq. (3a) may be either sines or cosines

$$X(x) = \Phi(k_x x)\,, \tag{6}$$

with k_x at this point still arbitrary. In order to satisfy the boundary conditions $X(\pm a/2) = 0$, we require for the cosine functions that

$$k_x a/2 = (n + \tfrac{1}{2})\pi\,. \tag{7}$$

Similarly, for the sin functions

$$k_x a/2 = n\pi\,. \tag{8}$$

Thus, the general form is

$$X_n(x) = \Phi_n(k_n x) \qquad n = 0, 1, \ldots, \tag{9}$$

with $k_n = (n+1)\pi/a$ and

$$\Phi_n(u) = \begin{cases} \cos(u) & \text{n even} \\ \sin(u) & \text{n odd} \end{cases}\,. \tag{10}$$

The $Y(y)$ direction follows by analogy

$$Y_m(y) = \Phi_m(k_m x) \qquad m = 0, 1, \ldots, \tag{11}$$

with $k_m = (m+1)\pi/a$. The solutions to the z direction are

$$Z(z) = e^{\pm\kappa z} \qquad \kappa = \sqrt{k_n^2 + k_m^2}\,. \tag{12}$$

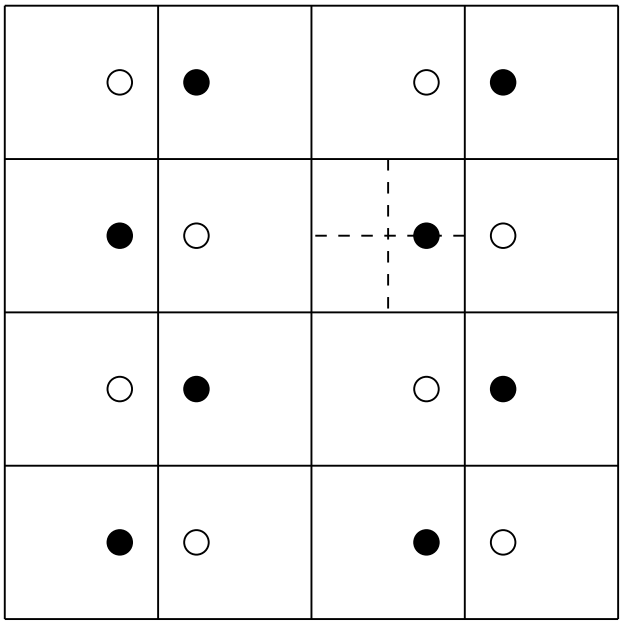


Figure 1: Arrangement of image charges. The black circles idicate plus charges, while the white circles indicate negative charges. The origin of the coordinate system is indicated by the dashed lines. The real charge is displaced by a distance b from the origin.

2. The image charges may be placed in a rectangular lattice as shown below. Their are four types of charges with coordinates

$$\boldsymbol{r}_1(n,m) = (b+2na)\hat{\boldsymbol{x}} + 2ma\hat{\boldsymbol{y}} \tag{13}$$
$$\boldsymbol{r}_2(n,m) = ((2n+1)a-b)\hat{\boldsymbol{x}} + 2ma\hat{\boldsymbol{y}} \tag{14}$$
$$\boldsymbol{r}_3(n,m) = (b+2na)\hat{\boldsymbol{x}} + (2m+1)a\hat{\boldsymbol{y}} \tag{15}$$
$$\boldsymbol{r}_4(n,m) = ((2n+1)a-b)\hat{\boldsymbol{x}} + (2m+1)a\hat{\boldsymbol{y}} \tag{16}$$

where n, m are integers. Then the potential is

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi}\sum_{n,m=0}^{\infty}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1(n,m)|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_2(n,m)|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_3(n,m)|} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_4(n,m)|} \tag{17}$$

3. For the potential at $\boldsymbol{r}$ due to a point charge at $\boldsymbol{r}_o = (b,0,0)$, we expand the potential as

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) = \left(\frac{2}{a}\right)^2 \sum_{n,m=0}^{\infty,\infty} X_n(x)X_n(b)\, Y_m(y)Y_m(0)\, g_{n,m}(z) \tag{18}$$

and substitute into the Poisson equation

$$-\nabla^2\varphi(\boldsymbol{r};\boldsymbol{r}_o) = q\delta(x-b)\delta(y)\delta(z)\,. \tag{19}$$

The leading factors $(2/a)^2$ arise from the fact that we have not normalized the eigenfunctions X and Y

$$\int_{-a/2}^{a/2} dx\, X_n(x)X_{n'}(x) = \frac{a}{2}\,\delta_{n,n'} \tag{20}$$

$$\int_{-a/2}^{a/2} dy\, Y_m(y)Y_{m'}(y) = \frac{a}{2}\,\delta_{m,m'} \tag{21}$$

If $g_{n,m}(z)$ satisfies

$$\left(k_n^2 + k_m^2 - \frac{\partial^2}{\partial z^2}\right) g_{n,m}(z) = q\delta(z)\,, \tag{22}$$

then using the completeness relation

$$\frac{2}{a}\sum_n X_n(x)X_n(x_o) = \delta(x - x_o) \tag{23}$$

$$\frac{2}{a}\sum_m Y_m(x)Y_m(x_o) = \delta(y - y_o) \tag{24}$$

it is not difficult to show that Eq. (19) is satisfied.

The solution to Eq. (22) is

$$g_{n,m}(z) = \begin{cases} Ae^{-\kappa_{n,m}z} & z > 0 \\ Ae^{\kappa_{n,m}z} & z < 0 \end{cases} \tag{25}$$

Integrating across the δ-fcn in Eq. (22) we have

$$-\left.\frac{dg}{dz}\right|_{z=0^+} + \left.\frac{dg}{dz}\right|_{0^-} = q \tag{26}$$

With this requirement $A = \frac{q}{2\kappa_{n,m}}$ and

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) = \frac{4q}{a^2}\sum_{n,m=0}^{\infty,\infty} X_n(x)X_n(b)\,Y_m(y)Y_m(0)\frac{e^{-\kappa_{n,m}|z|}}{2\kappa_{n,m}} \tag{27}$$

4. At asymptotic distances the terms with the smallest $\kappa_{n,m}$ dominate the sum. We then have only the contribution from $n = m = 0$ mode, and

$$\kappa_{0,0} = \sqrt{2}\pi/a\,. \tag{28}$$

The potential reads

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) \simeq \frac{4q}{a^2}\cos(\pi x/a)\cos(\pi b/a)\,\cos(\pi y/a)\frac{e^{-\kappa_{0,0}|z|}}{2\kappa_{0,0}} \tag{29}$$

or

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) \simeq \frac{\sqrt{2}q}{\pi a} cos(\pi x/a)\cos(\pi b/a)\ \cos(\pi y/a) e^{-\sqrt{2}\pi|z|/a} \tag{30}$$

Let us calculate the charge density on the bottom plate

$$\sigma = \boldsymbol{n}\cdot\boldsymbol{E} = -\partial_y\phi\big|_{y=-a/2}\ , \tag{31}$$

$$= -\frac{\sqrt{2}q}{a^2}\cos(\pi x/a)\cos(\pi b/a)\, e^{-\sqrt{2}\pi|z|/a}\,. \tag{32}$$

Finally, the force per area on the bottom plate is

$$\frac{F^y}{A} = \frac{\sigma^2}{2}\,, \tag{33}$$

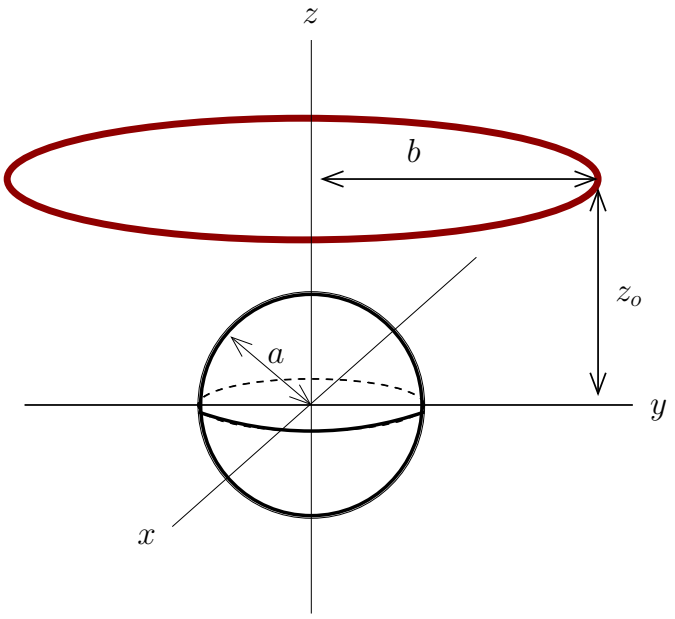
$$= \frac{q^2}{a^4}\cos^2(\pi x/a)\cos^2(\pi b/a)\, e^{-2\sqrt{2}\pi|z|/a}\,. \tag{34}$$

The direction of the force is into the tube. The other walls of the tube have the same force per area.

Electromagnetism 3

A ring and a sphere in a magnetic field

A sphere of radius a with magnetic permeability μ is placed in an external slowly varying homogeneous magnetic field (not shown), $\boldsymbol{B}_{\text{ext}}(t) = B_o(t)\,\hat{\boldsymbol{z}} = \mathcal{B}\cos(\omega t)\hat{\boldsymbol{z}}$. Placed above the sphere at height z_o is an ohmic ring of radius b and resistance $\mathcal{R}$. The center of the ring coincides with the z-axis and the plane of the ring is perpendicular to the z-axis (see below).



(a) (6 points) The induced magnetic moment of the sphere is proporitonal to the external field

$$\boldsymbol{m} = \alpha_B \boldsymbol{B}_{\text{ext}}\,. \tag{1}$$

Determine the polarizability, α_B. Neglect the fields from the currents induced in the ring.

(Hint: recall that for a permeable sphere in a constant external magnetic field, the magnetic field outside the sphere is that of an induced magnetic dipole plus the external field, while the magnetic field inside the sphere is constant, $\boldsymbol{B}_{\text{in}} = B_{\text{in}}\,\hat{\boldsymbol{z}}$. Determine α_B and B_{in} from the appropriate boundary conditions at the surface of the sphere.)

