Study of Correspondences in Supersymmetric Quantum Field Theories

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Saebyeok Jeong

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Saebyeok Jeong

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Nikita Nekrasov - Dissertation Advisor
Professor, Simons Center for Geometry and Physics

Martin Roček - Chairperson of Defense
Professor, C.N. Yang Institute for Theoretical Physics

Zohar Komargodski
Professor, Simons Center for Geometry and Physics

Thomas Allison
Assistant Professor, Department of Physics and Astronomy

Dennis Sullivan
Professor, Department of Mathematics

This dissertation is accepted by the Graduate School

Eric Wertheimer
Dean of the Graduate School

Abstract of the Dissertation

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In this dissertation we study correspondences of supersymmetric field theories with various objects in theoretical physics, by explicitly computing the relevant field theoretical quantities and investigating their mathematical properties.

In the first part, we consider a distinguished set of half-BPS observables in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories, called qq-characters. The regularity property of their gauge theory expectation values leads to the exact relations of four-dimensional $\mathcal{N} = 2$ gauge theories with quantum integrable systems, conformal field theories, and flat connections on Riemann surfaces. In particular, we investigate the splitting behavior of degenerate levels in quantum integrable system in the context of the correspondence with gauge theory, searching for its field theoretical implications. Also, we provide an exact derivation of the identity between the gauge theory partition functions and the conformal blocks of Liouville field theory in a specific subsector of the parameter space. Finally, we verify that the twisted superpotential which governs the effective dynamics of the $\mathcal{N} = 2$ theory subject to the two-dimensional Ω-background is equivalent to the generating function of a particular complex

Lagrangian submanifold, called the variety of opers, in the moduli space of flat connections on a Riemann surface.

In the second part, we attempt to re-assemble the constructs used in the first part in an algebraic point of view. We consider the five-dimensional uplift of the gauge theory defined on an orbifold. We introduce a new quantum toroidal algebra as a deformation of the quantum toroidal algebra of $\mathfrak{gl}(p)$. We show that it has the structure of a Hopf algebra, and present two representations, called *vertical* and *horizontal*, obtained by deforming respectively the Fock representation and Saito's vertex representations of the quantum toroidal algebra of $\mathfrak{gl}(p)$. We construct the vertex operator intertwining between these two types of representations. This object is identified with a deformation of the refined topological vertex, allowing us to reconstruct the partition function and the qq-characters of the quiver gauge theories.

At last, in the third part we investigate an alternative approach to the correspondence of four-dimensional $\mathcal{N} = 2$ superconformal theories and two-dimensional vertex operator algebras, in the framework of the Ω-deformation of supersymmetric gauge theories. The two-dimensional Ω-deformation of the holomorphic-topological theory on the product four-manifold is constructed at the level of supersymmetry variations and the action. The supersymmetric localization is performed to achieve a two-dimensional chiral CFT. The desired vertex operator algebra is recovered as the algebra of local operators of the resulting CFT. We also discuss the identification of the Schur index of the $\mathcal{N} = 2$ superconformal theory and the vacuum character of the vertex operator algebra at the level of their path integral representations, using our Ω-deformation point of view on the correspondence.

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List of Publications

[1] **“Splitting of surface defect partition functions and integrable system”**,
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[2] **“BPZ equations for higher degenerate fields and non-perturbative Dyson-Schwinger equations”**,
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[3] **“Opers, surface defects, and Yang-Yang functional”**,
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[4] **“SCFT/VOA correspondence via Ω-deformation”**,
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[5] **“New quantum toroidal algebras from 5d $\mathcal{N} = 1$ instantons on orbifolds”**,
J.-E. Bourgine and S. Jeong
`arXiv:1906.01625 [hep-th]`.

La lutte elle-même vers les sommets suffit à remplir un cœur d'homme;

il faut imaginer Sisyphe heureux.

Le Mythe de Sisyphe

Albert Camus

Chapter 1

Introduction

Quantum field theory is a fundamental framework in modern theoretical physics of understanding various physical systems, such as elementary particles and condensed matter systems. It is therefore important to study methodologies of computing field theoretical quantities, such as correlation functions of observables, to obtain better descriptions of those physical systems. When Lagrangian description is available, a primary methodology is given by the path integral over an infinite dimensional space of field configurations. A typical way to proceed is to work in a certain weak-coupling regime, applying the perturbation theory to express those quantities as series expansions in coupling constants of the given theory.

While the perturbation theory has been proven to have a good phenomenological prediction power, the study of non-perturbative dynamics is necessary to understand the full aspects of quantum field theory. When a theory admits multiple saddle-points of the action, for example, we have to take account for the instanton effect which corrects the correlation functions even in the weak-coupling regime with exponentially suppressed contributions. Even though for some cases in quantum mechanics it is known how to exactly determine those non-perturbative effects, for instance by the resurgence in trans-series expansions, these methods are not directly applicable to general quantum field theory, and the study of the non-perturbative dynamics mostly remains to be a difficult task so far.

The difficulty is significantly alleviated when we introduce some amount of supersymmetry. Supersymmetric field theories often allow exact evaluations of important field theoretical quantities such as indices, partition functions, and correlation functions of local and non-local observables, which effectively encode the non-perturbative dynamics of the theory. Moreover, supersymmetry facilitates embedding the quantum field theory into various string/M-theoretic setups, where the field theory is realized as the low energy effective theory. In this construction, the intricate and wealthy structures enjoyed in string theory and M-theory descend to the supersymmetric field theory, which in turn enrich the understanding of the dynamics of supersymmetric field theory with unexpected dualities and correspondences. The correspondences of supersymmetric field theories, especially, relate them even with seemingly distinct objects — e.g., (non-supersymmetric) conformal field theories, topological vertices, matrix models, integrable systems, quantum algebras, flat connections on Riemann surfaces, and isomonodromic deformations of Fuchsian systems — and sometimes the knowledge of supersymmetric field theories conversely provides new perspectives on these objects through the correspondences. Therefore, it is interesting to study dualities and correspondences in supersymmetric field theories, possibly motivated from the string/M-theoretic background, by exactly evaluating the relevant field theoretical quantities and investigating their mathematical properties. This is the main theme of this dissertation.

In particular, the dissertation is devoted to making attempts on discovering, verifying, and extending dualities of supersymmetric field theories and correspondences of them with other objects in theoretical physics, at least for several cases in which supersymmetric localization can be manipulated as a powerful tool. According to the objects responsible for the correspondence, the dissertation is divided into three parts: the Part I: Quantum integrable systems, conformal field theories, and classical symplectic geometry, the Part II: Quantum toroidal algebras, and the Part III: Vertex operator algebras. We briefly summarize the results and the plan of each part below.

1.1 Quantum integrable systems, conformal field theories, and classical symplectic geometry

The low-energy descriptions of four-dimensional $\mathcal{N}=2$ gauge theories of class $\mathcal{S}$ are associated with the algebraic integrable systems of Hitchin type. According to the Bethe/gauge correspondence, the quantization of the Hitchin integrable systems can be accomplished by putting the class $\mathcal{S}$ theory on the two-dimensional Ω-background, which retains the $\mathcal{N}=(2,2)$ supersymmetry. On the other hand, there is an already existing quantization procedure for the Hitchin integrable systems in the realm of the classical symplectic geometry of flat connections. Here, the holomorphic functions on a complex Lagrangian submanifold, spanned by certain differential operators called opers, in the moduli space of flat connections are identified with the (off-shell) spectra of quantum Hitchin Hamiltonians.

It is important to understand in which sense the aforementioned quantization procedures for the Hitchin integrable systems are equivalent, both to give a gauge theoretical appreciation on the mysterious quantum/classical duality and to have a concrete example of the quantization via gauge theory. Based on [1], we give a gauge theoretical derivation of a correspondence which reveals the precise relation between those quantization schemes. Firstly conjectured by Nekrasov, Rosly, and Shatashvili for the lowest rank case, the correspondence states that the effective twisted superpotential of the class $\mathcal{S}$ theory on the two-dimensional Ω-background is identical to the generating function for the space of opers in a specific Darboux coordinate system. The derivation involves the following key ingredients:

- The half-BPS codimension-two (surface) defects in the class $\mathcal{S}$ theories are used to construct the opers and their solutions, expressed in exact gauge theoretical terms.
- The surface defect partition functions in different convergence domains are connected to each other by analytic continuations, enabling the computation of monodromies of opers.

- A Darboux coordinate system on the moduli space of flat connections, which generalizes the NRS coordinate system to arbitrary ranks, is constructed.

The direct comparison between the holonomies of flat connections expressed in the proposed Darboux coordinates and the monodromies of opers expressed in the gauge theoretical terms establishes the desired equality.

The correspondence is mutually beneficial; not only do the gauge theories help us understand the quantization of integrable systems, the quantum integrable systems conversely improve our insights on the gauge theory dynamics. Based on [2], we study the Bethe/gauge correspondence at special loci of the Coulomb moduli space where the Nekrasov-Shatashvili limit of the partition function develops extra singularities. The effective twisted superpotential is not well-defined in the usual sense, but the corresponding quantum integrable system provides a hint for the resolution of the singularities. At the special loci, the integrable system develops degeneracies in the spectra of Hamiltonians which are split by the quantum effects. It is shown that the partition function in the presence of the regular surface defect, which provides solutions to the Schrödinger equations by the non-perturbative Dyson-Schwinger equations, splits correspondingly, recovering the Bethe/gauge correspondence from each split piece.

The six-dimensional point of view on the class $\mathcal{S}$ theories give rise to still another correspondence to two-dimensional non-supersymmetric conformal field theories. The class $\mathcal{S}$ theories are in general engineered by compactifying six-dimensional $\mathcal{N} = (0, 2)$ superconformal theories on Riemann surfaces. The famous AGT correpondence thereby identifies the four-sphere partition functions of the $d = 4$, $\mathcal{N} = 2$ gauge theories with the correlation functions of the primaries in the Liouville/Toda CFT. The main obstacle for the exact proof of the correspondence is the absence of the analytic control on the gauge theory partition functions. However, it is sometimes possible to sacrifice the generality, by restricting our attention to a particular sector in the parameter space, to achieve some analytic control in compensation. Based on [3], we show that the non-perturbative Dyson-Schwinger equations for the linear $SU(2)$ quiver gauge theory in the presence of a particular surface defect is identical to the null-vector decoupling equation for the Liouville correlation function with a next-to-simplest degenerate field. The result thus proves the AGT correspondence for this special sector in the parameter space.

1.2 Quantum toroidal algebras

Supersymmetric gauge theories can be enginnered as low-energy effective field theories in various string/M-theoretic setup. The five-dimensional uplifts of the four-dimensional $\mathcal{N} = 2$ gauge theories admit a very useful construction in the IIB string theory: the web of NS5/D5-branes. Here, all the branes on line segments are joined together via trivalent vertices, forming the brane web. The brane web can be thought of as the toric diagram for the Calabi-Yau threefold on which the topological string theory is compactified to engineer the five-dimensional $\mathcal{N} = 1$ gauge theory. Hence, the brane web construction establishes an equivalence between the gauge theory partition function and the topological string amplitude.

The very web diagram can be viewed in a slightly different point of view, linking quantum toroidal algebras to the story. For each edge of the diagram we associate suitable representation of the quantum toroidal algebra of $\mathfrak{gl}(1)$ (also known as Ding-Iohara-Miki algebra). More precisely, we introduce a *horizontal* representation for each NS5-brane and a *vertical* representation for each D5-brane. Then the topological vertex is replaced by an *intertwiner* which connect two horizontal representations and one vertical representation. The gauge theory partition function, or equivalently, the topological string amplitude, is then identified with the vacuum amplitude of these intertwiners.

Based on [4], we discuss the generalization of this correspondence with quantum toroidal algebras to the gauge theories on orbifolds. We introduce a new quantum toroidal algebra as a deformation of the quantum toroidal algebra of $\mathfrak{gl}(p)$, and prove its Hopf algebra structure. We construct analogs of the horizontal, the vertical representations, and the intertwiner for this case. The gauge theory partition function is recovered as the vacuum amplitude of the intertwiners. We also provide an algebraic representation of qq-characters and show the regularity of their expectation values.

1.3 Vertex operator algebras

Supersymmetric field theories enjoy non-trivial protected sectors of observables. For four-dimensional $\mathcal{N} = 2$ superconformal theories, there exists a protected sector defined as a cohomology of certain combination of supercharges, roughly in the form $\mathcal{Q} + \mathcal{S}$. This protected sector is particularly interesting since the local operators in the cohomology with the operator product expansion comprise a two-dimensional vertex operator algebra. A natural question is whether we can understand this vertex operator algebra as a non-commutative deformation of a commutative algebra of local operators in some conformal field theory.

Based on [5], we present here how this question can be answered at least for the Lagrangian $\mathcal{N} = 2$ superconformal theories. The key idea is to view the non-commutative deformation as being implemented by an Ω-deformation. There exists a holomorphic-topological twist of the $\mathcal{N} = 2$ superconformal theory, in which the algebra of protected local operators becomes a commutative chiral algebra on a plane. When the Ω-deformation is implemented at the level of supersymmetric variations of fields and the action, we can perform supersymmetric localization with respect to the Ω-deformed supercharge. The localization locus is two-dimensional field configurations whose target are given by certain gradient flows emanating from the fixed points of some superpotential. Thus, the four-dimensional path integral reduces to a path integral of two-dimensional CFT, whose algebra of local operators recovers the vertex operator algebra that we desired.

The correspondence with vertex operator algebras indicates intriguing consequences on the four-dimensional superconformal theory. We provide a path integral point of view on the identification between the Schur index of the four-dimensional theory and the vacuum character of the vertex operator algebra, which would imply, among other things, non-trivial modular property of the Schur index.