(b) (6 points) Determine the current induced in the ring.

(c) (2 points) Under what conditions can the induced magnetic fields from the ring be neglected in part (a)? Estimate.

(d) (6 points) Determine the force on the ring.

Solution

(a) The boundary conditions read

$$\boldsymbol{n} \times (\boldsymbol{H}_{\text{out}} - \boldsymbol{H}_{\text{in}}) = 0 \tag{2}$$
$$\boldsymbol{n} \cdot (\boldsymbol{B}_{\text{out}} - \boldsymbol{B}_{\text{in}}) = 0 \tag{3}$$

In terms of components

$$H_{\theta,\text{out}} - H_{\theta,\text{in}} = 0 \tag{4}$$
$$B_{r,\text{out}} - H_{r,\text{in}} = 0 \tag{5}$$

With the magnetic field of a dipole

$$\boldsymbol{B}_{\text{out}} = B_o\, \hat{\boldsymbol{z}} + \frac{3\hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^2} \tag{6}$$
$$\boldsymbol{B}_{\text{in}} = B_o\, \hat{\boldsymbol{z}} \tag{7}$$

we see that

$$B_{r,\text{out}} = \frac{2m\cos\theta}{4\pi a^3} + B_o \cos\theta \tag{8}$$
$$H_{\theta,\text{out}} = \frac{m\sin\theta}{4\pi a^3} - B_o \sin\theta \tag{9}$$

Inside we have

$$B_{r\,\text{in}} = B_{\text{in}} \cos\theta \tag{10}$$
$$H_{\theta\,\text{in}} = -\frac{1}{\mu} B_{\text{in}} \sin\theta \tag{11}$$

Putting together the ingredients we have

$$\frac{m}{4\pi a^3} - B_o + \frac{B_{\text{in}}}{\mu} = 0 \tag{12}$$
$$\frac{2m}{4\pi a^3} + B_o - B_{\text{in}} = 0 \tag{13}$$

Solving these equation for m and B_{in} we get

$$m = B_o(4\pi a^3)\frac{\mu - 1}{2 + \mu} \tag{14}$$
$$B_{\text{in}} = B_o \frac{3\mu}{2 + \mu} \tag{15}$$

(b) The flux through the loop has two contributions: the external magnetic field and the induced dipole. The external field contribution is simply

$$\Phi_{B,\text{ext}} = B_o(t)\pi b^2\,. \tag{16}$$

The dipole contribution is most easily found using the vector potential

$$\Phi_{B,\text{dip}} = \int \boldsymbol{B} \cdot d\boldsymbol{a} = \oint \boldsymbol{A} \cdot d\boldsymbol{\ell}\,. \tag{17}$$

With the vector potential of the dipole

$$\boldsymbol{A} = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{18}$$

we have

$$A_\phi = \frac{m \sin\theta}{4\pi(z^2+b^2)} \tag{19}$$

So with $\sin\theta = b/\sqrt{z^2+b^2}$ we have

$$\Phi_{B,\text{dip}} = \frac{m(t)}{2} \frac{b^2}{(z^2+b^2)^{3/2}} \tag{20}$$

$$= \alpha_B \frac{B_o(t)}{2} \frac{b^2}{(z^2+b^2)^{3/2}} \tag{21}$$

Thus the magnetic current is

$$I(t) = -\frac{1}{c\mathcal{R}} \partial_t \Phi_B(t) \tag{22}$$

Or

$$I(t) = \frac{-\dot{B}_o(t)\pi b^2}{c\mathcal{R}} \left[1 + \frac{\alpha_B}{2\pi} \frac{1}{(z^2+b^2)^{3/2}}\right] \tag{23}$$

(c) The current in the loop produces a field at the sphere of order $I(t)/[c(b^2+z^2)^{1/2}]$. We should compare this field to B_o, yielding the condition:

$$\frac{\omega B_o \pi b^2}{c^2\mathcal{R}} \frac{1}{(z^2+b^2)^{1/2}} \ll B_o\,. \tag{24}$$

Taking b and z the same order of magnitude $b \sim z$ as drawn in the figure,

$$\frac{\omega \pi b}{2\pi c^2 \mathcal{R}} \ll 1\,. \tag{25}$$

This is the answer.

It is useful to interpret the answer. The resistance is $\mathcal{R} = 2\pi b/(\sigma A)$ where A is the cross section of the wire and σ is the conductivity, yielding

$$\frac{\omega\sigma}{4\pi c^2} A \ll 1\,. \tag{26}$$

Recognizing the magnetic diffusion coefficient $D{=}c^2/\sigma$ of the wire and the skin depth $\delta(\omega) \sim \sqrt{D/\omega}$, we rewrite the condition as

$$\frac{A}{\pi\delta^2(\omega)} \ll 1\,. \tag{27}$$

(d) For the force we have the contribution of the constant field B_o and the field of the sphere $B_{\rm dip}$.

Using the right hand rule we see that the constant field produces no net force. All the forces of from the static field lie in the plane of the loop, tending to deform the ring but providing no net force.

From the dipole we have the Lorentz force

$$F^z = \int b d\phi \, \frac{I(t)}{c} \, \hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \boldsymbol{B}_{\rm dip}) \, . \tag{28}$$

With the dipole field,

$$\boldsymbol{B}_{\rm dip} = \frac{3\hat{\boldsymbol{r}} \cdot (\hat{\boldsymbol{r}} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3}, \tag{29}$$

the magnetic moment $m(t) = \alpha_B B_o(t) \hat{\boldsymbol{z}}$, the cross products

$$\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}}) = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\theta}} = -\sin\theta \, , \tag{30}$$

$$\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{z}}) = 0 \, , \tag{31}$$

we find

$$F^z = -\int b d\phi \, I(t)/c \, \frac{3 \sin\theta \cos\theta \, m(t)}{4\pi (z_o^2 + b^2)^{3/2}} \tag{32}$$

Thus

$$F^z = \left(-\frac{I(t) B_o(t) b}{c} \right) \, \frac{3}{4} \, \frac{\sin(2\theta) \alpha_B}{(z_o^2 + b^2)^{3/2}} \tag{33}$$

This is the answer after substituting the results of part (b).

After minor manipulations we find

$$F^z = \left(\frac{dB_o^2(t)}{dt} \frac{\pi b^3}{c^2 \mathcal{R}} \right) \, \frac{3}{8} \, \frac{\sin(2\theta) \, \alpha_B}{(z_o^2 + b^2)^{3/2}} \left[1 + \frac{\alpha_B}{2\pi (z_o^2 + b^2)^{3/2}} \right] \tag{34}$$

Quantum Mechanics 1

A particle in crossed $\boldsymbol{E}$ and $\boldsymbol{B}$ fields

Consider a particle of mass m and charge e moving in a 2-dimensional xy-plane in the presence of both a scalar potential $V(\vec{r})$ and a magnetic potential $\vec{A} = Bx\,\hat{y}$. For notational simplicity assume $e = \hbar = c = 1$.

a. (5 points) For $V(\vec{r}) = -ax$, identify the conserved momentum, and derive the energy spectrum of the system. If we were to trade $ax \to ay$, is there a conserved momentum?
b. (8 points) Define the following localized state for the potential $V(\vec{r}) = -ax$

$$|\Psi(0)\rangle = \int_{-\infty}^{+\infty} dk\, \Psi(k)\, |k\rangle_0\,,$$

where $|k\rangle_0$ is the lowest energy eigenstate with momentum k, and $\Psi(k)$ is any normalized momentum space wavefunction. Write down an expression for the time-evolved state, $|\Psi(t)\rangle$, and calculate its mean velocity along the conserved momentum direction. Give a physical interpretation of your result.
c. (7 points) Now consider instead, the hard wall potential

$$V(\vec{r}) = \begin{cases} 0 & x > 0 \\ \infty & x \leq 0 \end{cases} \tag{1}$$

with the same magnetic potential. Find the lowest energy of 3 indistinguishable particles, each at rest, that carry in addition a spin $\frac{1}{2}$ with a g-factor g, and write the corresponding wavefunction.

Solution:

a. Since the electric potential is y-independent, p_y is conserved. Thus

$$\begin{aligned} H = & \quad \frac{p_x^2}{2m} + \frac{(p_y - Bx)^2}{2m} - ax \\ = & \quad \frac{p_x^2}{2m} + \frac{B^2}{2m}\left(x - \frac{Bp_y + ma}{B^2}\right)^2 + \frac{1}{2m}\left(p_y^2 - \frac{(Bp_y + ma)^2}{B^2}\right) \end{aligned} \tag{2}$$

The energy spectrum is

$$E_n(p_y) = \left(n + \frac{1}{2}\right)\frac{B}{m} + \frac{1}{2m}\left(p_y^2 - \frac{(Bp_y + ma)^2}{B^2}\right) \tag{3}$$

If we trade $ax \to ay$, both $p_{x,y}$ do not commute with H. However, the combination $p_x - By$ commutes with H. So the conserved momentum is $\Pi_x = p_x - By$ with $[\Pi_x, H] = 0$.

b. The evolved state is $|\Psi(t)\rangle = e^{-iHt}\,|\Psi(0)\rangle$. The velocity $v_y(t)$ is

$$\begin{aligned} v_y(t) = & \quad \frac{d}{dt}\,\langle\Psi(t)|y|\Psi(t)\rangle = -i\,\langle\Psi(t)|[y,H]|\Psi(t)\rangle = \left\langle \Psi(t)\left|\frac{p_y - Bx}{m}\right|\Psi(t)\right\rangle \\ = & \quad \frac{1}{m}\int_{-\infty}^{+\infty} dk\,|\Psi(k)|^2\,(k - B\;{}_0\,\langle k|x|k\rangle_0) = -\frac{a}{B} \end{aligned} \tag{4}$$

We have used that for the lowest energy momentum eigenstate, i.e. $p_y|k\,\rangle_0 = k|k\,\rangle_0$, the spatial x-wavefunction is a normalized Gaussian with

$${}_0\,\langle k|x|k\rangle_0 = \frac{Bk + ma}{B^2} \tag{5}$$

This result is consistent with the classical interpretation of a charged particle subject to a zero net Lorentz force $\vec{F} = \vec{E} + \vec{v} \times \vec{B}$, i.e. $F_x = a + v_y B = 0$ or $v_y = -a/B$.

c. For the hard wall potential with $a = 0$ and for particles at rest with $p_y = 0$ we have $(x > 0)$

$$H \to \frac{p_x^2}{2m} + \frac{B^2 x^2}{2m} \tag{6}$$

This is a harmonic oscillator on the half-line. The energy spectrum is $(n + 1/2)B/m$ with only the odd wavefunctions with $n = 2l - 1$ and $l = 1, 2, ..$ vanishing at $x = 0$. Thus the energy spectrum for a spin $S_z = \pm 1/2$ particle at rest is

$$E_l^{\pm}(p_y = 0) = \left(2l - \frac{1}{2}\right)\frac{B}{m} \mp \frac{gB}{4m} \tag{7}$$

and its wavefunction is

$$\psi_{n,k=0}^{S=\pm}(x) = \phi_n(x - x_0)\chi(S = \pm) \tag{8}$$

with $x_0 = (Bk + ma)/B^2$ and $\phi_n(x)$ normalized harmonic oscillator wavefunctions, e.g.

$$\phi_0(x - x_0) = \left(\frac{B}{\pi}\right)^{\frac{1}{4}} e^{-B(x-x_0)^2/2} \tag{9}$$

For 3 fermions each at rest the lowest energy is

$$E(3) = 2\left(2 - \frac{1}{2}\right)\frac{B}{m} + \left(4 - \frac{1}{2}\right)\frac{B}{m} - \frac{gB}{4m} = \left(\frac{13}{2} - \frac{g}{4}\right)\frac{B}{m} \tag{10}$$

and their totally antisymmetric wavefunction is

$$\Psi(3) = \frac{1}{\sqrt{3!}} \begin{pmatrix} \phi_1(x_1 - x_0)\chi(+) & \phi_1(x_1 - x_0)\chi(-) & \phi_3(x_1 - x_0)\chi(+) \\ \phi_1(x_2 - x_0)\chi(+) & \phi_1(x_2 - x_0)\chi(-) & \phi_3(x_2 - x_0)\chi(+) \\ \phi_1(x_3 - x_0)\chi(+) & \phi_1(x_3 - x_0)\chi(-) & \phi_3(x_3 - x_0)\chi(+) \end{pmatrix} \tag{11}$$

Quantum Mechanics 2

BEC of Lithium-7

At about the same time that Ketterle, Wiemann, and Cornell formed Bose-Einstein Condensates (BEC) of rubidium-87 and sodium-23 atoms (and later won the Nobel prize), Randy Hulet's lab in Texas was trying to form a BEC of lithium-7 ($^7_3$Li) atoms in a harmonic trap. We will try to model Hulet's system with a mean-field approach, using the Gross-Pitaevskii equation — essentially a nonlinear generalization of Schrödinger's equation:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \frac{Ng}{2}|\psi|^2$$

where H acts on the wave function ψ. Here ψ is a single particle wave function. At low temperatures, all the N bosons should be in the same state, and the contribution of the interactions to the energy should scale as the local density of the wave function $|\psi|^2$, giving rise to the third term in our Hamiltonian. One very significant difference between these experiments is that lithium-7 atoms attract ($g < 0$) while rubidium-87 and sodium-23 repel ($g > 0$).

a. **(2 points)** Explain how lithium-7 is a boson.

b. **(7 points)** Using a Gaussian trial wave function $\psi = c\, e^{-r^2/2a^2}$, calculate the expectation value of the energy $E(a) = \langle H \rangle$ as a function of a. [Hint: The expectation value has the form $E(a) = A/a^2 + Ba^2 + C/a^3$ for some constants A, B, and C which you need to determine.]

c. **(5 points)** In the limit in which the interaction energy is large compared to the kinetic energy, minimize $E(a)$ as a function of a for the repulsive case $g > 0$. How do $a_{\min}$ and $\langle H \rangle_{\min}$ scale with N?

d. **(6 points)** The coupling, g, is proportional to the scattering length, ℓ, for the bosons

$$g = \frac{4\pi\hbar^2\ell}{m} \ .$$

Assume the trap has a frequency $\omega = 2\pi \times 145$ Hz and a scattering length $\ell = -1.5$ nm for lithium-7. What is the maximum number of lithium-7 atoms that can be placed in the trap?

BEC of Lithium-7: Solution

a. Lithium-7 has 3 protons, 4 neutrons, and 3 electrons. The total number of fermionic constituents is an even number, ten, and thus lithium-7 is a boson.

b. To simplify the algebra, we begin by integrating the kinetic energy term in $\langle H \rangle$ by parts:

$$\begin{aligned} \langle H \rangle &= \int d^3x \left(-\frac{\hbar^2}{2m} \psi^* \nabla^2 \psi + \frac{1}{2} m\omega^2 r^2 |\psi|^2 + \frac{Ng}{2} |\psi|^4 \right) \\ &= \int d^3x \left(\frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{1}{2} m\omega^2 r^2 |\psi|^2 + \frac{Ng}{2} |\psi|^4 \right) \ . \end{aligned}$$

The normalization condition for the trial wave function $\psi = c\, e^{-r^2/2a^2}$ is

$$1 = \int \psi^2 \, d^3x = 4\pi c^2 \int_0^\infty r^2 e^{-r^2/a^2} dr = c^2 \pi^{3/2} a^3 \ .$$

Now we are ready to compute $\langle H \rangle$:

$$\begin{aligned} E(a) &= \int d^3x \left(\frac{\hbar^2}{2m} (\nabla \psi)^2 + \frac{1}{2} m\omega^2 r^2 \psi^2 + \frac{Ng}{2} \psi^4 \right) \\ &= 4\pi c^2 \int_0^\infty \left[\frac{\hbar^2}{2ma^4} r^2 e^{-r^2/a^2} + \frac{m\omega^2}{2} r^2 e^{-r^2/a^2} + \frac{Ng}{2} c^2 e^{-2r^2/a^2} \right] r^2 \, dr \ . \end{aligned}$$

A few handy integrals are

$$\begin{aligned} \int_0^\infty r^4 e^{-r^2/a^2} dr &= \frac{3}{8} \sqrt{\pi} a^5 \ , \\ \int_0^\infty r^2 e^{-2r^2/a^2} dr &= \frac{1}{16} \sqrt{2\pi} a^3 \ . \end{aligned}$$

Thus

$$\begin{aligned} E(a) &= \frac{1}{4} \pi^{3/2} a^3 c^2 \left(\frac{3\hbar^2}{ma^2} + 3m\omega^2 a^2 + \frac{1}{\sqrt{2}} Ngc^2 \right) \\ &= \frac{1}{4} \left(\frac{3\hbar^2}{ma^2} + 3m\omega^2 a^2 + \frac{\sqrt{2} Ng}{2\pi^{3/2} a^3} \right) \ . \end{aligned}$$

The parameter a governs the size of the Gaussian. By the uncertainty principle, if the Gaussian is larger in position space, the average value of $\langle p^2 \rangle$ and hence the kinetic energy will be less. On the other hand, the larger the Gaussian, the larger the potential energy from the harmonic oscillator potential. The energy contribution from the interactions depends sensitively on the sign of g, favoring smaller a for negative g and larger a for positive g.

c. We should minimize the energy without the contribution from the kinetic term:

$$F(a) = \frac{1}{4}\left(3m\omega^2 a^2 + \frac{\sqrt{2}Ng}{2\pi^{3/2}a^3}\right) \ .$$

The condition $dF/da = 0$ implies

$$a_{\min} = \left(\frac{Ng}{(2\pi)^{3/2}m\omega^2}\right)^{1/5}$$

Thus the energy at the minimum is

$$F_{\min} = \frac{5}{4}\left(\frac{N^2g^2m^3\omega^6}{8\pi^3}\right)^{1/5} \ .$$

These scalings $a \sim N^{1/5}$ and $F \sim N^{2/5}$ have been verified experimentally.

d. When $g < 0$, the system can at best be meta-stable because $E(a) \to -\infty$ as $a \to 0$. However, for small enough N there will be a local maximum $a_{\max} > 0$ and a further local minimum $a_{\min} > a_{\max}$. A condensate with $a = a_{\min}$ can exist for a little while. As N is increased, these two critical points move together and eventually coalesce. The condition for this coalescence is that both $dE/da = 0$ and $d^2E/da^2 = 0$. In other words

$$\begin{aligned} \frac{\hbar^2}{ma^4} - m\omega^2 + \frac{\sqrt{2}Ng}{4\pi^{3/2}a^5} &= 0 \ , \\ \frac{3\hbar^2}{ma^4} + m\omega^2 + \frac{\sqrt{2}Ng}{\pi^{3/2}a^5} &= 0 \ . \end{aligned}$$

We solve these simultaneous equations for a and N yielding

$$a^4 = \frac{\hbar^2}{5m^2\omega^2} \quad \text{and} \quad N_{\max} = -\frac{8\sqrt{2}\pi^{3/2}\hbar^{5/2}}{5^{5/4}m^{3/2}\omega^{1/2}g} = \frac{2}{5^{5/4}|\ell|}\sqrt{\frac{2\pi\hbar}{m\omega}} \ .$$

Using the values suggested in the problem, we find that $N_{\max} \approx 1400$ atoms. Indeed Randy Hulet was never able to make large condensates of lithium-7 atoms.

Quantum Mechanics 3

Zitterbewegung and the Darwin Term

The phenomenon of Zitterbewegung ("quivering motion") for an electron was predicted by Schrödinger in 1928, and is a peculiar consequence of Dirac's 1928 relativistic quantum theory. In this problem we will use the Zitterbewegung to give a physical interpretation of the Darwin term, which is a relativistic correction to the non-relativistic Hamiltonian arising from an approximation of the Dirac equation.

(a) (5 points) For the electron in the hydrogen atom the Hamiltonian takes the following approximate form

$$H = \frac{p^2}{2m} + V(r) - \frac{p^4}{8m^3c^2} + \frac{\hbar^2}{4m^2c^2}\frac{1}{r}\frac{dV}{dr}\,\boldsymbol{\sigma}\cdot\boldsymbol{L} + \underbrace{\frac{\hbar^2}{8m^2c^2}\nabla^2 V(r)}_{\text{The Darwin term!}}\,, \tag{1}$$

where $V(r) = -e^2/r$ is the Coulomb potential, and m is the electron mass. The last term is known as the Darwin term.