Part I

Quantum Integrable Systems, Conformal Field Theories, and Classical Symplectic Geometry

Chapter 2

Generalities

2.1 $\mathcal{N} = 2$ supersymmetric quiver gauge theories

We give a brief review on the partition functions and the chiral observables of the four-dimensional $\mathcal{N} = 2$ supersymmetric quiver gauge theories. For more details on this subject, see [6, 7, 8, 9, 10].

2.1.1 Partition functions

For an oriented graph γ, we denote the sets of its vertices and edges and Vert_γ and Edge_γ, respectively. We define $s, t : \mathrm{Edge}_\gamma \to \mathrm{Vert}_\gamma$ as the maps which send an edge to its source and target, respectively. For each vertex we assign two integers,

$$\mathbf{n} = (n_\mathbf{i})_{\mathbf{i}\in\mathrm{Vert}_\gamma} \in \left(\mathbb{Z}^{>0}\right)^{\mathrm{Vert}_\gamma}, \quad \mathbf{m} = (m_\mathbf{i})_{\mathbf{i}\in\mathrm{Vert}_\gamma} \in \left(\mathbb{Z}^{\geq 0}\right)^{\mathrm{Vert}_\gamma}. \tag{2.1.1}$$

The $\mathcal{N} = 2$ quiver gauge theory associated to γ is the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory, whose gauge group is

$$G_g = \bigtimes_{\mathbf{i}\in\mathrm{Vert}_\gamma} U(n_\mathbf{i}), \tag{2.1.2}$$

and whose flavor group is

$$G_f = \left(\bigtimes_{\mathbf{i} \in \mathrm{Vert}_\gamma} U(\mathscr{m}_{\mathbf{i}}) \times U(1)^{\mathrm{Edge}_\gamma} \right) \Big/ U(1)^{\mathrm{Vert}_\gamma}. \tag{2.1.3}$$

Here the overall $U(1)^{\mathrm{Vert}_\gamma}$ transformation has been mod out due to the gauge symmetry,

$$(u_{\mathbf{i}})_{\mathbf{i} \in \mathrm{Vert}_\gamma} : \left((g_{\mathbf{i}})_{\mathbf{i} \in \mathrm{Vert}_\gamma}, (u_{\mathbf{e}})_{\mathbf{e} \in \mathrm{Edge}_\gamma} \right) \mapsto \left((u_{\mathbf{i}} g_{\mathbf{i}})_{\mathbf{i} \in \mathrm{Vert}_\gamma}, (u_{s(\mathbf{e})} u_{\mathbf{e}} u_{t(\mathbf{e})}^{-1})_{\mathbf{e} \in \mathrm{Edge}_\gamma} \right). \tag{2.1.4}$$

The field contents of the theory are the following: the vector multiplets $\mathbf{\Phi} = (\Phi_{\mathbf{i}})_{\mathbf{i} \in \mathrm{Vert}_\gamma}$ in the adjoint representation of G_g, the fundamental hypermultiplets $\boldsymbol{Q}_{\mathrm{fund}} = (Q_{\mathbf{i}})_{\mathbf{i} \in \mathrm{Vert}_\gamma}$ in the fundamental representation of G_g and the antifundamental representation of G_f, and finally the bifundamental hypermultiplets $\boldsymbol{Q}_{\mathrm{bifund}} = (Q_{\mathbf{e}})_{\mathbf{e} \in \mathrm{Edge}_\gamma}$ in the bifundamental representation $(\overline{n_{s(\mathbf{e})}}, n_{t(\mathbf{e})})$ of G_g. The $\mathcal{N} = 2$ supersymmetric action is then fixed up to the gauge couplings,

$$\mathfrak{q}_{\mathbf{i}} = \exp(2\pi i \tau_{\mathbf{i}}) \quad \left(\tau_{\mathbf{i}} = \frac{\vartheta_{\mathbf{i}}}{2\pi} + \frac{4\pi i}{g_{\mathbf{i}}^2} \right), \quad \mathbf{i} \in \mathrm{Vert}_\gamma, \tag{2.1.5}$$

and the masses of the hypermultiplets,

$$\begin{aligned} \boldsymbol{m} &= ((\mathbf{m}_{\mathbf{i}})_{\mathbf{i} \in \mathrm{Vert}_\gamma}, (m_{\mathbf{e}})_{\mathbf{e} \in \mathrm{Edge}_\gamma}), \\ \mathbf{m}_{\mathbf{i}} &= \mathrm{diag}(m_{\mathbf{i},1}, \cdots, m_{\mathbf{i},\mathscr{m}_{\mathbf{i}}}) \in \mathrm{End}(\mathbb{C}^{\mathscr{m}_{\mathbf{i}}}), \quad m_{\mathbf{e}} \in \mathbb{C}. \end{aligned} \tag{2.1.6}$$

The global symmetry group of the theory is

$$H = G_g \times G_f \times G_{\mathrm{rot}}, \tag{2.1.7}$$

where G_g (2.1.2) is the group of global gauge symmetry, G_f (2.1.3) is the group of flavor symmetry, and $G_{\mathrm{rot}} = SO(4)$ is the group of the Lorentz symmetry. We turn on equivariant parameters for the maximal torus $T_H \subset H$. The equivariant parameters for G_g is the vacuum expectation values of the complex scalars,

$$\langle \Phi_{\mathbf{i}} \rangle = \mathbf{a}_{\mathbf{i}}, \quad \mathbf{a}_{\mathbf{i}} = \mathrm{diag}(a_{\mathbf{i},1}, \cdots, a_{\mathbf{i},n_{\mathbf{i}}}) \in \mathrm{End}(\mathbb{C}^{n_{\mathbf{i}}}), \quad \mathbf{i} \in \mathrm{Vert}_\gamma. \tag{2.1.8}$$

The equivariant parameters for G_f is the masses of the hypermultiplets (2.1.6). Finally the equivariant parameters for G_{rot} is the Ω-deformation parameters $\varepsilon_1, \varepsilon_2$. The partition function of the theory is a function of these parameters $(\mathbf{a}, \boldsymbol{m}, \boldsymbol{\varepsilon}) \in \mathrm{Lie}(T_H)$. In expressing the partition function, we abuse our notation and denote the vector spaces and their T_H-equivariant characters in the same letters. Hence we write

$$N_{\mathbf{i}} = \sum_{\alpha=1}^{n_{\mathbf{i}}} e^{\beta a_{\mathbf{i},\alpha}}, \quad M_{\mathbf{i}} = \sum_{f=1}^{m_{\mathbf{i}}} e^{\beta m_{\mathbf{i},f}}. \tag{2.1.9}$$

It is helpful to use the following notation for abbreviated expressions,

$$\begin{aligned} &q_i \equiv e^{\beta \varepsilon_i}, \quad P_i \equiv 1 - q_i \quad i = 1, 2, \\ &q_{12} \equiv q_1 q_2, \quad P_{12} \equiv (1-q_1)(1-q_2). \end{aligned} \tag{2.1.10}$$

The action $\mathcal{S}_\gamma$ of the $\mathcal{N}=2$ supersymmetric γ-quiver gauge theory is given by

$$\begin{aligned} \mathcal{S}_\gamma = &-\frac{1}{8\pi^2} \sum_{\mathbf{i} \in \mathrm{Vert}_\gamma} i \mathrm{Re}\tau_{\mathbf{i}} \int_X \mathrm{Tr}_{N_{\mathbf{i}}} F_{A_{\mathbf{i}}} \wedge F_{A_{\mathbf{i}}} \\ &+ \mathrm{Im}\tau_{\mathbf{i}} \int_X \mathrm{Tr}_{N_{\mathbf{i}}} F_{A_{\mathbf{i}}} \wedge \star F_{A_{\mathbf{i}}} + \mathrm{Tr}_{N_{\mathbf{i}}} D_{A_{\mathbf{i}}} \Phi_{\mathbf{i}} \wedge \star D_{A_{\mathbf{i}}} \bar{\Phi}_{\mathbf{i}} + \mathrm{Tr}_{N_{\mathbf{i}}} [\Phi_{\mathbf{i}}, \bar{\Phi}_{\mathbf{i}}]^2 + \cdots \\ &+ \int_X \int d^2\theta \, \mathcal{W}_\gamma + c.c, \end{aligned} \tag{2.1.11}$$

where we suppressed the fermion terms. The superpotential contains the mass terms and the cubic coupling terms,

$$\begin{aligned} \mathcal{W}_\gamma = &\sum_{\mathbf{i} \in \mathrm{Vert}_\gamma} \mathrm{Tr}_{M_{\mathbf{i}}} \left(\mathbf{m}_{\mathbf{i}} Q_{\mathbf{i}} \tilde{Q}_{\mathbf{i}} \right) + \sum_{e \in \mathrm{Edge}_\gamma} m_e \mathrm{Tr}_{N_{s(e)}} \tilde{Q}_e Q_e \\ &+ \sum_{\mathbf{i} \in \mathrm{Vert}_\gamma} \mathrm{Tr}_{M_{\mathbf{i}}} \left(Q_{\mathbf{i}} \Phi_{\mathbf{i}} \tilde{Q}_{\mathbf{i}} \right) + \sum_{e \in \mathrm{Edge}_\gamma} \mathrm{Tr}_{N_{s(e)}} \left(\tilde{Q}_e \Phi_{t(e)} Q_e - \tilde{Q}_e Q_e \Phi_{s(e)} \right). \end{aligned} \tag{2.1.12}$$

It is possible to express the action in the $\mathcal{Q}$-cohomological field theory, where $\mathcal{Q}$ is the Donaldson-Witten topological supercharge, as

$$\mathcal{S}_{\gamma} = -\frac{i}{8\pi^2} \sum_{\mathbf{i} \in \mathrm{Vert}_{\gamma}} \tau_{\mathbf{i}} \int_X \mathrm{Tr}_{N_{\mathbf{i}}} F_{A_{\mathbf{i}}} \wedge F_{A_{\mathbf{i}}} + \mathcal{Q}\left(\cdots\right) \tag{2.1.13}$$

The Ω-deformation is implemented by modifying the theory to a $\mathcal{Q}_{\varepsilon}$-cohomological field theory, where $\mathcal{Q}_{\varepsilon}$ is the Ω-deformed supercharge which is suitably defined to square to the global symmetry generated by a generic element of $(\mathbf{a}, \boldsymbol{m}, \boldsymbol{\varepsilon}) \in \mathrm{Lie}(T_H)$. The Ω-deformed action can be simply written as

$$\mathcal{S}_{\gamma,\varepsilon} = -\frac{i}{8\pi^2} \sum_{\mathbf{i} \in \mathrm{Vert}_{\gamma}} \tau_{\mathbf{i}} \int_X \mathrm{Tr}_{N_{\mathbf{i}}} F_{A_{\mathbf{i}}} \wedge F_{A_{\mathbf{i}}} + \mathcal{Q}_{\varepsilon}\left(\cdots\right). \tag{2.1.14}$$

It is possible to understand this deformation as a supergravity background [7], but we do not elaborate on it here.

The partition function is an Euclidean path integral given by the Ω-deformed action:

$$\mathcal{Z}_{\gamma}(\mathbf{a}, \boldsymbol{m}, \boldsymbol{\varepsilon}, \mathfrak{q}) = \int DAD\Phi DQD\tilde{Q}[\cdots]\, e^{-\mathcal{S}_{\gamma,\varepsilon}} \tag{2.1.15}$$

with the boundary condition

$$\Phi(x) \longrightarrow \mathbf{a}, \quad x \longrightarrow \infty. \tag{2.1.16}$$

The partition function factors into the classical, the one-loop, and the instanton parts:

$$\mathcal{Z}_{\gamma}(\mathbf{a}, \boldsymbol{m}, \boldsymbol{\varepsilon}, \mathfrak{q}) = \mathcal{Z}_{\gamma}^{\text{classical}}\, \mathcal{Z}_{\gamma}^{\text{1-loop}}\, \mathcal{Z}_{\gamma}^{\text{inst}}. \tag{2.1.17}$$

The classical part is given by

$$\mathcal{Z}_{\gamma}^{\text{classical}}(\mathbf{a}, \boldsymbol{\varepsilon}, \mathfrak{q}) = \prod_{\mathbf{i} \in \text{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{-\frac{1}{2\varepsilon_1 \varepsilon_2} \sum_{\alpha=1}^{n_{\mathbf{i}}} a_{\mathbf{i},\alpha}^2}. \tag{2.1.18}$$

The one-loop part is obtained by integrating out the quadratic fluctuations around the trivial vacuum:

$$\begin{aligned} &\mathcal{Z}_{\gamma}^{\text{1-loop}}(\mathbf{a}, \boldsymbol{m}, \boldsymbol{\varepsilon}) \\ &= \epsilon \left[\frac{1}{(1-e^{-\beta\varepsilon_1})(1-e^{-\beta\varepsilon_2})} \left(\sum_{\mathbf{i} \in \text{Vert}_\gamma} (M_{\mathbf{i}} - N_{\mathbf{i}}) N_{\mathbf{i}}^* + \sum_{\mathbf{e} \in \text{Edge}_\gamma} e^{\beta m_{\mathbf{e}}} N_{t(\mathbf{e})} N_{s(\mathbf{e})}^* \right) \right], \end{aligned} \tag{2.1.19}$$

where the ϵ-symbol is defined by

$$\epsilon\left[\cdots\right] \equiv \exp\left[\left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} [\cdots] \right], \tag{2.1.20}$$

which converts a character into a product of weights. In particular, the ϵ-symbol regularizes an infinite product of weights such as (2.1.19) by the Barnes double gamma function,

$$\Gamma_2(x; \varepsilon_1, \varepsilon_2) \equiv \exp\left[-\left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} \frac{e^{-\beta x}}{(1-e^{-\beta\varepsilon_1})(1-e^{-\beta\varepsilon_2})} \right]. \tag{2.1.21}$$

The instanton part $\mathcal{Z}_{\gamma}^{\text{inst}}$ is computed by a T_H-equivariant integration over the instanton moduli space:

$$\mathcal{Z}_{\gamma}^{\text{inst}}(\mathbf{a}; \boldsymbol{m}; \boldsymbol{\varepsilon}; \mathfrak{q}) = \sum_{\mathbf{k}} \prod_{\mathbf{i} \in \text{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{k_{\mathbf{i}}} \int_{\mathcal{M}_\gamma(\mathbf{n}, \mathbf{k})} e_{T_H}\left(\text{Obs}_\gamma\right). \tag{2.1.22}$$

Given the vector of the instanton charges $\mathbf{k} = (k_{\mathbf{i}})_{\mathbf{i} \in \text{Vert}_\gamma} \in \mathbb{Z}^{\geq 0}$, the total framed noncommutative instanton moduli space of the quiver gauge theory for γ is

$$\mathcal{M}_\gamma(\mathbf{n}, \mathbf{k}) \equiv \bigtimes_{\mathbf{i} \in \text{Vert}_\gamma} \mathcal{M}(n_{\mathbf{i}}, k_{\mathbf{i}}), \tag{2.1.23}$$

where $\mathcal{M}(n_{\mathbf{i}}, k_{\mathbf{i}})$ is the ADHM moduli space

$$\mathcal{M}(n,k) = \left\{ \begin{array}{c|l} B_{1,2} : K \to K, & [B_1, B_2] + IJ = 0, \\ I : N \to K, J : K \to N & [B_1, {B_1}^\dagger] + [B_2, {B_2}^\dagger] + II^\dagger - J^\dagger J = \zeta \end{array} \right\} \Big/ U(k). \tag{2.1.24}$$

$(N = \mathbb{C}^n, K = \mathbb{C}^k)$

Here, $\zeta \in \mathbb{R}$ is a real parameter which originates from the non-commutativity of the spacetime $\mathbb{C}^2$. When a stability chamber is chosen as, say, $\zeta > 0$, solving the real moment map equation $[B_1, {B_1}^\dagger] + [B_2, {B_2}^\dagger] + II^\dagger - J^\dagger J = \zeta$ and dividing by the compact $U(k)$ is equivalent to imposing the stability condition and dividing by the complex group $GL(k)$,

$$\mathcal{M}(n,k) = \left\{ \begin{array}{c|c} B_{1,2} : K \to K, & [B_1, B_2] + IJ = 0, \\ I : N \to K, J : K \to N & K = \mathbb{C}[B_1, B_2]\, I(N) \end{array} \right\} \Big/ GL(k). \tag{2.1.25}$$

The obstruction sheaf Obs_γ over $\mathcal{M}_\gamma(\mathbf{n}, \mathbf{k})$ is defined by

$$\mathrm{Obs}_\gamma = R\pi_* \bigoplus_{e \in \mathrm{Edge}_\gamma} \mathrm{Hom}\left(\mathcal{E}_{s(e)}, \mathcal{E}_{t(e)}\right) \oplus \bigoplus_{\mathbf{i} \in \mathrm{Vert}_\gamma} \mathrm{Hom}\left(\mathcal{E}_{\mathbf{i}}, M_{\mathbf{i}}\right), \tag{2.1.26}$$

where $\mathcal{E}_{\mathbf{i}}$ is the universal $\mathbf{i}$'th sheaf over $\mathcal{M}_\gamma(\mathbf{n}, \mathbf{k}) \times \mathbb{P}^2$ and $\pi : \mathcal{M}_\gamma(\mathbf{n}, \mathbf{k}) \times \mathbb{P}^2 \to \mathcal{M}_\gamma(\mathbf{n}, \mathbf{k})$ is the projection. $e_{T_H}(\cdots)$ denotes the T_H-equivariant Euler class.

The T_H-equivariant integration over the instanton moduli space (2.1.23) localizes on the set of fixed points of T_H-action, $\mathcal{M}_\gamma(\mathbf{n}, \mathbf{k})^{T_H}$, which is the set of colored partitions $\boldsymbol{\lambda} = ((\lambda^{(\mathbf{i},\alpha)})_{\alpha=1}^{n_{\mathbf{i}}})_{\mathbf{i} \in \mathrm{Vert}_\gamma}$, where each $\lambda^{(\mathbf{i},\alpha)}$ is a partition,

$$\lambda^{(\mathbf{i},\alpha)} = \left(\lambda_1^{(\mathbf{i},\alpha)} \geq \lambda_2^{(\mathbf{i},\alpha)} \geq \cdots \geq \lambda_{l(\lambda^{(\mathbf{i},\alpha)})}^{(\mathbf{i},\alpha)} > \lambda_{l(\lambda^{(\mathbf{i},\alpha)})+1}^{(\mathbf{i},\alpha)} = \cdots = 0 \right), \tag{2.1.27}$$

with the size $|\lambda^{(\mathbf{i},\alpha)}| = \sum_{i=1}^{l(\lambda^{(\mathbf{i},\alpha)})} \lambda_i^{(\mathbf{i},\alpha)} = k_{\mathbf{i},\alpha}$ constrained by $k_{\mathbf{i}} = \sum_\alpha k_{\mathbf{i},\alpha} = |\boldsymbol{\lambda}^{(\mathbf{i})}|$ [6, 7]. At each fixed point $\boldsymbol{\lambda}$, the vector space $K_{\mathbf{i}}$ carrys a representation of T_H with the weights given by the formula

$$K_{\mathbf{i}}[\boldsymbol{\lambda}] = \sum_{\alpha=1}^{n_{\mathbf{i}}} \sum_{\square \in \lambda^{(\mathbf{i},\alpha)}} e^{\beta c_\square}, \tag{2.1.28}$$

where we have defined the content of the box,

$$c_\square = a_{\mathbf{i},\alpha} + \varepsilon_1(i-1) + \varepsilon_2(j-1) \quad \text{for} \quad \square = (i,j) \in \lambda^{(\mathbf{i},\alpha)} \iff 1 \leq j \leq \lambda_i^{(\mathbf{i},\alpha)}. \tag{2.1.29}$$

The tangent bundle and the matter bundle comprise the character

$$\begin{aligned} \mathcal{T}_\gamma[\boldsymbol{\lambda}] = & \sum_{\mathbf{i} \in \mathrm{Vert}_\gamma} \left(N_{\mathbf{i}} K_{\mathbf{i}}^* + q_{12} N_{\mathbf{i}}^* K_{\mathbf{i}} - P_{12} K_{\mathbf{i}} K_{\mathbf{i}}^* - M_{\mathbf{i}}^* K_{\mathbf{i}} \right) \\ & - \sum_{\mathbf{e} \in \mathrm{Edge}_\gamma} e^{\beta m_{\mathbf{e}}} \left(N_{t(\mathbf{e})} K_{s(\mathbf{e})}^* + q_{12} N_{s(\mathbf{e})}^* K_{t(\mathbf{e})} - P_{12} K_{t(\mathbf{e})} K_{s(\mathbf{e})}^* \right), \end{aligned} \tag{2.1.30}$$

assoicated to each fixed point $\boldsymbol{\lambda} \in \mathcal{M}_\gamma(\mathbf{n},\mathbf{k})^{T_H}$. At last the instanton part of the partition function is evaluated by

$$\mathcal{Z}_\gamma^{\mathrm{inst}}(\mathbf{a}; \boldsymbol{m}; \boldsymbol{\varepsilon}; \mathfrak{q}) = \sum_{\boldsymbol{\lambda}} \prod_{\mathbf{i} \in \mathrm{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{|\boldsymbol{\lambda}^{(\mathbf{i})}|} \, \epsilon \left[\mathcal{T}_\gamma[\boldsymbol{\lambda}] \right], \tag{2.1.31}$$

where we have used the ϵ-symbol (2.1.20). Note that the one-loop part and the instanton part can be combined into

$$\begin{aligned} & \mathcal{Z}_\gamma^{\text{1-loop}}(\mathbf{a}, \boldsymbol{m}, \boldsymbol{\varepsilon}) \, \mathcal{Z}_\gamma^{\mathrm{inst}}(\mathbf{a}; \boldsymbol{m}; \boldsymbol{\varepsilon}; \mathfrak{q}) \\ & = \sum_{\boldsymbol{\lambda}} \prod_{\mathbf{i} \in \mathrm{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{|\boldsymbol{\lambda}^{(\mathbf{i})}|} \, \epsilon \left[\frac{1}{(1-e^{-\beta\varepsilon_1})(1-e^{-\beta\varepsilon_2})} \left(\sum_{\mathbf{i} \in \mathrm{Vert}_\gamma} (M_{\mathbf{i}} - S_{\mathbf{i}}) S_{\mathbf{i}}^* + \sum_{\mathbf{e} \in \mathrm{Edge}_\gamma} e^{\beta m_{\mathbf{e}}} S_{t(\mathbf{e})} S_{s(\mathbf{e})}^* \right) \right], \end{aligned} \tag{2.1.32}$$

with the character $S_{\mathbf{i}} \equiv N_{\mathbf{i}} - P_{12} K_{\mathbf{i}}$.

We conclude this section with a remark on the identity that the 1-loop part of the A_1-

theory partition function satisfies, which will be useful in section 5.6. The formula (2.1.19) tells that

$$\mathcal{Z}_{A_1}^{\text{1-loop}} = \frac{\prod_{\alpha,\beta=1}^{N} \Gamma_2(a_\alpha - a_\beta; \varepsilon_1, \varepsilon_2)}{\prod_{\alpha,\beta=1}^{N} \Gamma_2(a_\alpha - a_{0,\beta}; \varepsilon_1, \varepsilon_2)\Gamma_2(a_{3,\alpha} - a_\beta; \varepsilon_1, \varepsilon_2)}. \tag{2.1.33}$$

Note that we have the following identity,

$$\frac{\partial}{\partial x}\left(\lim_{\varepsilon_2\to 0} \varepsilon_2 \log \Gamma_2(x; \varepsilon_1, \varepsilon_2)\right) = -\log \Gamma_1(x; \varepsilon_1), \tag{2.1.34}$$

where we have defined

$$\Gamma_1(x; \varepsilon_1) \equiv \exp\left[-\frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} \frac{e^{-\beta x}}{1 - e^{-\beta \varepsilon_1}}\right] = \frac{\sqrt{2\pi/\varepsilon_1}}{\varepsilon_1^{\frac{x}{\varepsilon_1}} \Gamma\left(\frac{x}{\varepsilon_1}\right)}. \tag{2.1.35}$$

Thus for the 1-loop part of the effective twisted superpotential,

$$\widetilde{\mathcal{W}}_{A_1}^{\text{1-loop}} \equiv \lim_{\varepsilon_2\to 0} \varepsilon_2 \log \mathcal{Z}_{A_1}^{\text{1-loop}}, \tag{2.1.36}$$

we derive the identity,

$$\left(\frac{\partial}{\partial a_\alpha} - \frac{\partial}{\partial a_\beta}\right) \widetilde{\mathcal{W}}_{A_1}^{\text{1-loop}} = \log \prod_{\alpha' \neq \alpha} \frac{\Gamma\left(\frac{a_\alpha - a_{\alpha'}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{\alpha'} - a_\alpha}{\varepsilon_1}\right)} \prod_{\beta' \neq \beta} \frac{\Gamma\left(\frac{a_{\beta'} - a_\beta}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_\beta - a'_\beta}{\varepsilon_1}\right)} \prod_{\gamma=1}^{N} \frac{\Gamma\left(\frac{a_{3,\gamma} - a_\alpha}{\varepsilon_1}\right)\Gamma\left(\frac{a_\beta - a_{0,\gamma}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_\alpha - a_{0,\gamma}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,\gamma} - a_\beta}{\varepsilon_1}\right)}, \tag{2.1.37}$$

which is used in section 5.6 to absorb the 1-loop contribution $\widetilde{\mathcal{W}}^{\text{1-loop}}$ into $\widetilde{\mathcal{W}}^{\text{full}}$.

2.1.2 Chiral observables

The Coulomb branch chiral observables are generated by the gauge invariant polynomials

$$\mathcal{O}_{\mathbf{i},k} \equiv \mathrm{Tr}_{N_\mathbf{i}} \Phi_\mathbf{i}^k, \quad \mathbf{i} \in \mathrm{Vert}_\gamma,\ k \geq 1. \tag{2.1.38}$$

The gauge theory expectation values of them can be computed again by equivariant localization,

$$\left\langle \mathcal{O}_{\mathbf{i},k} \right\rangle_\gamma = \frac{1}{\mathcal{Z}_\gamma^{\text{inst}}} \sum_{\boldsymbol{\lambda}} \mathcal{O}_{\mathbf{i},k}[\boldsymbol{\lambda}] \prod_{\mathbf{i}\in\text{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{|\boldsymbol{\lambda}^{(\mathbf{i})}|} \, \epsilon\left[\mathcal{T}_\gamma[\boldsymbol{\lambda}]\right], \tag{2.1.39}$$

where the chiral observables are represented on the colored partitions as

$$\mathcal{O}_{\mathbf{i},k}[\boldsymbol{\lambda}] = \sum_{\alpha=1}^{n_{\mathbf{i}}} \left[a_{\mathbf{i},\alpha}^k + \sum_{\square\in\lambda^{(\mathbf{i},\alpha)}} \Big((c_\square + \varepsilon_1)^k + (c_\square + \varepsilon_2)^k - c_\square^k - (c_\square + \varepsilon)^k \Big) \right]. \tag{2.1.40}$$

Consider the following regularized characteristic polynomials of the adjoint scalars, called the $\mathcal{Y}$-observables:

$$\mathcal{Y}_{\mathbf{i}}(x) \equiv x^{n_{\mathbf{i}}} \exp \sum_{l=1}^{\infty} -\frac{1}{l x^l} \text{Tr}\ \Phi_{\mathbf{i}}^l|_0, \tag{2.1.41}$$

Their expressions at the fixed point $\boldsymbol{\lambda}$ are written as

$$\mathcal{Y}_{\mathbf{i}}(x)[\boldsymbol{\lambda}] = \prod_{\alpha=1}^{n_{\mathbf{i}}} \left((x - a_{\mathbf{i},\alpha}) \prod_{\square\in\lambda^{(\mathbf{i},\alpha)}} \frac{(x - c_\square - \varepsilon_1)(x - c_\square - \varepsilon_2)}{(x - c_\square)(x - c_\square - \varepsilon)} \right). \tag{2.1.42}$$

which shows that upon the regularization, the instanton contribution makes the characteristic polynomials into rational functions of the auxiliary variable x. The $\mathcal{Y}$-observable can be simply written as

$$\mathcal{Y}_{\mathbf{i}}(x)[\boldsymbol{\lambda}] = \beta^{-n_{\mathbf{i}}} \, \epsilon[-e^{\beta x} S_{\mathbf{i}}^*]. \tag{2.1.43}$$

Note that the $\mathcal{Y}$-observables are the generating functions for the chiral observables.