(i) Briefly state the origin of the the p^4 term and the $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ term (no long derivations).

(ii) Determine the energy shift δE from the Darwin term to the $1s$ state of hydrogen with wave function $\psi_{1s}(r) = e^{-r/a_o}/\sqrt{\pi a_0^3}$.

(iii) Evaluate the magnitude of the shift $\delta E/E_{1s}$ numerically.

(iv) What is the shift for the $2p$ state? Explain.

The origin of the Darwin term can be understood with the Dirac Hamiltonian

$$H = c\,\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta\, mc^2\,. \tag{2}$$

Here $\boldsymbol{\alpha}$ and β are 4×4 Dirac matrices which (in the Schrödinger picture) are[4]

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}\,, \qquad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\,, \tag{3}$$

where σ^k are the Pauli matrices and $\mathbb{I}$ is the unit matrix. In the Heisenberg representation these matrices are time-dependent operators, $\hat{\boldsymbol{\alpha}}(t)$ and $\hat{\beta}(t)$. They satisfy the anti-commutation relations

$$\left\{\alpha^i, \alpha^j\right\} = 2\,\delta^{ij}\cdot\mathbb{I}_{4\times4}, \qquad \{\beta,\beta\} = 2\cdot\mathbb{I}_{4\times4}, \qquad \{\alpha,\beta\} = 0. \tag{4}$$

(b) (2 points) Using the Heisenberg picture evaluate the velocity operator $\hat{v}^k = \mathrm{d}\hat{x}^k(t)/\mathrm{d}t$, and show that the eigenvalues of $\hat{v}^z$ are $\pm c$.

[4] These matrices are given in the Dirac representation.

(c) (5 points) Now evaluate $\mathrm{d}\hat{v}^k/\mathrm{d}t$ in the Heisenberg picture, and write the result in terms of $\hat{p}^k$ and $\hat{v}^k$ and H. Why are $\hat{p}^k$ and H time independent?

(i) First for $p^k \simeq 0$, determine $\hat{v}^k(t)$ and then $\hat{x}^k(t)$ by integrating the Heisenberg equations of motion.

(ii) Now for general p^k, determine $\hat{v}^k(t)$ and then $\hat{x}^k(t)$ by integrating the Heisenberg equations of motion.

You should find a term linear in time and an oscillatory term. Interpret the linear term. The oscillating motion in $\hat{x}^k(t)$ is known as Zitterbewegung or "quivering motion".

(d) (6 points) Now consider the following electron wave function with $\langle\vec{p}\rangle \simeq 0$ in a specific superposition of states at time $t = 0$

$$\psi(\vec{x}) = \frac{\psi_0(\vec{x})}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \tag{5}$$

Here $\psi_0(\vec{x})$ is a normalized Gaussian wave packet with a large spatial width, so that the uncertainty in $\vec{p}$ is negligibly small.

(i) Using the results of part (c), determine the mean position and velocity of the electron in the z direction, $\langle\hat{z}(t)\rangle$ and $\langle\hat{v}^z(t)\rangle$.

(ii) Numerically determine the amplitude of the oscillating term of $\langle\hat{z}(t)\rangle$ in meters. Compare the amplitude of the Zitterbewegung oscillatory motion to the Bohr radius a_o.

(iii) Using microscopy, could it be possible in the future to directly detect the oscillatory motion of the electron? Explain.

(e) (2 points) Show that the Darwin correction to the Hamiltonian can be qualitatively, and even quantitatively, explained as the average potential experienced by an electron undergoing Zitterbewegung oscillations around its equilibrium position $\vec{r}$.

Solution

(a) (i) The p^4 term arises from an expansion of the energy $E = \sqrt{(cp)^2 + (mc^2)^2}$. The spin orbit term $\boldsymbol{\sigma} \cdot \boldsymbol{L}$ reflects the coupling between the spin and the magnetic field in the rest frame of the electron.

(ii) Examining the Darwin term for the Coulomb potential

$$-\nabla^2 \frac{1}{r} = 4\pi\delta^3(\boldsymbol{r})\,, \tag{6}$$

the perturbing Hamiltonian reduces to

$$H = \frac{\pi e^2\hbar^2}{2m^2c^2}\delta^3(\boldsymbol{r})\,. \tag{7}$$

Thus the energy shift is

$$\delta E = \int d^3\boldsymbol{r}\,\psi^*(\boldsymbol{r})H\psi(\boldsymbol{r}) = \frac{\pi e^2\hbar^2}{2m^2c^2}|\psi(\boldsymbol{0})|^2\,. \tag{8}$$

For the $1s$ state

$$|\psi_{1s}(0)|^2 = \frac{1}{\pi a_0^3}\,, \tag{9}$$

and the shift is

$$\delta E = \frac{e^2\hbar^2}{2m^2c^2a_0^3}\,. \tag{10}$$

(iii) To evaluate the shift numerically we note that

$$|E_{1s}| = \frac{e^2}{2a_0} = \frac{\hbar^2}{2ma_0^2} = \tfrac{1}{2}mc^2\alpha^2\,, \tag{11}$$

where $\alpha = e^2/\hbar c = 1/137$. Thus

$$\frac{\delta E}{|E_{1s}|} = \frac{1}{mc^2}\frac{e^2}{a_o} = \alpha^2\,, \tag{12}$$

which is consistent with a non-relativistic approximation of order $(v/c)^2$.

(iv) For the $2p$ state the shift is zero since the wave function vanishes at the origin.

(b) To evaluate $\dot{x}^k$ we note that

$$\frac{\mathrm{d}\hat{x}^k}{\mathrm{d}t} = \frac{-i}{\hbar}[\hat{x}^k, H] = c\hat{\alpha}^k \tag{13}$$

Since

$$\alpha^k\alpha^k = \mathbb{I}_{4\times4} \quad \text{(no sum over } k\text{)} \tag{14}$$

The eigenvalues of $\boldsymbol{\alpha}$ are therefore ±1, and thus the eigenvalues of $\hat{v}^k = c\hat{\alpha}^k$ are $\pm c$.

(c) Now we evaluate

$$\frac{\mathrm{d}\hat{v}^k}{\mathrm{d}t} = \frac{-i}{\hbar}[\hat{v}^k, H] = \frac{-i}{\hbar}\left(\hat{v}^k H - H\hat{v}^k\right) . \tag{15}$$

Anti-commuting $\hat{v}^k$ across the Hamiltonian ($H = \boldsymbol{v}\cdot\boldsymbol{p} + \beta mc^2$) using the anti-commutation relations,

$$Hv^k = -v^k H + 2c^2 p^k , \tag{16}$$

we find

$$\frac{\mathrm{d}\hat{v}^k}{\mathrm{d}t} + \frac{2i}{\hbar}\hat{v}^k H = \frac{2i}{\hbar}c^2\hat{p}^k . \tag{17}$$

where H and $\boldsymbol{p}$ are constants of the motion.

(i) The homogeneous equation is

$$\frac{\mathrm{d}v^k}{\mathrm{d}t} + \frac{2i}{\hbar}v^k H = 0 , \tag{18}$$

and has solution[5]

$$v^k(t) = v^k(0)e^{-\frac{i}{\hbar}2Ht} . \tag{19}$$

Then integrating we find

$$x^k(t) = x^k(0) + v^k(0)\frac{i\hbar}{2H}\left(e^{-\frac{i}{\hbar}2Ht} - \mathbb{I}\right) . \tag{20}$$

(ii) For $p^k \neq 0$, the rhs of Eq. (17) acts as a driving term. Multiplying Eq. (17) by e^{i2Ht} we find

$$\frac{\mathrm{d}(v^k e^{i2Ht})}{\mathrm{d}t} = \frac{i}{\hbar}2c^2 p^k e^{i2Ht} . \tag{21}$$

Integrating we both sides we have

$$v^k(t)e^{i2Ht} - v^k(0) = \frac{c^2 p^k}{H}e^{i2Ht} - \frac{c^2 p^k}{H} , \tag{22}$$

or

$$v^k(t) = \frac{c^2 p^k}{H} + \left(v^k(0) - \frac{c^2 p^k}{H}\right)e^{-\frac{i}{\hbar}2Ht} . \tag{23}$$

Integrating once again we have

$$x^k(t) = x^k(0) + \left(\frac{c^2 p^k}{H}\right)t + \left(v^k(0) - \frac{c^2 p^k}{H}\right)\frac{i\hbar}{2H}\left(e^{-\frac{i}{\hbar}2Ht} - \mathbb{I}\right) . \tag{24}$$

[5]Here and below we will drop the "hats" in $\hat{v}^k(t)$ when confusion can not arise.

(d) (i) Now we evaluate Eq. (19) and Eq. (20) for a specific state. First we note that the given wave function

$$\psi = \frac{\psi_0(\vec{x})}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{25}$$

is in an eigenstate of the $\hat{v}^z$ operator

$$\hat{v}^z(0)\psi = c\psi\,, \tag{26}$$

We also note that since

$$H \simeq \begin{pmatrix} mc^2\,\mathbb{I} & 0 \\ 0 & -mc^2\,\mathbb{I} \end{pmatrix}, \tag{27}$$

we find

$$e^{-\frac{2i}{\hbar}Ht}\psi = \frac{\psi_0(\vec{x})}{\sqrt{2}} \begin{pmatrix} e^{-\frac{2i}{\hbar}mc^2t} \\ 0 \\ e^{+\frac{2i}{\hbar}mc^2t} \\ 0 \end{pmatrix}. \tag{28}$$

So

$$\langle \hat{v}^z(t) \rangle = \int d^3\vec{x}\, \psi^\dagger\, \hat{v}^z(t)\, \psi = c\cos\left(\frac{2mc^2t}{\hbar}\right), \tag{29}$$

and integrating we find

$$\langle \hat{z}(t) \rangle = \text{const} + \frac{\hbar}{2mc}\sin\left(\frac{2mc^2t}{\hbar}\right). \tag{30}$$

(ii) The amplitude is half the electron Compton wavelength

$$\text{amplitude} = \frac{\hbar c}{2mc^2} = \frac{1}{2}\frac{200\,\text{eV}\cdot\text{nm}}{0.5\times10^6\,\text{eV}} \simeq 2\times10^{-13}\,\text{m}\,. \tag{31}$$

This is significantly smaller than the Bohr radius which is half an Angstrom

$$a_0 = \frac{\hbar}{mc\alpha} \sim 0.05\,\text{nm}\,, \tag{32}$$

where $\alpha = e^2/(\hbar c) = 1/137$ is the fine structure constant. Thus

$$\frac{\text{amplitude}}{a_o} \sim \frac{1}{2\cdot 137}\,. \tag{33}$$

(iii) In order to see the Zitterbewegung oscillations with microscopy, the scattered photons would need a wavelength of order half a Compton wavelength

$$\lambda \sim \frac{h}{2mc}\,. \tag{34}$$

At this point the energy of the photon is

$$E \sim \frac{hc}{\lambda} \sim 2mc^2, \tag{35}$$

which is sufficient to produce electron-positron pairs. This essentially makes direct detection of Zitterbewegung impossible.

(e) The Coulomb potential including small oscillations of the position of the electron $\boldsymbol{x} \to \boldsymbol{x} + \delta\boldsymbol{x}$ is

$$V(\boldsymbol{x} + \delta\boldsymbol{x}) = V(\boldsymbol{x}) + \frac{\partial V}{\partial x^i}\,\delta x^i + \frac{1}{2}\frac{\partial^2 V}{\partial x^i \partial x^j}\delta x^i \delta x^j \tag{36}$$

Averaging over time with the statistics

$$\left\langle \delta x^i(t) \right\rangle_{\text{time}} = 0\,, \qquad \left\langle \delta x^i(t)\delta x^j(t) \right\rangle_{\text{time}} = \left(\frac{\hbar}{2mc}\right)^2 \delta^{ij}\,, \tag{37}$$

yields the Darwin term

$$\left\langle V(\boldsymbol{x} + \delta\boldsymbol{x}) \right\rangle_{\text{time}} = V(\boldsymbol{x}) + \frac{\hbar^2}{8m^2c^2}\frac{\partial^2 V}{\partial x^i \partial x^i}\,. \tag{38}$$

It is perhaps fortuitous that this qualitative derivation of the Darwin term precisely reproduces the coefficient of $\nabla^2 V$ term.

Statistical Mechanics 1

Blackbody radiation and its fluctuations

This problem addresses properties of the spontaneous electromagnetic radiation at thermal equilibrium.

(a) [2 points] Calculate the probability for a one-dimensional quantum harmonic oscillator of eigenfrequency ω to be on its n^{th} energy level, in thermal equilibrium at temperature T.

(b) [3 points] Calculate the average energy, the free energy, and the entropy of the oscillator, and discuss their dependences on temperature.

(c) [4 points] Calculate the variance (dispersion) of fluctuations of oscillator's energy, and express it via the average energy and $\hbar\omega$.

(d) [2 points] Calculate the number of electromagnetic standing-wave modes in a large, closed free-space volume V, with frequencies within a narrow interval $[\omega, \omega + d\omega]$, where $d\omega$ is much smaller than ω, but still large enough to contain many modes. Briefly explain why each mode may be treated as a one-dimensional quantum harmonic oscillator.

(e) [6 points] Calculate the average total energy of the electromagnetic field in volume V (including all essential modes), and the variance of its fluctuations. Express the variance via the average energy and temperature, and find the dependence of the relative r.m.s. fluctuation of the energy on temperature T and volume V.

(f) [3 points] How large should volume V be for your results to be qualitatively valid? Evaluate the condition for room temperature.

Hint: You may like to use the following table integral: $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$.

Statistical Mechanics 1

Solutions

(a) [2 points] We may calculate the probability W_n of the oscillator being on n^{th} energy level, using the canonical (Gibbs) distribution:

$$W_n = \frac{1}{Z}\exp\left\{-\frac{E_n}{T}\right\}, \tag{1}$$

where E_n is the energy of the n^{th} eigenstate, $T \equiv k_B T_K$ is temperature in energy units, and Z is the statistical sum

$$Z = \sum_n \exp\left\{-\frac{E_n}{T}\right\}. \tag{2}$$

For a quantum oscillator, it is convenient to refer the energy to the (temperature-independent) ground state energy $\hbar\omega/2$, then $E_n = n\hbar\omega$ (with $n = 0, 1, 2,\ldots$), and Eq. (2) yields

$$Z \equiv \sum_{n=0}^{\infty}\exp\left\{-\frac{n\hbar\omega}{T}\right\} = \sum_{n=0}^{\infty}\lambda^n, \quad \text{where } \lambda \equiv \exp\left\{-\frac{\hbar\omega}{T}\right\} \le 1. \tag{3}$$

This series is just the infinite geometric progression, whose sum is well known:

$$Z = \frac{1}{1-\lambda} = \frac{1}{1-e^{-\hbar\omega/T}}, \tag{4}$$

so that Eq. (1) gives

$$W_n = \left(1-e^{-\hbar\omega/T}\right)e^{-n\hbar\omega/T}.$$

(b) [3 points] This average energy of the oscillator may be calculated in any of two ways: either directly:

$$\langle E\rangle = \sum_{n=0}^{\infty} E_n W_n = \left(1-e^{-\hbar\omega/T}\right)\sum_{n=0}^{\infty} n\hbar\omega\, e^{-n\hbar\omega/T}$$

(where the sum may be readily calculated by the differentiation of Eqs. (3) and (4) over the reciprocal temperature $\beta \equiv 1/T$), or using the well-known general relation (see *Appendix 1*)

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z\,. \tag{5}$$

Both methods give the same result:

$$\langle E\rangle = \frac{\hbar\omega}{e^{\hbar\omega/T}-1}. \tag{6}$$

The free energy of the oscillator is

$$F = T\ln\frac{1}{Z} = T\ln\left(1 - e^{-\hbar\omega/T}\right),$$

and now its entropy may be found from thermodynamics: either as $S = -\partial F/\partial T$, or (even more easily) as $S = (\langle E\rangle - F)/T$. Both relations give, of course, the same result:

$$S = \frac{\hbar\omega}{T}\frac{1}{e^{\hbar\omega/T} - 1} - \ln\left(1 - e^{-\hbar\omega/T}\right).$$

At $T \to 0$, all these averages ($\langle E\rangle$, F, and S) tend to zero, while in the classical limit ($T >> \hbar\omega$), their magnitudes grow: $\langle E\rangle$ tends to the equipartition-theorem value T, $F \to -T\ln(T/\hbar\omega)$, and $S \to \ln(T/\hbar\omega)$. (The last trend reflects the growth of the number $M \sim T/\hbar\omega$ of the levels with energies not exceeding T too much, and hence having substantial occupancy.)

(c) [4 points] The variance of energy fluctuations $\widetilde{E} \equiv E - \langle E\rangle$ may be found from the following general relation (which may be readily derived from the Gibbs distribution – see *Appendix 2*):

$$\left\langle \widetilde{E}^2 \right\rangle = -\frac{\partial\langle E\rangle}{\partial\beta}. \tag{7}$$

With Eq. (5), this expression yields

$$\left\langle \widetilde{E}^2 \right\rangle = -\frac{\partial}{\partial\beta}\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} = \left(\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\right)^2 e^{\beta\hbar\omega} \equiv \left(\frac{\hbar\omega}{e^{\hbar\omega/T} - 1}\right)^2 e^{\hbar\omega/T}. \tag{8}$$

It is straightforward to verify that this expression is simply related to Eq. (6):[1]

$$\left\langle \widetilde{E}^2 \right\rangle = \hbar\omega\langle E\rangle + \langle E\rangle^2.$$

This result shows that r.m.s. fluctuation of energy,

$$\delta E \equiv \left\langle \widetilde{E}^2 \right\rangle^{1/2},$$

is always larger than its average value, and approaches it only in the classical limit T, when, as was discussed above, $\langle E\rangle \to T >> \hbar\omega$.

[1] Note, however, that this relation is valid only if E is referred to ground state energy $\hbar\omega/2$ of the oscillator. It was first obtained in 1909 by A. Einstein from the Planck's blackbody radiation formula (which does not involve the ground state energy), and is reproduced in some textbooks without proper qualification. If the energy is referred to the minimum of the oscillator's potential energy, then the relation is different: $\left\langle \widetilde{E}^2 \right\rangle = \langle E'\rangle^2 - (\hbar\omega/2)^2$, where $E' \equiv E + \hbar\omega/2$.

(d) [2 points] If the volume V is large enough (see below), we may apply the well-known general result for the number of different standing wave modes with the wave numbers within small interval $dk \ll k$:

$$dN = \frac{gV}{(2\pi)^3} d^3k = \frac{gV}{(2\pi)^3} 4\pi k^2 dk ,$$

regardless of the boundary conditions on the walls limiting the volume. For the electromagnetic waves in free space, $k = \omega/c$, and the degeneracy factor g equals 2, due to two independent polarizations of these transverse waves. As a result, we get

$$dN = \frac{2V}{(2\pi)^3} 4\pi \frac{\omega^2 d\omega}{c^3} = V \frac{\omega^2}{\pi^2 c^3} d\omega. \qquad (9)$$

The variable separation in the source-free Maxwell equations of classical electrodynamics shows that, the amplitude of each spatial mode obeys the same temporal equation as a one-dimensional harmonic oscillator, so that the transition to from classical to quantum mechanics may be performed exactly as for a mechanical one-dimensional oscillator, so that all the above formulas are applicable to each mode.

(e) [6 points] The field oscillators at different frequencies (and at different polarizations at the same frequency) are independent, and the fluctuations of their energies are uncorrelated, so that their average energies and the average squares (i.e. variances) of their fluctuations just add up. As a result, the average total energy E_t of the field may be calculated as

$$\langle E_t \rangle = \int_{\omega=0}^{\omega=\infty} \langle E \rangle dN = \int_0^\infty \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} V \frac{\omega^2}{\pi^2 c^3} d\omega = \frac{VT^4}{\hbar^3 \pi^2 c^3} \int_0^\infty \frac{x^3 dx}{e^x - 1},$$

where $x \equiv \hbar\omega/T$. Using the table integral provided in the Hint, we get the famous result

$$\langle E_t \rangle = \frac{\pi^2}{15} \frac{VT^4}{\hbar^3 c^3}, \qquad (10)$$

which is directly and simply related to the Stefan-Boltzmann law for the blackbody radiation power.