The qq-characters for the quiver gauge theories are given as certain Laurent polynomials (or Laurent power series) of the $\mathcal{Y}$-observables, as we now describe below.

2.2 qq-characters

In this section we introduce an important class of half-BPS chiral observables in the $\mathcal{N} = 2$ γ-quiver gauge theory: the qq-characters [10]. As the name suggests, the qq-characters can be thought of as generalizations of Yangian q-characters of finite dimensional representations of Yangian $Y(\mathfrak{g}_\gamma)$, constructed for finite γ in [11], in the sense that the qq-characters reduce to those Yangian q-characters in the limit $\varepsilon_2 \to 0$. An analogous story is present for the K-theoretic uplift, namely, the q-characters for the quantum affine algebras $U_q(\mathfrak{g}_\gamma)$ constructed for finite γ in [12] and for affine γ in [13]. The qq-characters of the five-dimensional uplifts of the γ-quiver gauge theories compactified on a circle reduce to those q-characters in the form appearing in [9], in the limit $q_2 \to 1$ and $q_1 = q$.

2.2.1 Crossed instantons

The physical origin of the qq-characters is the mutually transversally intersecting $D3$-branes in type-IIB string theory [10], which we briefly describe below. Consider IIB string theory on the ten-dimensional manifold $\mathbb{R}^2 \times X \times Y/\Gamma$, where $X = \mathbb{R}^4$, $Y = \mathbb{R}^4$, and Γ is a McKay ADE subgroup of $SU(2)$. We can introduce a stack of N $D3$-branes at $0 \times X \times 0$, to realize the $\mathcal{N} = 2$ supersymmetric γ-quiver gauge theories as the low energy effective theory on the $D3$-branes, with affine ADE quivers γ corresponding to Γ. To construct the qq-characters, we need to introduce an additional stack of $D3$-branes lying along Y/Γ, so that they intersect with the previous stack of $D3$-branes at the origin of $X \times Y/\Gamma$. To fully specify the gauge theory living on this additional stack of $D3$-branes, we need to choose the holonomy of its gauge field along the non-contractible loops on the boundary at infinity, S^3/Γ. This is equivalent to the choice of $\mathbf{w} = (w_\mathbf{i})_{\mathbf{i}\in\mathrm{Vert}_\gamma} \in \mathbb{Z}_{\geq 0}^{\mathrm{Vert}_\gamma}$. Also, we go to the Coulomb branch of this theory by choosing non-zero positions of the $D3$-branes in $\mathbb{R}^2$ as $\boldsymbol{\nu} = (\vec{\nu}_\mathbf{i} + x)_{\mathbf{i}\in\mathrm{Vert}_\gamma}$ where $\vec{\nu}_\mathbf{i} \in \mathbb{C}^{w_\mathbf{i}}$ (in other words, x is the *center of mass* position in $\mathbb{R}^2$).

The qq-character $\mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(x)$ is the local observable in the original gauge theory on the

$D3$-branes along X, which is obtained by integrating out the degrees of freedom on all the transversal $D3$-branes, in the vacuum determined by the asymptotic holonomy $\mathbf{w}$ of the gauge field and the vacuum expectation values $\boldsymbol{\nu}$ of the scalars in the vector multiplet on Y/Γ.

As for usual $\mathcal{N}=2$ partition functions, we need to introduce a suitable Ω-deformation to regularize the infrared divergence in the expectation values of qq-characters. The subgroup of $Spin(8)$ of $X \times Y$ which commutes with the action of Γ generically has rank two, which enhances to three for Γ of A-type. We will only deal with Γ of A-type throughout this dissertation, so we restrict our attention to this case from now on. The Ω-deformation parameters corresponding to the rank three Cartan torus of the preserved subgroup of $Spin(8)$ can be introduced as complex numbers $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{C}^3$. It is convenient to further introduce ε_4 so that we have $\sum_{a=1}^4 \varepsilon_a = 0$. We also often denote $\varepsilon = \varepsilon_1 + \varepsilon_2$.

Also, we need to properly turn on the B-field which makes $D(-1)$-instantons bound to the two orthogonal stacks of $D3$-branes. Such field configurations, instantons bound to two transversal gauge theories, are called *crossed instantons*. In the point of view of the worldvolume theory on the $D(-1)$-instantons, the Higgs branch opens up when they become bound to the $D3$-branes, which is identified with the moduli space of crossed instantons. The crossed instanton partition function is precisely the equivariant integration over this moduli space. As shown in [14], the moduli space of crossed instantons can be viewed as a certain fibration over the moduli space of ordinary instantons of the gauge theory on X. Recalling the definition of the qq-character as integrating out the degrees of freedom on the transversal $D3$-branes, we obtain the qq-character $\mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(x)$ of the quiver gauge theory on X by performing the equivariant integration only along the fiber. The further equivariant integration on the base, which results in the full crossed instanton partition function by construction, is nothing but taking the gauge theory expectation value $\left\langle \mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(x) \right\rangle$ of the very qq-character.

As a remark, we point out here that two transversal stacks of $D3$-brane configurations just described can be generalized at most to mutually transversally intersecting six stacks of $D3$-branes by inserting one for each choice of two-planes inside $\mathbb{C}^4$, $\binom{4}{2} = 6$. The $D(-1)$-instantons bound to these six stacks of $D3$-branes are called *spiked instantons* [10, 14]. The spiked instanton partition functions enjoy the same regularity property that the crossed instanton partition function possesses [14], and thus produce useful half-BPS observables on the gauge theory on one of the stack of $D3$-branes. However, we do not make use of spiked instantons throughout this dissertation, and refer to [14, 15, 16] for further studies of them. We also remark that we can recover finite quiver gauge theories from the affine ADE quiver gauge theories discussed in this section by taking various limits of parameters. For example, the A_r linear quiver gauge theory with the identical ranks of gauge groups at all vertices, which is the main concern of this dissertation, can be obtained from the $\widehat{A}_{r+1}$-quiver gauge theory by taking the limit of the gauge couplings $\mathfrak{q}_0 \to 0$ and $\mathfrak{q}_{r+1} \to 0$. We refer to [10] for further discussions.

2.2.2 The main property

The characteristic property of the qq-character $\mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(x)$ is that its expectation value $\left\langle \mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(x) \right\rangle$ is regular in x. Physically, this means that the combined system of transversal stacks of $D3$-branes does not experience any phase transition or runaway zero-mode at any value of x, in the presence of the Ω-deformation. Mathematically, the suggested regularity follows from the compactness of the moduli space of crossed instantons, which is rigorously proven in [14].

More specifically, $\mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(x)$ is given by a Laurent polynomial (or a Laurent power series for affine γ) of $\mathcal{Y}$-observables, with possibly shifted arguments of $\mathcal{Y}_{\mathbf{i}}(x)$, which begins with

$$\mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}\left(\mathcal{Y}(x)\right) = \prod_{\mathbf{i}\in\mathrm{Vert}_\gamma} \prod_{l=1}^{w_{\mathbf{i}}} \mathcal{Y}_{\mathbf{i}}(x + \nu_{\mathbf{i},l} + \varepsilon) + \mathcal{O}(\mathfrak{q}). \tag{2.2.1}$$

The main property described above is that the γ-quiver gauge theory expectation value,

$$\left\langle \mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(x) \right\rangle_\gamma = \frac{1}{\mathcal{Z}_\gamma^{\text{inst}}} \sum_{\boldsymbol{\lambda}} \mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(\mathcal{Y}[\boldsymbol{\lambda}]) \prod_{\mathbf{i}\in\text{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{|\boldsymbol{\lambda}^{(\mathbf{i})}|} \, \epsilon\left[\mathcal{T}_\gamma[\boldsymbol{\lambda}]\right] \equiv T_{\mathbf{w},\boldsymbol{\nu}}(x), \tag{2.2.2}$$

is a polynomial in x. The degree of the polynomial is given by

$$\deg T_{\mathbf{w},\boldsymbol{\nu}}(x) = \sum_{\mathbf{i}\in\text{Vert}_\gamma} w_{\mathbf{i}} n_{\mathbf{i}}. \tag{2.2.3}$$

From the regularity of the expectation value (2.2.2), it follows that when it is expanded in large x the coefficients of negative powers of x identically vanish:

$$[x^{-n}] \left\langle \mathcal{X}_{\mathbf{w},\boldsymbol{\nu}}(x) \right\rangle_\gamma = 0, \quad n \geq 1. \tag{2.2.4}$$

These identities contain non-trivial analytic information of the partition functions, and we call them *non-perturbative Dyson-Schwinger equations.*

For the case of $\mathbf{w} = (\delta_{\mathbf{i},\mathbf{j}})_{\mathbf{j}\in\text{Vert}_\gamma}$ and $\boldsymbol{\nu} = 0$, we denote the corresponding qq-character by $\mathcal{X}_{\mathbf{i}}(x)$ and call it the $\mathbf{i}$*'th fundamental* qq-character. Throughout the dissertation, we will only concern the fundamental qq-characters of the A_r-quiver gauge theory, which can be simply expressed as follows. Define $r+1$ complex numbers $z_{\mathbf{i}}$, $\mathbf{i} = 0, 1, \cdots, r$ by

$$z_{\mathbf{i}} = z_0 \mathfrak{q}_1 \cdots \mathfrak{q}_{\mathbf{i}}, \quad \mathbf{i} = 1, \cdots, r. \tag{2.2.5}$$

Let us also define

$$\Xi_{\mathbf{i}}(x) = \frac{\mathcal{Y}_{\mathbf{i}+1}(x+\varepsilon)}{\mathcal{Y}_{\mathbf{i}}(x)}. \tag{2.2.6}$$

The $\mathbf{i}$'th fundamental qq-character $\mathcal{X}_{\mathbf{i}}(x)$ of the A_r-quiver gauge theory can be written as

[10]

$$\mathcal{X}_{\mathbf{i}}(x) = \frac{\mathcal{Y}_0\left(x+\varepsilon\left(1-\mathbf{i}\right)\right)}{z_0 z_1 \cdots z_{\mathbf{i}-1}} \sum_{\substack{I\subset[0,r]\\ |I|=\mathbf{i}}} \prod_{\mathbf{j}\in I} \left[z_{\mathbf{j}}\ \Xi_{\mathbf{j}}\left(x+\varepsilon\left(h_I(\mathbf{j})+1-\mathbf{i}\right)\right)\right], \quad \mathbf{i}=1,\cdots,r, \tag{2.2.7}$$

where $[0,r]=\{0,1,2,\cdots,r\}$ and $h_I(\mathbf{i})$ is the number of elements in I which is less than $\mathbf{i}$, namely,

$$h_I(\mathbf{i}) = |\{\mathbf{i}' \,|\, \mathbf{i}' \in I,\, \mathbf{i}' < \mathbf{i}\}|. \tag{2.2.8}$$

Note that the regularity of the expectation values of the fundamental qq-characters of the A_r-quiver gauge theory can be directly proven without using the compactness of the moduli space of crossed instantons, by studying the pole cancellation between the measure factor and the qq-character in (2.2.2). For more complicated theories, however, such a direct method would not be applicable.

2.3 Surface defects

Surface (codimension-two) defects in four-dimensional supersymmetric field theories are non-local observables supported on two-dimensional submanifolds in the four-dimensional spacetime. The non-locality of surface defects makes them very interesting objects to study. Their appearance in the path integral is fundamentally different from how local observables enter, and sometimes their correlation functions contain significant information on the non-perturbative dynamics of the bulk theory in four-dimension.

Surface defects in four-dimensional gauge theories are special, since the dimension of the support D of the surface defect in the four-dimensional spacetime X is the same with its codimension, and it is also identical to the degree of the gauge curvature 2-form F. Hence when the gauge group is $U(1)$, the surface defects are characterized by the continuous parameters (α, η) [17]. Here α determines the singular behavior of the gauge field around D,

$$A = \alpha d\theta + \cdots , \tag{2.3.1}$$

where (r, θ) are local radial coordinates in the plane normal to $D \subset X$, while η determines the contribution of the flux on D to the path integral by

$$\exp\left(i\eta \int_D F \right) . \tag{2.3.2}$$

Note that (2.3.1) assumes that the codimension of D is two while (2.3.2) assumes that the dimension of D is two. When the gauge group G_g is non-abelian, the surface defects are further characterized by the choice of the subgroup $\mathbb{L} \subset G_g$ preserved in its presence. The continuous parameters should be suitably generalized to $\mathcal{W}_{\mathbb{L}}$-invariant pair [17]

$$(\alpha, \eta) \in T_g \times {}^L T_g, \tag{2.3.3}$$

where T_g is a maximal torus in G_g, ${}^L T_g$ is a maximal torus of the Langlands dual group ${}^L G_g$, and $\mathcal{W}_{\mathbb{L}}$ is the Weyl group of $\mathbb{L}$.