Similarly, the variance of fluctuations of the total energy E_t,

$$\langle \tilde{E}_t^2 \rangle = \int_{\omega=0}^{\omega=\infty} \langle \tilde{E}^2 \rangle dN ,$$

may be calculated by the direct integration of Eq. (8); however, in order to avoid working out another integral, it is beneficial to use Eq. (7) to write

$$\langle \tilde{E}_t^2 \rangle = -\int_{\omega=0}^{\omega=\infty} \frac{\partial \langle E \rangle}{\partial \beta} dN \equiv -\int_0^\infty \frac{\partial \langle E \rangle}{\partial \beta} \frac{dN}{d\omega} d\omega .$$

Since, according to Eq. (9), the density of states $dN/d\omega$ is temperature-independent, we may take the partial derivative $\partial/\partial\beta$ out of the integral, and notice that the remaining integral is just the average total energy, which was already calculated above – see Eq. (10):

$$\left\langle \widetilde{E}_t^2 \right\rangle = -\frac{\partial}{\partial\beta}\int_0^\infty \langle E\rangle dN \equiv -\frac{\partial}{\partial\beta}\langle E_t\rangle = -\frac{\pi^2}{15}\frac{V}{\hbar^3 c^3}\frac{d\left(\beta^{-4}\right)}{d\beta} = \frac{\pi^2}{15}\frac{V}{\hbar^3 c^3}4\beta^{-5} = 4T\langle E_t\rangle . \qquad (11)$$

The relative r.m.s. fluctuation of the total energy is

$$\frac{\delta E_t}{\langle E_t\rangle} \equiv \frac{\left\langle \widetilde{E}_t^2 \right\rangle^{1/2}}{\langle E_t\rangle} = 2\left(\frac{T}{\langle E_t\rangle}\right)^{1/2} \propto \frac{1}{T^{3/2}V^{1/2}} .$$

Somewhat counter-intuitively, the relative fluctuation decreases with temperature. The result shows also that by increasing the volume V, we can always make the fluctuation much smaller than 1. (The last result has important implications for accurate measurements of the fundamental anisotropy of the cosmic background radiation.)

(f) [3 points] Eq. (9) is strictly valid only if $dN >> 1$, i.e. volume V is much larger that than the cube of the wavelength: $V >> c^3/\omega^3$ for all substantial frequencies. In thermal equilibrium, the right-hand side of this relation should be evaluated for the frequencies that dominate integral (11), namely $\hbar\omega \sim T$, so that the required condition is

$$V >> \left(\frac{c}{\omega}\right)^3 \sim \left(\frac{\hbar c}{T}\right)^3 .$$

For room temperature ($T \equiv k_B T_K \approx 1.38\times10^{-23}\text{J/K}\times300\text{K} \approx 4\times10^{-21}\text{J}$) the right-hand side of this relation is of the order of $10^{-15}\text{m}^3 = (10\ \mu\text{m})^3$, i.e. small on the human scale, but not quite microscopic.

Appendix 1

For a system with the general energy spectrum E_n, we may calculate the average energy as

$$\langle E\rangle = \sum_n E_n W_n = \frac{1}{Z}\sum_n E_n e^{-E_n/T} \equiv \frac{1}{Z}\sum_n E_n e^{-\beta E_n} ,$$

On the other hand, using Eq. (2) to write

$$-\ln Z = -\ln\sum_n \exp\left\{-\frac{E_n}{T}\right\} \equiv -\ln\sum_n e^{-\beta E n} ,$$

and differentiating both parts of this relation over β, we get

$$-\frac{\partial}{\partial\beta}\ln Z = -\frac{\partial}{\partial\beta}\ln\sum_n e^{-\beta E_n} = \frac{1}{\sum_n e^{-\beta E n}}\sum_n E_n e^{-\beta E_n} \equiv \frac{1}{Z}\sum_n E_n e^{-\beta E_n} = \langle E\rangle ,$$

thus proving Eq. (5).

Appendix 2

The average square of energy may be calculated as

$$\left\langle E^2 \right\rangle = \sum_n W_n E_n^2 = \frac{1}{Z} \sum_n E_n^2 e^{-\beta E n} \ .$$

On the other hand, a double differentiation of the statistical sum over β yields

$$\frac{\partial^2 Z}{\partial \beta^2} = \frac{\partial^2}{\partial \beta^2} \sum_n e^{-\beta E_n} = \sum_n E_n^2 e^{-\beta E_n} \ .$$

A comparison of these two formulas show that

$$\left\langle E^2 \right\rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2},$$

so that using Eq. (5), for the energy fluctuation variance we may write

$$\left\langle \widetilde{E}^2 \right\rangle = \left\langle E^2 \right\rangle - \left\langle E \right\rangle^2 = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \frac{1}{Z^2} \left(\frac{\partial Z}{\partial \beta} \right)^2 = -\frac{\partial}{\partial \beta} \left(-\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) = -\frac{\partial \left\langle E \right\rangle}{\partial \beta},$$

thus proving Eq. (7).

Statistical Mechanics 2

Ising model on a triangle

The Ising model on a triangle is described by the energy:

$$E = -J(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - h(\sigma_1 + \sigma_2 + \sigma_3)\,.$$

Here J and h are known parameters: exchange energy and external magnetic field, respectively. The Ising spins $\sigma_{1,2,3}$ are the only degrees of freedom in the problem and they are taking values ± 1. Assume that the temperature of the system is T.

(a) [2 points] Compute the partition function of the model.

(b) [3 points] Compute the free energy and the entropy of the model.

(c) [3 points] Compute the specific heat at temperature T and $h = 0$. Plot qualitative dependence of the specific heat as a function of T.

(d) [4 points] Compute the magnetization $M = \langle\sigma\rangle \equiv \langle\sigma_1+\sigma_2+\sigma_3\rangle$ at given h and $T \ll J$. What is the behavior of the magnetic susceptibility $\chi = \frac{\partial M}{\partial h}\Big|_{h\to 0}$ at low temperature?

(e) [8 points] Find the fluctuation of magnetization $\langle(\sigma - M)^2\rangle$ at $T \ll J$.

Solution

It is convenient to rewrite the energy in terms of the total spin $\sigma = \sigma_1 + \sigma_2 + \sigma_3$. We have

$$E(\sigma) = -\frac{J}{2}(\sigma^2 - 3) - h\sigma\,.$$

There is 1 state with $\sigma = 3$, 3 states with $\sigma = 1$, 3 states with $\sigma = -1$ and 1 state with $\sigma = -3$.

(a) [2 points] Taking into account the weights of states $\exp(-E/T)$ with their multiplicities we have for the partition function

$$\begin{aligned} Z &= e^{\frac{3J+3h}{T}} + 3e^{-\frac{J-h}{T}} + 3e^{-\frac{J+h}{T}} + e^{\frac{3J-3h}{T}} \\ &= 2e^{\frac{3J}{T}} \cosh\left(3\frac{h}{T}\right) + 6e^{-\frac{J}{T}} \cosh\left(\frac{h}{T}\right)\,. \end{aligned}$$

(b) [3 points] The free energy is given by the standard expression $F = -T \log Z$ or

$$F = -T \log\left[2e^{\frac{3J}{T}} \cosh\left(3\frac{h}{T}\right) + 6e^{-\frac{J}{T}} \cosh\left(\frac{h}{T}\right)\right]\,.$$

(c) [3 points] At $h = 0$ we have for free energy

$$F(h=0) = -T \log\left[2e^{\frac{3J}{T}} + 6e^{-\frac{J}{T}}\right]$$

and we obtain the entropy

$$S = -\frac{\partial F}{\partial T} = \log\left[2e^{\frac{3J}{T}} + 6e^{-\frac{J}{T}}\right] - \frac{J}{T}\frac{3e^{\frac{3J}{T}} - 3e^{-\frac{J}{T}}}{e^{\frac{3J}{T}} + 3e^{-\frac{J}{T}}}\,.$$

Here we can either calculate the specific heat as $c = T\frac{dS}{dT}$ or first calculate the energy

$$E = F + TS = -J\frac{3e^{\frac{3J}{T}} - 3e^{-\frac{J}{T}}}{e^{\frac{3J}{T}} + 3e^{-\frac{J}{T}}} = -3J\frac{1 - e^{-\frac{4J}{T}}}{1 + 3e^{-\frac{4J}{T}}} = -3J + 12J\frac{e^{-\frac{4J}{T}}}{1 + 3e^{-\frac{4J}{T}}}$$

(this expression is, of course, can be written right away as the average of energy) and then specific heat as

$$c = \frac{dE}{dT} = \frac{48J^2}{T^2}\frac{e^{-\frac{4J}{T}}}{\left(1 + 3e^{-\frac{4J}{T}}\right)^2}$$

At $T \ll J$ we have $c \sim \frac{48J^2}{T^2}e^{-\frac{4J}{T}}$ and at $T \gg J$ we have $c \sim \frac{12J^2}{T^2}$. The maximum is at $T \sim J$. The plot is very easy to make now.

(d) [4 points] The magnetization is given by

$$M = -\frac{\partial F}{\partial h} = 3\frac{e^{\frac{3J}{T}}\sinh\left(\frac{3h}{T}\right) + e^{-\frac{J}{T}}\sinh\left(\frac{h}{T}\right)}{e^{\frac{3J}{T}}\cosh\left(\frac{3h}{T}\right) + 3e^{-\frac{J}{T}}\cosh\left(\frac{h}{T}\right)} \approx 3\tanh\left(\frac{3h}{T}\right) .$$

We used here low temperature condition $T \ll J$. For susceptibility we have

$$\chi = \frac{\partial M}{\partial h}\Big|_{h\to 0} \approx \frac{9}{T} ,$$

that is Curie's law.