We have described a way of constructing surface defects in which the singular behavior of the gauge field around the surface is explicitly prescribed. In this case, the field configurations for which the path integral is performed are modified by the presence of the surface defect. There is another way of constructing surface defects where such modifications of field configurations do not manifestly occur. It is coupling two-dimensional theory to the bulk four-dimensional gauge theory, by gauging a flavor symmetry of the two-dimensional theory to the bulk [17, 18]. To summarize, we discussed two distinct methods of introducing surface defects:

- Prescribing singular boundary conditions for the four-dimensional gauge field along a surface

- Coupling a two-dimensional theory living on a surface to the bulk theory by gauging the flavor symmetry with the bulk gauge field

It is interesting to study the relations between the two constructions and whether the surface defect of one type can be realized as the other. We address some of these questions in the Part I.

When the surface defect is half-BPS, i.e., preserving half of the supersymmetries present in the bulk theory, the amount of supersymmetry after the insertion of the surface defect is usually sufficient to apply supersymmetric localization to various path integral computations regarding the bulk theory with the surface defect insertion. It is the half-BPS surface defects in four-dimensional $\mathcal{N}=2$ gauge theories which play crucial roles throughout the discussion in the Part I, and we will focus on these from now on.

There are several ways to construct half-BPS surface defects in four-dimensional $\mathcal{N}=2$ gauge theories. Practically, our main concern would be the partition function of the $\mathcal{N}=2$ theory in the presence of the surface defect on the Ω-deformed $\mathbb{R}^4 = \mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\varepsilon_2}$, and there are two constructions of half-BPS surface defects which nicely fit to the equivariant integration for the $\mathcal{N}=2$ partition functions. These methods can be roughly described as [19, 20, 21]:

- Orbifold: Placing the bulk theory on an orbifold $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_p$
- Quiver: Starting with a large quiver gauge theory and partially higging some of the gauge groups by imposing constraints, in such a way that some two-dimensional degrees of freedom in the gauge field survive

For the orbifold construction, the surface defect is constructed as a prescription of performing the path integral only over the $\mathbb{Z}_p$-invariant field configurations. Indeed, under the map $(z_1, z_2) \mapsto (\tilde{z}_1 \equiv z_1, \tilde{z}_2 \equiv z_2^p)$, the orbifold $\mathbb{C}_{\varepsilon_1} \times (\mathbb{C}_{\varepsilon_2}/\mathbb{Z}_p)$ is mapped to $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{p\varepsilon_2}$, and the field configurations are allowed to be singular along the surface $\tilde{z}_2 = 0$. Therefore, the resulting theory on $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{p\varepsilon_2}$ can be interpreted as having a surface defect of singular boundary conditions.

For the quiver construction, the IIA brane realization [22] of the quiver gauge theory immediately tells that such higgsing of gauge group leads to the emergence of two-dimensional theory coupled to the remaining bulk theory via the Hanany-Witten transition of branes [23]. Therefore, the resulting theory can be interpreted as a surface defect of 2d-4d coupled system.

The parameters characterizing the surface defects, i.e., the subgroup $\mathbb{L} \subset G_g$ and the continuous parameters $(\alpha,\ \eta)$, are determined by the choice of orbifolding action and the constraints of higgsing, respectively. We will see in the following chapters how these parameters natually appear in the process of evaluating their partition functions.

The qq-characters have to be properly generalized in the presence of surface defects. For the orbifold surface defect, each qq-character fractionalizes into p pieces according to the p charges of the $\mathbb{Z}_p$-action. For the quiver surface defect, the qq-characters of the larger quiver theory restrict to the qq-characters of the 2d-4d coupled theory once the mentioned constraints on the gauge theory parameters are imposed. In the following chapters, we will present the exact expressions of these qq-characters and investigate the implications of their non-perturbative Dyson-Schwinger equations on various correspondences of the four-dimensional $\mathcal{N} = 2$ gauge theories.

Chapter 3

Splitting of surface defect partition functions and integrable systems

3.1 Introduction

Supersymmetric gauge theories in various dimensions exhibit diverse connections with integrable systems. The four-dimensional gauge theory with $\mathcal{N} = 2$ supersymmetry is one of the interesting cases to consider. The common feature of this class of theories is that the low-energy description achieved in [24, 25] naturally reveals the structure of an algebraic integrable system [26, 27]. The correspondence was promoted to the quantum level in [28], by putting the $\mathcal{N} = 2$ gauge theory into the general framework of the Bethe/gauge correspondence [29, 30]. When subject to the Nekrasov-Shatashvili limit of the Ω-deformation ($\varepsilon_1 = \hbar, \varepsilon_2 \to 0$), the four-dimensional $\mathcal{N} = 2$ gauge theory effectively becomes a two-dimensional theory with $\mathcal{N} = (2, 2)$ supersymmetry. The general Bethe/gauge correspondence states the chiral ring is the set of quantum Hamiltonians, while the set of supersymmetric vacua is identified with the (Hilbert) space of the corresponding quantum integrable system,

$$|\mathrm{eigen}\rangle \longleftrightarrow \mathrm{vac}. \tag{3.1.1}$$

In particular, the spectrum of the Hamiltonian is calculated as the gauge theory vacuum expectation value of the corresponding chiral observable in the Nekrasov-Shatashvili limit,

$$\langle \mathrm{eigen}|H_{\mathcal{O}}|\mathrm{eigen}\rangle = \langle \mathcal{O}\rangle|_{\varepsilon_2\to 0,\mathrm{vac}}. \tag{3.1.2}$$

The chiral ring is spanned by the gauge-invariant observables $\mathcal{O}_k = \mathrm{Tr}\phi^k$, where ϕ is the complex scalar in the $\mathcal{N}=2$ vector multiplet. In generic case, their vacuum expectation values are finite in the Nekrasov-Shatashvili limit, since they reduce to the vacuum expectation values of the twisted chiral observables in the effective two-dimensional $\mathcal{N}=(2,2)$ theory. Therefore the right hand side of the dictionary (3.1.2) is well-defined, providing a way to compute the spectrum of the quantum Hamiltonian from gauge theory perspective. Note that the partition function of the gauge theory shows the asymptotic behavior $\log\mathcal{Z} = \frac{\widetilde{\mathcal{W}}}{\varepsilon_2} + \mathcal{O}(\varepsilon_2^0)$ in the Nekrasov-Shatashvili limit, where $\widetilde{\mathcal{W}}$ is the effective twisted superpotential of the effective two-dimensional theory.

The equations which describe the vacua in the low-energy theory correspond to the quantization conditions on the integrable system side. The Nekrasov-Shatashvili limit of the four-dimensional $\mathcal{N}=2$ gauge theory leads to several inequivalent quantization schemes, in particular, the type A and the type B quantizations [28, 31]. In the present chapter we mainly focus on the type B quantization, in which we impose the condition

$$\exp\left(2\pi i\frac{a_\alpha}{\varepsilon_1} - i\theta_\alpha\right) = 1, \quad \theta_\alpha \in [0,2\pi). \tag{3.1.3}$$

Note that the θ-angles can be introduced in a gauge-invariant fashion. Namely, for given values of the gauge-invariant coordinates on the Coulomb moduli space, $\langle\mathcal{O}_k\rangle = \langle\mathrm{Tr}\phi^k\rangle$, the

Coulomb moduli a_α are determined up to the permutations with each other. Therefore in the real slice that we are choosing in the type B quantization, $\frac{a_\alpha}{\varepsilon_1} \in \mathbb{R}$, the θ-angles are determined up to the permutations with each other. For the quantization condition (3.1.3), we look for the eigenfunctions which are quasi-periodic with the Bloch angles (θ_α). For example, for the pure $\mathcal{N}=2$ theory and for the $\mathcal{N}=2^*$ theory with the gauge group $U(N)$, the formula (3.1.2) under the condition (3.1.3) computes the spectrum of the Hamiltonians of the N-particle periodic Toda system and the N-particle elliptic Calogero-Moser system respectively, whose eigenfunctions are quasi-periodic with the Bloch angles (θ_α). Note that the spectrum would have been $(N!)$-fold degenerate in the non-interacting limit had we tuned all the Bloch angles to be the same. For generic values of Bloch angles, the S_N-symmetry of the 0-th order wavefunctions is completely broken, leaving non-degenerate level for each spectrum.

We can revive some of the degenerate levels at the 0-th order by tuning the corresponding Bloch angles, e.g. as $\theta_\alpha = \theta_\beta$. The integrable system is still well-defined, and the eigenvalues are expected to be non-degenerate. However, we observe the missing link in the correspondence with the gauge theory. According to the condition (3.1.3), tuning the Bloch angles as $\theta_\alpha = \theta_\beta$ is equivalent to investigaing the special locus of Coulomb moduli, $\left\{\frac{a_{\alpha\beta}}{\varepsilon_1} \in \mathbb{Z} \setminus \{0\}\right\}$.[1] At the locus, the formula (3.1.2) breaks down since the right hand side becomes divergent due to the additional singularities in $\varepsilon_2 \to 0$. The asymptotic behavior of the partition function is no longer $\log\mathcal{Z} = \frac{\widetilde{\mathcal{W}}}{\varepsilon_2} + \mathcal{O}(\varepsilon_2^0)$, and the effective twisted superpotential cannot be properly obtained by just taking $\widetilde{\mathcal{W}} = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log\mathcal{Z}$. Inspired by the well-established correspondence for the generic value of the Coulomb moduli, we now may attempt to recover the correspondence at the special locus, especially by first investigating the perturbative series in the integrable system side. This is the main subject of the present chapter.

We may try to approach the special locus of Coulomb moduli from the gauge theory with partial Ω-deformation and partial noncommutativity. Instead of turning on both Ω-

[1] We have excluded $a_{\alpha\beta} = 0$ since in this case the splitting of the degeneracy at the 0-th order does not occur and (3.1.2) works as it is.

deformation parameters and then taking the Nekrasov-Shatashvili limit, we can from the beginning turn on one of the parameters ε_1 only. When the noncommutativity along the ε_1-plane is turned on, the four-dimensional $\mathcal{N} = 2$ theory can be described by a two-dimensional $\mathcal{N} = (2, 2)$ theory with an infinite dimensional gauge group. The investigation shows that the only massless modes around the trivial vacuum are the diagonal components of the gauge multiplet, which is consistent with the expectation that the low-energy effective theory is in Coulomb phase without any matter. However, when the Coulomb moduli assume the special values as $\frac{a_{\alpha\beta}}{\varepsilon_1} \in \mathbb{Z} \setminus \{0\}$, additional massless matter multiplets seem to arise, signifying the failure of the effective description.

The surface defect provides a tool for the investigation. The four-dimensional gauge theory with a half-BPS surface defect can be viewed as the theory on an orbifold. The equivariant localization computation applied for the bulk theory immediately generalizes to compute the surface defect partition function [20]. The gauge theory observables are also naturally generalized to the theory in the presence of the surface defect. In particular, an important class of observables, called the qq-character, has its fractionalized counterpart in the theory with the surface defect [10]. In [15, 21] the qq-characters with and without the surface defect were realized as the orbifolded crossed instanton partition functions. The compactness theorem proved in [14] implied a certain vanishing theorem for the expectation value of the qq-characters. The vanishing equations, called the non-perturbative Dyson-Schwinger equations, can be used to derive the KZ equation satisfied by the surface defect partition function of quiver gauge theory [32]. In this chapter, we show that the Dyson-Schwinger equation in the presence of the surface defect produces a Schrödinger-type equation satisfied by the orbifold surface defect partition function of the pure $U(N)$ gauge theory. Therefore the surface defect partition function provides a constructive approach to the eigenstate wavefunctions as well as the spectra of the Hamiltonians of the corresponding quantum integrable system.

The main observation of this chapter is that the orbifold surface defect partition function at the special locus $\left\{\frac{a_{\alpha\beta}}{\varepsilon_1} \in \mathbb{Z} \setminus \{0\}\right\}$ splits into parts, schematically,

$$\mathbf{\Psi} = \sum_{\gamma} \mathbf{\Psi}_{\gamma}. \tag{3.1.4}$$

This behavior accounts for the level splitting on the integrable system side. Each part of the surface defect partition function shows the proper asymptotic behavior of $\log \mathbf{\Psi}_{\gamma} = \frac{\widetilde{\mathcal{W}}_{\gamma}}{\varepsilon_2} + \mathcal{O}(\varepsilon_2^0)$, and the dictionary (3.1.2) is recovered to reproduce the spectrum of each split level. It should be noted that each split part $\mathbf{\Psi}_{\gamma}$ of the surface defect partition function shows the series expansion in fractional powers of the gauge coupling, which correctly accounts for the series expansions of the spectra of the split levels.

The rest of the chapter is organized as follows. In section 3.2, we explain the Bethe/gauge correspondence and two inequivalent types of quantization. In section 3.3, we study the special locus of Coulomb moduli in the four-dimensional gauge theory with partial Ω-deformation and partial noncommutativity. The investigation reveals the emergence of additional massless modes, which indicates a failure of the effective description of the theory. In section 3.4, we review the orbifold constructions of half-BPS surface defect, and compute the surface defect partition function. We study the non-perturbative Dyson-Schwinger equations in the presence of surface defects. We verify that the partition function of the A_1-theory with a regular orbifold surface defect satisfies the Schrödinger-type differential equations. In section 3.5, we observe that at the special locus of the Coulomb moduli, the surface defect partition function splits into parts, recovering the correspondence with the quantum integrable system. We conclude in section 3.6 with possible generalizations and discussions.

3.2 Bethe/gauge correspondence

It has been known that the low-energy effective theory of (un-deformed) four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories can be described by classical integrable systems [26, 27]. A well-established example is the correspondence between the class $\mathcal{S}$ theories and the Hitchin integrable systems [33, 34, 35]. Setting the 6-dimensional $\mathcal{N} = (0,2)$ superconformal theory on $\mathbb{R}^3 \times S^1 \times \mathcal{C}_{g,n}$, where $\mathcal{C}_{g,n}$ is the Riemann surface with g genus and n punctures, and reducing on $S^1 \times \mathcal{C}_{g,n}$ in two different orders, we observe that the total space of the fibration of the Jacobian of the Seiberg-Witten curve on the Coulomb moduli space of the class $\mathcal{S}$ theory is identical to the phase space of the Hitchin integrable system on $\mathcal{C}_{g,n}$. The correspondence can be extended to more general four-dimensional $\mathcal{N} = 2$ gauge theories with less hypermultiplets by taking proper decoupling limits. In this chapter we are mainly interested in the pure $U(N)$ gauge theory. It is well-known that the corresponding integrable system is the N-particle periodic Toda system [26, 36].

The N-periodic Toda system is the algebraic integrable system of N non-relativistic particles in one dimension with the interaction

$$V(x_1, \cdots, x_N) = \Lambda^2 \sum_{i=1}^{N} e^{x_i - x_{i+1}}, \tag{3.2.1}$$

and the periodicity $x_{N+1} = x_1$. The Lax operator for this system can be written as

$$L(z) = \begin{pmatrix} p_1 & \Lambda^2 e^{x_1-x_2} & 0 & \cdots & \cdots & \Lambda^N z^{-1} \\ 1 & p_2 & \Lambda^2 e^{x_2-x_3} & 0 & \cdots & 0 \\ 0 & 1 & p_3 & \Lambda^2 e^{x_3-x_4} & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & p_{N-1} & \Lambda^2 e^{x_{N-1}-x_N} \\ \Lambda^{2-N} e^{x_N - x_1} z & 0 & \cdots & 0 & 1 & p_N \end{pmatrix}, \tag{3.2.2}$$

from which we define the spectral curve

$$\Sigma(x,z): \quad 0 = \mathrm{Det}(x - L(z)) = -\Lambda^N (z + z^{-1}) + x^N + u_1 x^{N-1} + u_2 x^{N-2} + \cdots + u_N. \tag{3.2.3}$$

The standard Lax formalism tells that the (classical) Hamiltonians,

$$u_1 = -\sum_{i=1}^{N} p_i, \quad u_2 = -\sum_{i<j} p_i p_j + \Lambda^2 \sum_i e^{x_i - x_{i+1}}, \cdots , \tag{3.2.4}$$

mutually commute with respect to the Poisson bracket $\{p_i, x_j\} = \delta_{ij}$, and thus establishes the classical integrability. Note that the spectral curve (3.2.3) is precisely the Seiberg-Witten curve of the pure $U(N)$ gauge theory, in which $\{u_k = \langle \mathcal{O}_k \rangle | k = 1, \cdots , N\}$ spans the Coulomb branch of the vacua. Therefore we observe the correspondence between the low-energy description of the pure $U(N)$ gauge theory and the classical N-particle periodic Toda system. (See also [37, 38] for the earlier work in the case of Toda/pure $\mathcal{N} = 2$.)

In [28] the correspondence between the vacua of $\mathcal{N} = 2$ theories and integrable systems was promoted further to the quantum level. Let us turn on the Ω-deformation and take the Nekrasov-Shatashvili limit ($\varepsilon_1 \neq 0, \varepsilon_2 \to 0$). Since we have used one of the two orthogonal rotations to deform the theory, the theory can be now effectively described as a two-dimensional theory with $\mathcal{N} = (2, 2)$ supersymmetry. The low-energy effective action of this two-dimensional theory contains the twisted F-term[2] from the effective twisted superpotential $\widetilde{\mathcal{W}}(\mathbf{a}, \varepsilon_1, \mathfrak{q})$, which can be computed by the supersymmetric localization for generic $(\mathbf{a}, \varepsilon_1)$ as

$$\widetilde{\mathcal{W}}(\mathbf{a}, \varepsilon_1, \mathfrak{q}) = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \mathcal{Z}(\mathbf{a}, \boldsymbol{\varepsilon}, \mathfrak{q}). \tag{3.2.5}$$

The effective twisted superpotential becomes important for determining vacua and expectation values of the twisted chral observables, as we shall see below.

The space of vacua of the effective theory is a representation of the twisted chiral ring, which is spanned by the gauge-invariant polynomials of the complex adjoint scalar, (2.1.40).[3]

[2]In the reduction from the four-dimensional $\mathcal{N} = 2$ to the two-dimensional $\mathcal{N} = (2, 2)$, we are choosing the convention in which $\mathcal{N} = (2, 2)$ gauge multiplet is described by the twisted chiral superfield. Note that the complex adjoint scalar in the $\mathcal{N} = 2$ vector multiplet becomes the one in the $\mathcal{N} = (2, 2)$ twisted chiral multiplet under this reduction. See section 3.3.

[3]See footnote 2 and section 3.3. The *chiral* observables in the four-dimensional gauge theory are reduced

In [29, 30], it was shown that the twisted chiral ring of a two-dimensional $\mathcal{N} = (2, 2)$ gauge theory is identified with the Hamiltonians of the corresponding integrable system. Namely, the problem of quantization becomes the spectral problem, with the identification

$$\langle \mathcal{O}_k \rangle|_{\varepsilon_2 \to 0, \mathbf{a} \in vac} = E_k(\mathbf{a}, \varepsilon_1), \tag{3.2.6}$$

the eigenvalue of the corresponding quantum Hamiltonian $\hat{H}_k$. Here the equation for the vacua of the two-dimensional effective theory corresponds to the quantization condition of the integrable system. As noted in [28, 31], the Nekrasov-Shatashvili limit of the $\mathcal{N} = 2$ supersymmetric gauge theory leads to several quantization conditions and correspondingly to different quantum integrable systems. The choice of quantization condition becomes manifest in the topological sigma model description of the quantization. We can interpret the Nekrasov-Shatashvili limit of the Ω-deformation as the cigar metric $\mathbb{R} \times S^1 \times D_R$, in which the cigar has the asymptotic behavior of $D_R \sim I \times S^1$ with $I = [0, R]$. Then by reducing the four-dimensional $\mathcal{N} = 2$ gauge theory on $\mathbb{R} \times I$, the theory is reduced to the topological A-model with the worldsheet with the boundaries and the target space being the complexified phase space. We can make use of the brane quantization picture from this topological A-model description [18]. In particular, the quantization is realized by choosing the boundary condition at $0 \in I$ to be the canonical coisotropic A-brane and the boundary condition at $R \in I$ to be the Lagrangian A-brane. There are two classes of the Lagrangian A-branes that can be chosen, which lead to two different types of the quantization:

$$\text{Type A:} \qquad \exp\left(2\pi \frac{\partial \widetilde{\mathcal{W}}}{\partial a_\alpha} - i\theta_\alpha\right) = 1, \tag{3.2.7a}$$

$$\text{Type B:} \qquad \exp\left(2\pi i \frac{a_\alpha}{\varepsilon_1} - i\theta_\alpha\right) = 1, \quad \theta_\alpha \in [0, 2\pi). \tag{3.2.7b}$$

In the original four-dimensional gauge theory on $\mathbb{R} \times S^1 \times D_R$, they correspond to the choices

to the *twisted chiral* observables in the effective two-dimensional theory.

of the supersymmetric boundary conditions at $R \in I$. In particular, the type A condition corresponds to the Neumann boundary condition for the vector multiplet. In this case the four-dimensional vector multiplet is reduced to the two-dimensional vector multiplet in the effective theory on $\mathbb{R} \times S^1$, which is $\mathcal{N} = (2,2)$ abelian gauge theory so that the vacua are determined by the effective twisted superpotential as (3.2.7a) (we included the θ-shift). The type B condition corresponds to the Dirichlet condition for the vector multiplet. The gauge symmetry is completely broken and both vector multiplets and hypermultiplets of the four-dimensional theory are reduced to chiral multiplets of the effective two-dimensional theory. We impose the vanishing condition for the holonomy around the boundary ∂D_R to preserve the supersymmetry, yielding the quantization condition (3.2.7b). See [31] for more detail.

For the case of the pure $U(N)$ gauge theoy, type A and B reality conditions correspond to the following formulations of quantum periodic Toda system. In the type A quantization, we are taking the real slice of $x_i \in \mathbb{R}$. After decoupling the motion of the center of mass, we look for the L^2-normalizable eigenfunctions with real and discrete spectra. It was shown that the vacuum equation (3.2.7a) precisely leads to the Gutzwiller quantization condition for this type of spectral problem [39]. See also [40, 41, 42, 43, 44] for previous works on the type A periodic Toda system.

In this chapter, we mainly focus on the type B quantization of the periodic Toda system, which shows quite a different interesting feature. Here we have (quasi-)periodic eigenfunctions with the period $2\pi i$. The spectra of the Hamiltonians are complex but still discrete. With the θ-shift, the quantization condition is

$$a_\alpha = \left(n_\alpha + \frac{\theta_\alpha}{2\pi} \right) \varepsilon_1, \quad n_\alpha \in \mathbb{Z}, \tag{3.2.8}$$

where θ_α is precisely the Bloch angle for the shift of x_α by the period $2\pi i$. The spectra of the Hamiltonians can be computed as the expectation value of the observables in the twisted chiral ring, under the Nekrasov-Shatashvili limit with the condition (3.2.8) imposed:

$$E_k(\mathbf{a}, \varepsilon_1) = \langle \mathcal{O}_k \rangle|_{\varepsilon_2 \to 0,(3.2.8)} = \frac{1}{\mathcal{Z}^{\text{inst}}} \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} \mathcal{O}_k[\boldsymbol{\lambda}] \boldsymbol{\mu}_{\boldsymbol{\lambda}}(\mathbf{a}, \boldsymbol{\varepsilon}) \Bigg|_{\varepsilon_2 \to 0,(3.2.8)}, \tag{3.2.9}$$

where the statistical model form of the observable $\mathcal{O}_k[\boldsymbol{\lambda}]$ is given in (2.1.40). In particular, the spectra of two lowest order Hamiltonians $\mathcal{O}_2$ and $\mathcal{O}_3$ take simple form:

$$E_2(\mathbf{a}, \varepsilon_1) = \left[\sum_\alpha a_\alpha^2 - \frac{1}{N} \varepsilon_1 \Lambda \frac{\partial \widetilde{\mathcal{W}}}{\partial \Lambda} \right] \Bigg|_{(3.2.8)}, \tag{3.2.10a}$$

$$E_3(\mathbf{a}, \varepsilon_1) = \left[\sum_\alpha a_\alpha^3 - \frac{3\varepsilon_1^2}{2N} \Lambda \frac{\partial \widetilde{\mathcal{W}}}{\partial \Lambda} - 6\varepsilon_1 \lim_{\varepsilon_2 \to 0} \varepsilon_2 \left\langle \sum_{\square \in K} c_{\hat{\square}} \right\rangle \right] \Bigg|_{(3.2.8)}. \tag{3.2.10b}$$

For example, in the case of $N = 2$ the type B quantum periodic Toda system is reduced to the Mathieu system, whose discrete energy spectrum has been well-studied. For generic value of the Coulomb moduli $a_{12} = a_1 - a_2$ (on the integrable system side, generic value of $\theta_1 - \theta_2$) the gauge theory computation of the spectrum (3.2.10a) precisely reproduces the known perturbative computation, order by order in the series of Λ^4. We also checked that the perturbative spectra of $N = 3$ periodic Toda system are reproduced by (3.2.10). The generalization of the computation to the higher N is straightforward.

However, when the Coulomb moduli assume special values $\frac{a_{\alpha\beta}}{\varepsilon_1} \in \mathbb{Z} \setminus \{0\}$, the correspondence breaks down as we now describe. A relation among the equivariant parameters implies that the maximal torus T_H used for the equivariant localization becomes smaller than generic cases. When the torus becomes smaller, the set of fixed points $\mathcal{M}(N)^{T_H}$ in general becomes larger; as noted in [10], one may find a copy of $\mathbb{P}^1$'s or a even more complicated subvariety instead of isolated set of fixed points with the reduction of symmetry group.

It can be shown that for the specific case at hand, $\frac{a_{\alpha\beta}}{\varepsilon_1} \in \mathbb{Z} \setminus \{0\}$, $\mathcal{M}(N)^{T_H}$ actually contains products of $\mathbb{P}^1$'s. Recall that before taking $\frac{a_{\alpha\beta}}{\varepsilon_1} \in \mathbb{Z} \setminus \{0\}$ the isolated fixed points $\mathcal{M}(N)^{T_H}$ are classified by N-tuples of Young diagrams $\{\boldsymbol{\lambda}\}$. The boxes in these Young diagrams encode the weights of linearly independent vectors in the space $K[\boldsymbol{\lambda}]$ in terms of Coulomb moduli and Ω-deformation parameters, and these weights are all distinct. However, once we introduce the new constraint $\frac{a_{\alpha\beta}}{\varepsilon_1} \in \mathbb{Z} \setminus \{0\}$, the weights now may overlap (or in terms of the Young diagrams, two boxes in different Young diagrams may collide). This implies two isolated fixed points disappear into an emergent fixed point set $\mathbb{P}^1$ (so that when the symmetry group action is refined by an extra $U(1)$ as it used to be, we recover two isolated fixed points on the emergent $\mathbb{P}^1$). Since we get an emergent $\mathbb{P}^1$ whenever this overlap occurs, the fixed point set $\mathcal{M}(N)^{T_H}$ now contains a product of mutiple $\mathbb{P}^1$'s.

Hence the integral that provides the instanton partition function remains finite due to the compactness of $\mathcal{M}(N)^{T_H}$. Nevertheless, the integral over the emergent $\mathbb{P}^1$'s gives additional poles in ε_2, altering the asymptotic behavior of the instanton partition function in the limit $\varepsilon_2 \to 0$. Most importantly, the effective twisted superpotential is not properly obtained by taking $\widetilde{\mathcal{W}} = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \mathcal{Z}$ since the expression becomes divergent. Therefore we see that (3.2.10) cannot work as it is stated. The main subject of the present chapter is to recover the correspondence at this special locus.