(e) [8 points] The fluctuation is given by

$$\langle\langle\sigma^2\rangle\rangle = \langle(\sigma - M)^2\rangle = -T\frac{\partial^2 F}{\partial h^2} = T\frac{\partial M}{\partial h} = \frac{9}{\cosh^2(3h/T)} .$$

At $h = 0$ it becomes equal 9 which is pretty obvious as σ is fluctuating with equal probability between 3 and -3 at this point.

Statistical Mechanics 3

Bose-Einstein condensation

This problem addresses the Bose-Einstein condensation (BEC) of the gas of $N >> 1$ indistinguishable, noninteracting bosons of mass m, in various confining potentials.

(a) [4 points] Calculate the critical temperature T_c of the condensation in a rectangular, hard-wall box of volume $V = a \times b \times c$, with all linear sizes of the same order. What is the exact value of the chemical potential μ at $T < T_c$?

(b) [3 points] Now one of the box sizes (say, c) is slowly reduced, while other two dimensions are increased to keep the volume V constant. Estimate the value c_0 at which T_c becomes substantially affected by the change.

(c) [4 points] Can the BEC take place at $c << c_0$? If yes, calculate the corresponding T_c. If not, provide a proof.

(d) [7 points] Now the same particles are placed into a soft, spherically-symmetric potential well, whose potential may be approximated as $U = m\omega^2 r^2/2 \equiv m\omega^2(x^2 + y^2 + z^2)/2$. Can the BEC take place in this system? If yes, calculate the T_c.

(e) [2 points] Suggest (and justify by estimations) a simple way of experimental detection of the BEC in the case of a soft confining potential.

Hints: You may use the following table integrals,

$$\int_{a>0}^{\infty} \frac{dx}{e^x - 1} = \ln \frac{1}{1 - e^{-a}}; \qquad \int_0^{\infty} \frac{x^{s-1} dx}{e^x - 1} = \Gamma(s)\zeta(s), \quad \text{for } s > 1,$$

and treat particular values of the gamma-function $\Gamma(s)$ and the Riemann zeta-function $\zeta(s)$ as known numbers. (For $s \sim 1$, they are of the order of 1 as well, for example, $\Gamma(3/2) = \pi^{1/2}/2$, $\zeta(3/2) \approx 2.612$; $\Gamma(2) = 1$, $\zeta(2) = \pi^2/6$, etc.)

Statistical Mechanics 3

Solutions

(a) [4 points] The standard quantum-mechanical analysis of a single particle placed in a rectangular, rigid-wall box yields the following energy spectrum:

$$\varepsilon_{n_a,n_b,n_c} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_a^2}{a^2}+\frac{n_b^2}{b^2}+\frac{n_c^2}{c^2}\right),$$

with the quantum numbers n_a, n_b, and n_c taking all positive integer values starting from 1. Now we may use the use the Bose-Einstein distribution

$$\langle N_{\mathbf{n}}\rangle = \frac{1}{\exp\{(\varepsilon_{\mathbf{n}}-\mu)/T\}}, \qquad \text{with } \mathbf{n} = \{n_a, n_b, n_c\},$$

to calculate the average number of particles in the case when the box is in a thermal and chemical equilibrium with the environment with temperature T (in energy units, $T \equiv k_B T_{\text{Kelvin}}$), and chemical potential μ:

$$\langle N\rangle = g\sum_{\mathbf{n}}\langle N_{\mathbf{n}}\rangle = \sum_{n_a,n_b,n_c=1}^{\infty}\frac{1}{\exp\left\{\dfrac{\varepsilon_{n_a,n_b,n_c}-\mu}{T}\right\}-1}$$

$$= g\sum_{n_a,n_b,n_c=1}^{\infty}\frac{1}{\exp\left\{\dfrac{\pi^2\hbar^2}{2mT}\left(\dfrac{n_a^2}{a^2}+\dfrac{n_b^2}{b^2}+\dfrac{n_c^2}{c^2}\right)-\dfrac{\mu}{T}\right\}-1},$$

where g is the internal (say, spin) degeneracy of the energy levels.

Generally, this expression is only valid for the grand canonical ensemble, in which the number N of particles in the box is not fixed. However, in the canonical ensemble of systems with the number N of particles in the box fixed but very large, we may still use this formula, with the replacement $\langle N\rangle \to N$, for the calculation of the relation between the *average* values of T and μ, neglecting the small fluctuations of the chemical potential, whose r.m.s. value scales as $1/N^{1/2} << 1$. In particular, in order to calculate T_c, we have to take μ equal to the ground-state energy of a single particle, in this case

$$\varepsilon_g \equiv \varepsilon_{1,1,1} = \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right),$$

i.e. solve the following equation:

$$N = g\sum_{n_a,n_b,n_c=1}^{\infty}\frac{1}{\exp\left\{\dfrac{\pi^2\hbar^2}{2mT_c}\left(\dfrac{n_a^2-1}{a^2}+\dfrac{n_b^2-1}{b^2}+\dfrac{n_c^2-1}{c^2}\right)\right\}-1}. \qquad (*)$$

If the box sizes a, b, and c are all comparable, such a sum converges as soon as the magnitude of the argument under the exponent becomes much larger than 1, i.e. at $n_a \sim n_b \sim n_c \sim n_{\max} \equiv (T_c/\varepsilon_g)^{1/2}$ Since the first, most significant terms of the sum are of the order of 1, the sum itself may be estimated as $n^3_{\max}$, so that, by the order of magnitude, Eq. (*) gives

$$N \sim g n_{\max}^3 \equiv \left(\frac{T_c}{\varepsilon_g}\right)^{3/2}, \qquad \text{i.e. } T_c \sim \varepsilon_g \left(\frac{N}{g}\right)^{2/3}.$$

But since $N >> 1$ and $g \sim 1$, this means that T_c is much larger than ε_g, which is the scale of the distance between the adjacent energy levels (which differ by a unit change of one of the quantum numbers). Hence at $T \sim T_c$, many lower levels are populated, so that in the sums in Eq. (*), the terms (-1) may be neglected, and the sum as a whole may be approximated by an integral. As a result, the equation for T_c takes the form

$$N = g\int_0^\infty dn_a \int_0^\infty dn_b \int_0^\infty dn_c \frac{1}{\exp\left\{\frac{\pi^2\hbar^2}{2mT_c}\left(\frac{n_a^2}{a^2}+\frac{n_b^2}{b^2}+\frac{n_c^2}{c^2}\right)\right\}-1}$$

$$\equiv \frac{gV}{\pi^3\hbar^3}\int_0^\infty dp_a \int_0^\infty dp_b \int_0^\infty dp_c \frac{1}{\exp\left\{\frac{p_a^2+p_b^2+p_c^2}{2mT_c}\right\}-1}, \qquad \text{where } p_a \equiv \frac{\pi\hbar n_a}{a}, \text{ etc.},$$

and $V \equiv abc$ is the volume of the confining box. Making the transition from the Cartesian coordinates in the momentum space $\mathbf{p} = \{p_a, p_b, p_c\}$ with the elementary volume $d^3p = dp_a dp_b dp_c$, to the spherical coordinates in the same space, so that $d^3p = 4\pi p^2 dp$, and $p_a^2 + p_b^2 + p_c^2 = p^2$, we get

$$N = \frac{gV}{8\pi^3\hbar^3} 4\pi \int_0^\infty \frac{p^2 dp}{\exp\left\{\frac{p^2}{2mT_c}\right\}-1} \equiv \frac{gV}{\pi^2\hbar^3}(2mT_c)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x-1}.$$

where $x \equiv p^2/2mT_c$, and the front factor 1/8 compensates the 8-fold increase of the original integration volume $p_a, p_b, p_c > 0$. Now using the second of the provided table integrals, with $s = 3/2$, we get

$$N = \frac{gV}{\pi^2\hbar^3}(2mT_c)^{3/2}\Gamma(3/2)\zeta(3/2), \qquad \text{so that } T_c = \frac{\hbar^2}{2m}\left(\frac{\pi^2 n}{\Gamma(3/2)\zeta(3/2)g}\right)^{2/3} \sim \frac{\hbar^2}{2m}\left(\frac{N}{V}\right)^{2/3}.$$

This is the standard expression for T_c, usually derived using the well-known formula for the 3D density of states, and valid for any form of the confining box. However, the more specific derivation, carried out above, is more convenient for addressing the next tasks of the problem. It also indicates the actual exact value of the chemical potential of the condensate:

$$\mu = \varepsilon_g = \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \sim \frac{\hbar^2}{2mV^{2/3}}, \qquad \text{at } T \le T_c..$$

Since at $N >> 1$, this $\mu << T_c$, so that the result of the standard treatment, $\mu = 0$, is not entirely wrong.

More effective approaches, for $N >> 1$, i.e. $a, b >> c_0$, include the replacement of the sums over n_a and n_b with integrals, very similarly to what has been done in part (c). Another useful trick is to use the well-known formula,

$$\frac{1}{e^x - 1} = \sum_{k=1}^{\infty} e^{-kx} ,$$

to trade off the inconvenient denominator in the above formulas for an additional summation, reducing some sums to either the geometric progression or its derivatives - see, e.g., the original analysis of the BEC by R. Fowler and H. Jones, *Proc. Camb. Phil. Soc*. **34**, 573 (1928).

(b) [3 points] If c is reduced, the distances between the energy levels due to the quantization in this direction,

$$\varepsilon_c = \frac{\pi^2 \hbar^2 n_c^2}{2mc^2} \sim \frac{\pi^2 \hbar^2}{2mc^2} ,$$

grow. However, the above result for T_c shows that if the particle density N/V is kept constant (say, by increasing the base area $A \equiv ab$ of the box), this decrease does not affect the critical temperature until ε_c becomes comparable with T_c, i.e. until

$$c \sim c_0 \equiv \frac{\hbar}{(2mT_c)^{1/2}} \sim \left(\frac{V}{N}\right)^{1/3} = \left(\frac{Ac_0}{N}\right)^{1/3} .$$

Solving this (approximate) equation for the critical thickness c_0 of the box, we get the following simple estimate:

$$c_0 \sim \left(\frac{A}{N}\right)^{1/2} .$$

If $c \sim c_0$, we may use the above expression for the number of particles,

$$N = g \sum_{n_a, n_b, n_c = 1}^{\infty} \frac{1}{\exp\left\{\left[\frac{\pi^2 \hbar^2}{2mT}\left(\frac{n_a^2}{a^2} + \frac{n_b^2}{b^2} + \frac{n_c^2}{c^2}\right) - \frac{\mu}{T}\right]\right\} - 1} ,$$

for a numerical calculation of $\mu(T)$. In the conceptually simplest (though not the most computationally efficient) algorithm, for each temperature T we would gradually change the magnitude of the dimensionless fraction $\mu /T < 0$, at each step using this formula to calculate N and compare it with the given number of particles. (Their balance gives us the actual value of the chemical potential.)