3.3 Gauge theory with partial Ω-deformation and partial noncommutativity

To explore the gauge theoretical meaning of the special locus of Coulomb moduli, let us study the pure $\mathcal{N} = 2$ $U(N)$ gauge theory with partial Ω-deformation and partial noncommutativity. The four-dimensional $\mathcal{N} = 2$ supersymmetry can be described by the super-covariant derivatives in the covariant basis,

$$\begin{aligned}
&\{\boldsymbol{\nabla}^A_\alpha, \bar{\boldsymbol{\nabla}}_{B\dot{\alpha}}\} = -i\delta^A{}_B \boldsymbol{\nabla}_{\alpha\dot{\alpha}} \\
&\{\boldsymbol{\nabla}^A_\alpha, \boldsymbol{\nabla}^B_\beta\} = i\epsilon^{AB}\epsilon_{\alpha\beta}\bar{\boldsymbol{\Phi}} \\
&\{\bar{\boldsymbol{\nabla}}_{A\dot{\alpha}}, \bar{\boldsymbol{\nabla}}_{B\dot{\beta}}\} = i\epsilon_{AB}\epsilon_{\dot{\alpha}\dot{\beta}}\boldsymbol{\Phi},
\end{aligned} \tag{3.3.1}$$

where we are using the convention $\sigma^{\mu}_{\alpha\dot{\alpha}} = (\mathbb{1}_{\alpha\dot{\alpha}}, \vec{\tau}_{\alpha\dot{\alpha}})$. Here $\mathbf{\Phi}$ is the $\mathcal{N}=2$ chiral superfield constrained by $\boldsymbol{\nabla}^{\alpha}_{A}\boldsymbol{\nabla}_{B\alpha}\mathbf{\Phi} = -\bar{\boldsymbol{\nabla}}_{B\dot{\alpha}}\bar{\boldsymbol{\nabla}}^{\dot{\alpha}}_{A}\bar{\mathbf{\Phi}}$ due to the Bianchi identities. The action for the pure $\mathcal{N}=2$ gauge theory can be written in the $\mathcal{N}=2$ chiral superspace as

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\int d^4\theta \frac{1}{2}\tau \mathrm{Tr}\mathbf{\Phi}^2. \tag{3.3.2}$$

The partial Ω-deformation ($\varepsilon_1 \neq 0$, $\varepsilon_2 = 0$) breaks the $\mathcal{N}=2$ supersymmetry, but preserves a $\mathcal{N}=(2,2)$ subalgebra on the (x^0, x^3)-plane,

$$\begin{aligned}\{\boldsymbol{\nabla}^1_+, \bar{\boldsymbol{\nabla}}_{1\dot{+}}\} &= -i\boldsymbol{\nabla}_{+\dot{+}} = -i\left(\boldsymbol{\nabla}_0 + \boldsymbol{\nabla}_3\right)\\ \{\boldsymbol{\nabla}^2_-, \bar{\boldsymbol{\nabla}}_{2\dot{-}}\} &= -i\boldsymbol{\nabla}_{-\dot{-}} = -i\left(\boldsymbol{\nabla}_0 - \boldsymbol{\nabla}_3\right).\end{aligned} \tag{3.3.3}$$

Let us choose the following convention for the reduced algebra

$$\begin{aligned}\boldsymbol{\nabla}^1_+ &\equiv \boldsymbol{\nabla}_+, \quad \boldsymbol{\nabla}^2_- \equiv \bar{\boldsymbol{\nabla}}_-,\\ \bar{\boldsymbol{\nabla}}_{1\dot{+}} &\equiv \bar{\boldsymbol{\nabla}}_+, \quad \bar{\boldsymbol{\nabla}}_{2\dot{-}} \equiv \boldsymbol{\nabla}_-,\end{aligned} \tag{3.3.4}$$

so that the restriction of the $\mathcal{N}=2$ chiral superfield $\Sigma \equiv \mathbf{\Phi}| = i\{\bar{\boldsymbol{\nabla}}_+, \boldsymbol{\nabla}_-\}$ is a twisted chiral superfield in the reduced $\mathcal{N}=(2,2)$ supersymmetry. Note that Σ contains the complex scalar of the $\mathcal{N}=2$ vector multiplet as its component field. Also it is important that we have the following relations from the Bianchi identities,

$$[\bar{\boldsymbol{\nabla}}_{\pm}, \boldsymbol{\nabla}_{-\dot{+}}] = 0. \tag{3.3.5}$$

The $\mathcal{N}=2$ superspace action is reduced to the $\mathcal{N}=(2,2)$ superspace,

$$\mathcal{L} = -\frac{1}{2g^2}\int d^4\theta \mathrm{Tr}\bar{\Sigma}\Sigma - \mathrm{Im}\left[\frac{\tau}{8\pi}\int d^2\tilde{\theta}\mathrm{Tr}\left(i\Sigma[\boldsymbol{\nabla}_{+\dot{-}}, \boldsymbol{\nabla}_{-\dot{+}}] - [\boldsymbol{\nabla}_-, \boldsymbol{\nabla}_{-\dot{+}}][\bar{\boldsymbol{\nabla}}_+, \boldsymbol{\nabla}_{+\dot{-}}]\right)\right]. \tag{3.3.6}$$

Now let us turn on the noncommutativity on the (x^1, x^2)-plane, $[x^1, x^2] = i\zeta$, while leaving the (x^0, x^3)-plane commutative. Define the raising and the lowering operators:

$$c = \frac{1}{\sqrt{2\zeta}}(x^1 + ix^2), \quad c^\dagger = \frac{1}{\sqrt{2\zeta}}(x^1 - ix^2), \quad [c, c^\dagger] = 1. \tag{3.3.7}$$

The effect of the noncommutativity is that the *covariant coordinate*

$$\Phi \equiv -i\frac{1}{\sqrt{\zeta}}c - \frac{1}{\sqrt{2}}(A_1 + iA_2) \tag{3.3.8}$$

can act by commutator as the covariant derivative along the noncommutative direction [45, 46]. Namely, we can make a substitution $\boldsymbol{\nabla}_{-\dot{+}} \to \sqrt{2}\Phi$ except in the commutator of two such covariant derivatives,

$$[\boldsymbol{\nabla}_{-\dot{+}}, \boldsymbol{\nabla}_{+\dot{-}}] = 2[\Phi, \bar{\Phi}] - \frac{2}{\zeta}, \tag{3.3.9}$$

where we have the extra term from the commutator of c and $c^\dagger$. Note that Φ is an adjoint chiral superfield in the $\mathcal{N} = (2, 2)$ supersymmetry by the relation (3.3.5). The fields are now promoted to endomorphisms of the Fock space $\mathcal{H}$ that represents the algebra (3.3.7), on which the dependence of the fields on the noncommutative coordinates are encoded. The integration along the noncommutative directions is replaced by the trace over the Fock space,

$$\int dx^1 dx^2 (\cdots) = \zeta \mathrm{Tr}_{\mathcal{H}}(\cdots). \tag{3.3.10}$$

Thus, with the Wick rotation, we arrive at the Euclidean two-dimensional $\mathcal{N} = (2, 2)$ superspace action of the four-dimensional theory with the partial noncommutativity

$$\mathcal{L} = \frac{i}{8\pi}\left(\tau \int d^2\tilde{\theta} \mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^N} \Sigma + c.c\right) + \frac{\zeta}{g^2}\int d^4\theta \mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^N}\left[-\frac{1}{2}\bar{\Sigma}\Sigma + \bar{\Phi}e^V\Phi\right], \tag{3.3.11}$$

with the following superfield contents[4]

$$\text{Twisted chiral}: \quad \Sigma = (\sigma, \lambda_+, \bar{\lambda}_-, iD + F_{43}) \tag{3.3.12a}$$

$$\text{Adjoint chiral}: \quad \Phi = (\phi, \psi_\pm, F). \tag{3.3.12b}$$

As is apparent from the definition of Φ as the covariant coordinate, the $U(1) = SO(2)_{12} \subset G_{\mathrm{rot}}$ spacetime rotation becomes the flavor symmetry rotating the chiral multiplet Φ. The partial Ω-deformation ($\varepsilon_1 \neq 0, \varepsilon_2 = 0$) is simply weakly gauging this $U(1)$ flavor symmetry to generate the twisted mass for the chiral multiplet,

$$\widetilde{V}_{\varepsilon_1} = -\varepsilon_1 \theta^- \bar{\theta}^+ - \bar{\varepsilon}_1 \theta^+ \bar{\theta}^-. \tag{3.3.13}$$

Thus the final form of the action is

$$\mathcal{L} = \frac{i}{8\pi}\left(\tau \int d^2\tilde{\theta} \mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^N} \Sigma + c.c\right) + \frac{\zeta}{g^2}\int d^4\theta \mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^N}\left[-\frac{1}{2}\bar{\Sigma}\Sigma + \bar{\Phi} e^{V+\widetilde{V}_{\varepsilon_1}}\Phi\right], \tag{3.3.14}$$

which can be expanded to an x-space action,

$$\begin{aligned}
\mathcal{L} = \frac{\zeta}{g^2}\mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^N}\Big[&\frac{1}{2}F_{43}^2 + D_\mu\sigma^\dagger D^\mu\sigma + \frac{1}{2}D^2 - iD([\phi,\phi^\dagger] - \frac{1}{\zeta}) + \frac{1}{2}[\sigma,\sigma^\dagger]^2 + FF^\dagger \\
&+ D_\mu\phi^\dagger D^\mu\phi + |[\sigma,\phi] + \varepsilon_1\phi|^2 + |[\sigma,\phi^\dagger] - \varepsilon_1\phi^\dagger|^2 \\
&+ 2i\bar{\lambda}_+ D_z\lambda_+ - 2i\bar{\lambda}_- D_{\bar{z}}\lambda_- + 2i\bar{\psi}_+ D_z\psi_+ - 2i\bar{\psi}_- D_{\bar{z}}\psi_- \\
&+ \sqrt{2}\lambda_+[\sigma,\bar{\lambda}_-] - \sqrt{2}[\sigma^\dagger,\lambda_-]\bar{\lambda}_+ + \sqrt{2}\bar{\psi}_+([\sigma^\dagger,\psi_-] + \bar{\varepsilon}_1\psi_-) + \sqrt{2}\bar{\psi}_-([\sigma,\psi_+] + \varepsilon_1\psi_+) \\
&- i\sqrt{2}\bar{\psi}_+[\bar{\lambda}_-,\phi] + i\sqrt{2}\bar{\psi}_-[\bar{\lambda}_+,\phi] - i\sqrt{2}[\phi^\dagger,\lambda_+]\psi_- + i\sqrt{2}[\phi^\dagger,\lambda_-]\psi_+\Big] \\
-\frac{i\vartheta}{8\pi^2}&\mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^N} F_{43}.
\end{aligned} \tag{3.3.15}$$

[4]Here we are denoting the complex scalar which descends from the $\mathcal{N} = 2$ vector multiplet as σ, which has been denoted as ϕ so far. The convention may be confusing but is more traditional in $\mathcal{N} = (2,2)$ context.

The bosonic part of the action can be written as

$$
\begin{aligned}
\mathcal{L}_{\text{bos}} = & -\frac{i\tau}{4\pi}\mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^N} F_{43} \\
& + \frac{\zeta}{g^2}\mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^N}\left[\frac{1}{2}\left(F_{43}+[\phi,\phi^\dagger]-\frac{1}{\zeta}\right)^2 + 4|D_{\bar{z}}\phi|^2 + D_\mu\sigma^\dagger D^\mu\sigma + FF^\dagger\right. \\
& \left. +\frac{1}{2}\left(D-i\left([\phi,\phi^\dagger]-\frac{1}{\zeta}\right)\right)^2 + |[\sigma,\phi]+\varepsilon_1\phi|^2 + |[\sigma,\phi^\dagger]-\varepsilon_1\phi^\dagger|^2 + \frac{1}{2}[\sigma,\sigma^\dagger]^2\right],
\end{aligned}
\tag{3.3.16}
$$

from which we read off the vaccum equations

$$
\begin{aligned}
& F_{43}+[\phi,\phi^\dagger]-\frac{1}{\zeta}=0, \quad D_{\bar{z}}\phi=0, \quad D-i\left([\phi,\phi^\dagger]-\frac{1}{\zeta}\right)=0, \\
& D_\mu\sigma=0, \quad [\sigma,\sigma^\dagger]=0, \quad [\sigma,\phi]+\varepsilon_1\phi=[\sigma,\phi^\dagger]-\varepsilon_1\phi^\dagger=0.
\end{aligned}
\tag{3.3.17}
$$