More effective approaches, for $N >> 1$, i.e. $a, b >> c_0$, include the replacement of the sums over n_a and n_b with integrals, very similarly to what has been done in part (c). Another useful trick is to use the well-known formula,

$$\frac{1}{e^x - 1} = \sum_{k=1}^{\infty} e^{-kx} ,$$

to trade off the inconvenient denominator in the above formulas for an additional summation, reducing some sums to either the geometric progression or its derivatives - see, e.g., the original analysis of the BEC by R. Fowler and H. Jones, *Proc. Camb. Phil. Soc*. **34**, 573 (1928).

(c) [4 points] If $c << c_0$, all energy levels corresponding to $n_c > 1$ are much higher than the T_c evaluated above, so that at $T \sim T_c$ their population is negligible. Hence we may redo the calculations of part (a), excluding the sum over n_c. As a result, Eq. (*) is reduced to

$$N = g \sum_{n_a,n_b=1}^{\infty} \frac{1}{\exp\left\{\frac{\pi^2\hbar^2}{2mT_c}\left(\frac{n_a^2-1}{a^2}+\frac{n_b^2-1}{b^2}\right)\right\}-1}.$$

Reviewing the above estimate, we see that at $N >> 1$ we still may replace the remaining double sum by a 2D integral:

$$N = g\int_0^{\infty} dn_a \int_0^{\infty} dn_b \frac{1}{\exp\left\{\frac{\pi^2\hbar^2}{2mT_c}\left(\frac{n_a^2}{a^2}+\frac{n_b^2}{b^2}\right)\right\}-1} = \frac{gA}{\pi^2\hbar^2}\int_0^{\infty} dp_a \int_0^{\infty} dp_b \frac{1}{\exp\left\{\frac{p_a^2+p_b^2}{2mT_c}\right\}-1}.$$

Transitioning from the Cartesian coordinates $\mathbf{p} = \{p_a, p_b\}$, with the elementary area $d^2p = dp_a dp_b$ to the polar coordinates with $d^2p = 2\pi p dp$, and $p_a^2 + p_b^2 = p^2$, we get

$$N = \frac{gA}{4\pi^2\hbar^2} 2\pi \int_0^{\infty} \frac{pdp}{\exp\left\{\frac{p^2}{2mT_c}\right\}-1} \equiv \frac{gA}{4\pi\hbar^2} 2mT_c \int_0^{\infty} \frac{dx}{e^x-1}, \qquad (**)$$

where $x \equiv p^2/2mT_c$. This integral diverges at the lower limit, indicating that in this case $T_c = 0$, i.e. that the hard-confined 2D gas does not exhibit the Bose-Einstein condensation.

Note, however, that this conclusion is strictly valid only in the limit $N \to \infty$ (and hence $A \to \infty$), because the divergence of integral (**) at the lower limit is very weak (logarithmic), and is cut off by virtually any factor. In particular, a finite area $A = ab$, with $a \sim b$, keeps the particle energy quantized on a small scale $\varepsilon \sim \pi^2\hbar^2/2mA$, corresponding to $x_{\min} \sim \pi^2\hbar^2/2mAT_c << 1$. With this modification, Eq. (**), together with the first of the provided table integrals, shows that T_c is nonvanishing and has to calculated from the following transcendent equation

$$N = \frac{gAmT_c}{2\pi\hbar^2} \ln \frac{2mAT_c}{\pi^2\hbar^2},$$

where the argument of the logarithm is approximate. (More exact calculations require numerical methods.)

(d) [7 points] A well-known quantum-mechanical analysis of single particle's motion in such a potential well (frequently called the *3D harmonic oscillator*) yields the following energy spectrum:

$$\varepsilon_{n_x,n_y,n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right),$$

with quantum numbers n_x, n_y, and n_z taking all non-negative integer values starting from 0. Just as was done for the hard-wall box, we may use the Bose-Einstein distribution to calculate the average number of particles in the case when the gas is in a thermal and chemical equilibrium with the environment with temperature T and chemical potential μ:

$$\langle N \rangle = g \sum_{n_x,n_y,n_z=0}^{\infty} \left[\exp\left\{ \frac{\varepsilon_{n_x,n_y,n_z} - \mu}{T} \right\} - 1 \right]^{-1} = g \sum_{n_x,n_y,n_z=0}^{\infty} \left[\exp\left\{ \frac{\hbar\omega\left(n_x + n_y + n_z + 3/2\right) - \mu}{T} \right\} - 1 \right]^{-1},$$

where g is the spin degeneracy of each "orbital" state. Using the standard arguments for the transfer from the grand canonical to the canonical ensemble, quantitatively correct in the limit $N >> 1$, and taking the chemical potential μ equal to the ground state energy ε_g (in our current case, equal to $\varepsilon_{0,0,0} = (3/2)\hbar\omega$), we get the following equation for the critical temperature T_c:

$$N = g \sum_{n_x,n_y,n_z=0}^{\infty} \frac{1}{\exp\left\{ \frac{\hbar\omega}{T_c}\left(n_x + n_y + n_z\right) \right\} - 1}. \qquad (***)$$

Such a sum converges as soon as the magnitude of the argument under the exponent becomes much larger than 1, i.e. at $n \equiv n_x + n_y + n_z \sim n_{\max} \equiv T_c/\hbar\omega$. Since the first, significant terms of the sum are of the order of 1, the sum itself may be estimated as $n^3_{\max}$, giving

$$N \sim g n^3_{\max} \equiv \left(\frac{T_c}{\hbar\omega} \right)^3, \qquad \text{i.e. } T_c \sim \hbar\omega \left(\frac{N}{g} \right)^{1/3}.$$

But since $N >> 1$ and $g \sim 1$, so that $(N/g)^{1/3}$ is much larger than 1 as well, this means that T_c is much larger than $\hbar\omega$, which is the scale of the distance between the adjacent energy levels (which differ by a unit change of one of the quantum numbers). Hence at $T \sim T_c$, many lower levels are populated, so that the sum (**) may be well approximated by an integral.

As a result, the equation for T_c takes the form

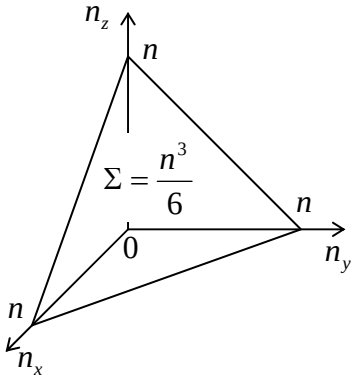
$$N = g \int_0^{\infty} dn_x \int_0^{\infty} dn_y \int_0^{\infty} dn_z \frac{1}{\exp\left\{ \frac{\hbar\omega}{T_c}\left(n_x + n_y + n_z\right) \right\} - 1} = g \int_{n_x,n_y,n_z \geq 0} \frac{d\Sigma}{\exp\left\{ \frac{\hbar\omega}{T_c}\left(n_x + n_y + n_z\right) \right\} - 1},$$

where $d\Sigma$ is an elementary volume of the state number space $\{n_x, n_y, n_z\}$. Since the function under the integral depends only on one linear combination, $n \equiv n_x + n_y + n_z$, of the Cartesian coordinates of this space, is beneficial to select the differential $d\Sigma$ in the form

$$d\Sigma = d\left(\frac{n^3}{6} \right) \equiv \frac{n^2}{2} dn$$

(see Fig. on the right), so that our 3D integral reduces to a 1D one:

$$N = g \int_0^{\infty} \left[\exp\left\{ \frac{\hbar\omega}{T_c} n \right\} - 1 \right]^{-1} \frac{n^2 dn}{2}.$$



With the integration variable replacement $x \equiv (\hbar\omega/T_c)n$, this equation takes the form

$$N = \frac{g}{2}\left(\frac{T_c}{\hbar\omega}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1}.$$

Using the provided table integral with $s = 3$, we get

$$N = \zeta(3) g \left(\frac{T_c}{\hbar\omega}\right)^3, \qquad \text{i.e. } T_c = \hbar\omega \left(\frac{N}{g\zeta(3)}\right)^{1/3}.$$

(e) [2 points] The average distance R of a 3D classical particle from the bottom of a quadratic potential may be found from the equipartition theorem:

$$\langle U \rangle = \frac{m\omega^2 R^2}{2} = \frac{3T}{2}.$$

Hence the optically visible area of all the gas cloud above T_c, and of its uncondensed fraction below the critical temperature, is of the order of

$$A \sim R^2 \sim \frac{T}{m\omega^2} \sim \frac{T_c}{m\omega^2} \sim \frac{\hbar}{m\omega}\left(\frac{N}{g}\right)^{1/3}.$$

However, all particles of the condensed fraction of the gas, at $T < T_c$, are in their ground state, with energy $\varepsilon_g \equiv \varepsilon_{0,0,0} = (3/2)\hbar\omega$, so that the radius R_c of their cloud should be estimated from the relation

$$U_c \sim 3\frac{m\omega^2 R_c^2}{2} \sim \varepsilon_g = 3\frac{\hbar\omega}{2},$$

giving the visible area

$$A_c \sim R_c^2 \sim \frac{\hbar}{m\omega} \sim \frac{A}{(N/g)^{1/3}} << A.$$

Hence, the most direct manifestation of the Bose-Einstein condensation at the soft confinement is the appearance, at $T = T_c$, of a small, dense "blob" on the background of a larger gas cloud image. Some spectacular images of this appearance are available online - see, e.g., https://en.wikipedia.org/wiki/Bose-Einstein_condensate.