We focus on the trivial sector where $F_{43}=0$. Then the vaccum equations are solved by

$$
\begin{aligned}
& D=0, \\
& \sigma=\varepsilon_1 c^\dagger c\otimes\mathbb{1}_{\mathbb{C}^N}+\mathbb{1}_{\mathcal{H}}\otimes\mathrm{diag}(a_1,a_2,\cdots,a_N), \\
& \phi=\frac{1}{\sqrt{\zeta}}c\otimes\mathbb{1}_{\mathbb{C}^N}, \quad \phi^\dagger=\frac{1}{\sqrt{\zeta}}c^\dagger\otimes\mathbb{1}_{\mathbb{C}^N},
\end{aligned}
\tag{3.3.18}
$$

where a_α are moduli that parametrize the vacua. Since σ is the complex scalar in the $\mathcal{N}=2$ vector multiplet, a_α are nothing but the Coulomb moduli in the four-dimensional perspective. The low-energy effective action is obtained by integrating out all the massive modes and high energy modes around the vaccum (3.3.18). Thus we split the vacuum expectation value and the quantum fluctuation,

$$
\sigma=\sigma_0+\hat{\sigma}, \quad \phi=\phi_0+\hat{\phi},
\tag{3.3.19}
$$

and expand the action in fluctuation modes. We introduce the following gauge fixing term

$$\mathcal{L}_{\text{fix}} = \frac{\zeta}{2g^2}\text{Tr}_{\mathcal{H}\otimes\mathbb{C}^N}\left[\partial_\mu A^\mu - i[\sigma_0^\dagger, \hat{\sigma}] - i[\sigma_0, \hat{\sigma}^\dagger] - i[\phi_0^\dagger, \hat{\phi}] - i[\phi_0, \hat{\phi}^\dagger]\right]^2, \tag{3.3.20}$$

to cancel the mixing terms in the quadratic order. Then we are left with

$$\begin{aligned}
&\mathcal{L}_{\text{bos}} + \mathcal{L}_{\text{fix}} \\
=&\frac{\zeta}{g^2}\text{Tr}_{\mathcal{H}\otimes\mathbb{C}^N}\left[\frac{1}{2}F_{43}^2 + |[A_\mu, \sigma_0]|^2 + |[A_\mu, \phi_0]|^2 + D_\mu\hat{\sigma}^\dagger D^\mu\hat{\sigma} + D_\mu\hat{\phi}^\dagger D^\mu\hat{\phi} + \frac{1}{2}(\partial_\mu A^\mu)^2 + FF^\dagger\right. \\
&\qquad - iD[\hat{\phi}, \hat{\phi}^\dagger] + \frac{1}{2}\left(D - i([\phi_0, \hat{\phi}^\dagger] + [\hat{\phi}, \phi_0^\dagger])\right)^2 + 2|[\hat{\phi}, \phi_0^\dagger]|^2 + \frac{1}{2}[\hat{\sigma}, \hat{\sigma}^\dagger]^2 \\
&\qquad + [\hat{\sigma}, \hat{\sigma}^\dagger]\left([\sigma_0, \hat{\sigma}^\dagger] + [\hat{\sigma}, \sigma_0^\dagger]\right) + 2|[\hat{\sigma}, \sigma_0^\dagger]|^2 - [A_\mu, \sigma_0^\dagger][A^\mu, \hat{\sigma}] - [A_\mu, \hat{\sigma}^\dagger][A^\mu, \sigma_0] \\
&\qquad - [A_\mu, \phi_0^\dagger][A^\mu, \hat{\phi}] - [A_\mu, \hat{\phi}^\dagger][A^\mu, \phi_0] + |[\sigma, \hat{\phi}] + \varepsilon_1\hat{\phi}|^2 + |[\sigma, \hat{\phi}^\dagger] - \varepsilon_1\hat{\phi}^\dagger|^2 \\
&\qquad \left. +|[\hat{\sigma}, \phi_0]|^2 + |[\hat{\sigma}, \phi_0^\dagger]|^2 + [\hat{\sigma}, \hat{\phi}][\phi_0^\dagger, \hat{\sigma}^\dagger] + [\hat{\phi}^\dagger, \hat{\sigma}^\dagger][\hat{\sigma}, \phi_0] + [\hat{\sigma}, \hat{\phi}^\dagger][\phi_0, \hat{\sigma}^\dagger] + [\hat{\phi}, \hat{\sigma}^\dagger][\hat{\sigma}, \phi_0^\dagger]\right] \\
&- \frac{i\vartheta}{8\pi^2}\text{Tr}_{\mathcal{H}\otimes\mathbb{C}^N}F_{43}
\end{aligned} \tag{3.3.21}$$

For generic values of Coulomb moduli, the only massless fluctuations are the modes of the abelian twisted chiral multiplet,

$$\hat{\Sigma} = \hat{\sigma} + \cdots = \mathbb{1}_{\mathcal{H}} \otimes \text{diag}(\Sigma_1, \Sigma_2, \cdots, \Sigma_N). \tag{3.3.22}$$

All the other modes are integrated out in the effective theory, possibly contributing to the effective twisted superpotential $\widetilde{\mathcal{W}}(\Sigma_\alpha)$. Therefore the effective two-dimensional theory is a pure abelian gauge theory of rank N with a certain effective twisted superpotential.

However, we discover that additional massless modes emerge at the special locus of Coulomb moduli, $\{a_{\alpha\beta} = m\varepsilon_1 \mid m \in \mathbb{Z} \setminus \{0\}\}$. Namely, the mass term for the chiral multiplet mode

$$\hat{\Phi} \equiv \hat{\phi} + \cdots = \begin{cases} (c^\dagger)^{m-1} \otimes E_{\beta,\alpha} \Phi_{\alpha\beta}, & \text{if} \quad m > 0 \\ (c^\dagger)^{-m-1} \otimes E_{\alpha,\beta} \Phi_{\alpha\beta}, & \text{if} \quad m < 0 \end{cases} \tag{3.3.23}$$

vanishes at the locus. Here, $E_{\alpha,\beta}$ is the $N \times N$ matrix whose elements are all 0 except 1 for the element in the αth row and the βth column. A massless mode of chiral multiplet is generated for each such a pair of (α, β). The emergent massless modes signify the failure of the effective description of the theory. In [47], it was argued that this failure is cured by the appearance of solitonic particles, which prevent the massless modes to occur through the wall-crossing. It would be nice to directly see how this wall-crossing phenomenon interplays with the insertion of surface defects discussed in the following sections.

3.4 Surface defect

3.4.1 Construction

As non-local gauge-invariant observables, the surface defects enrich the study of $\mathcal{N} = 2$ supersymmetric gauge theories and Bethe/gauge correspondence. As discussed in section 2.3, there are two ways of constructing the half-BPS surface defects in the context of the $\mathcal{N} = 2$ gauge theory. One of them is orbifolding the four-dimesional spacetime with respect to the action of the cyclic group $\mathbb{Z}_p$ as $\mathbb{C}_{\varepsilon_1} \times (\mathbb{C}_{\varepsilon_2}/\mathbb{Z}_p)$. This type of surface defect is referred as the orbifold surface defect. The second way is inserting a degenerate gauge vertex in the quiver which defines the quiver gauge theory of interest. Even though these constructions seem to be distinct, we shall see in the Chapter 5 that for some cases there is an exact equivalence between the two types of surface defects, which generalizes the IR duality of [48] (at least in the A_1 case) between the two types of surface defect that descends from the M-theory brane transition. We mainly utilize the orbifold surface defect for the purpose of this Chapter, so we will only discuss the orbifold construction in this chapter. More results regarding the quiver surface defects will follow in the subsequent chapters.

3.4.1.1 Orbifold construction

Throughout the discussion, let us restrict our attention to the pure $U(N)$ gauge theory. The orbifold surface defect $\mathcal{D}_{\mathbb{Z}_p,\rho}$ is constructed by specifying the embedding

$$\rho : \mathbb{Z}_p \longrightarrow H = G_g \times G_{\text{rot}}, \tag{3.4.1}$$

from which we define the surface defect as the prescription of performing the path integral over the space of $\mathbb{Z}_p$-invariant fields. The rotation group part of the embedding is always chosen to be

$$\Omega(\zeta) : (z_1, z_2) \mapsto (z_1, \zeta z_2), \quad \text{for} \quad \zeta = \exp\left(\frac{2\pi i}{p}\right). \tag{3.4.2}$$

To fully characterize the surface defect we need to further specify the gauge group part of the embedding ρ. It is assigned by the coloring function

$$c : [N] = \{0, \cdots, N-1\} \longrightarrow \mathbb{Z}_p, \tag{3.4.3}$$

from which we define the gauge group part of the embedding ρ such that the vector space N decomposes as

$$N = \sum_\alpha e^{\beta a_\alpha} \mathcal{R}_{c(\alpha)} = \sum_{\omega \in \mathbb{Z}_p} N_\omega \mathcal{R}_\omega \quad \Longrightarrow \quad N_\omega = \sum_{\alpha \in c^{-1}(\omega)} e^{\beta a_\alpha}, \tag{3.4.4}$$

where $\mathcal{R}_\omega$ is the one-dimensional irreducible representation of $\mathbb{Z}_p$ of weight ω,

$$\begin{aligned} \mathbb{Z}_p &\longrightarrow \mathrm{End}(\mathcal{R}_\omega) \\ \zeta &\longmapsto \zeta^\omega. \end{aligned} \tag{3.4.5}$$

Then we also decompose

$$K = \sum_{\omega\in\mathbb{Z}_p} K_\omega \mathcal{R}_\omega, \quad \text{where} \quad K_\omega = \sum_\alpha \sum_{\substack{(i,j)\in\lambda^{(\alpha)} \\ c(\alpha)+j-1\equiv\omega \bmod p}} e^{\beta(a_\alpha+\varepsilon_1(i-1)+\varepsilon_2(j-1))}. \tag{3.4.6}$$

We can identify the spacetime $\mathbb{C}^2$ with the orbifold $\mathbb{C}^2/\mathbb{Z}_p$ through the map $(z_1, z_2) \mapsto (\tilde{z}_1 = z_1, \tilde{z}_2 = z_2^p)$. This map is singular along the surface $z_2 = 0$. Therefore the path integral over the space of the $\mathbb{Z}_p$-invariant fields on (z_1, z_2)-space is interpreted as the path integral over the $(\tilde{z}_1, \tilde{z}_2)$-space with the insertion of a defect along the surface $\tilde{z}_2 = 0$.

An orbifold surface defect is called *regular* for the special case when $p = N$ and $c \in S_N$, where S_N is the permutation group of $[N] = \{0, \cdots N-1\}$. This special kind of surface defects plays an important role in constructing the eigenstate wavefunctions of the integrable system in section 3.4.2.1 and section 3.5.

3.4.1.2 $\mathcal{N} = 2$ supersymmetric gauge theory with orbifold surface defect

We now investigate the $\mathcal{N} = 2$ gauge theory in the presence of the orbifold surface defect. In the presence of the surface defect, the coupling constant is fractionalized

$$\mathfrak{q} \mapsto \mathfrak{q}_\omega \equiv \Lambda^2 \frac{z_\omega}{z_{\omega-1}}, \quad \omega \in \mathbb{Z}_p, \tag{3.4.7}$$

with $z_{\omega+p} \equiv z_\omega$. The surface defect partition function is the path integral over the space of $\mathbb{Z}_p$-invariant fields, which can be easily obtained from the bulk partition function. From

(2.1.31), the instanton part of the surface defect partition function is immediately obtained

$$\Psi_c^{\text{inst}}(\mathbf{a}, \boldsymbol{\varepsilon}, \mathfrak{q}, \mathbf{z}) = \sum_{\boldsymbol{\lambda}} \prod_{\omega \in \mathbb{Z}_p} \mathfrak{q}_\omega^{k_\omega} \epsilon(\mathcal{T}[\boldsymbol{\lambda}]^{\mathbb{Z}_p, c}), \tag{3.4.8}$$

where $k_\omega[\boldsymbol{\lambda}] = \dim K_\omega[\boldsymbol{\lambda}]$ is the fractionalized instanton number and $(\cdots)^{\mathbb{Z}_p,c}$ is the prescription of keeping the $\mathbb{Z}_p$-invariant piece for the given coloring function c only. The $\mathbb{Z}_p$-invariant piece of the character (2.1.30) is given by

$$\mathfrak{T}[\boldsymbol{\lambda}]^{\mathbb{Z}_p,c} = \sum_{\omega \in \mathbb{Z}_p} \left[N_\omega K_\omega^* + q_1 q_2 N_\omega^* K_{\omega-1} - (1-q_1) K_\omega K_\omega^* + q_2 (1-q_1) K_\omega K_{\omega+1}^* \right]. \tag{3.4.9}$$

In the special case that the coloring function $c : [N] \to \mathbb{Z}_p$ is chosen to be surjective, (3.4.8) is identical to the computation from the chain-saw quiver [20]. Note that the instanton part of the surface defect partition function also defines a statistical model on the set of colored partitions $\{\boldsymbol{\lambda}\}$, with the measure $\boldsymbol{\mu}_{\boldsymbol{\lambda}}^{\mathbb{Z}_p,c}(\mathbf{a}, \boldsymbol{\varepsilon}) = \prod_{\omega \in \mathbb{Z}_p} \mathfrak{q}_\omega^{k_\omega} \epsilon(\mathcal{T}[\boldsymbol{\lambda}]^{\mathbb{Z}_p,c})$.

3.4.2 Consequences of the non-perturbative Dyson-Schwinger equations

We now derive the differential equations that surface defect partition functions satisfy, using the non-perturbative Dyson-Schwinger equations. For generic quiver gauge theories with half-BPS surface defects, the non-perturbative Dyson-Schwinger equations derived in [21] can be used to prove the KZ equation and the BPZ equation satisfied by the partition functions [32]. In this chapter, we study the surface defects on the pure $U(N)$ gauge theory which is relevant to the periodic Toda system. The orbifold surface defect partition function is shown to satisfy the Schrödinger-type equation, while the degenerate gauge vertex partition function satisfies the Baxter-type equation. Note that those differential equations are valid for all values of $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, as the fact will be crucial for investigating the special locus of the Coulomb moduli.

3.4.2.1 A_1-theory with orbifold surface defect

Let us consider the A_1-theory with the gauge group $U(N)$ in the presence of the regular orbifold surface defect $\mathcal{D}_{\mathbb{Z}_N,\rho}$, with the coloring function $s \in S_N$. With respect to the representations of $\mathbb{Z}_N$, the $\mathcal{Y}$-observable factors as:

$$\mathcal{Y}(x) = \prod_{\omega\in\mathbb{Z}_N} \mathcal{Y}_\omega(x), \tag{3.4.10}$$

where

$$\mathcal{Y}_\omega(x)[\boldsymbol{\lambda}] = (x - a_{s^{-1}(\omega)}) \prod_{\square\in K_\omega} \frac{x - c_{\hat{\square}} - \varepsilon_1}{x - c_{\hat{\square}}} \prod_{\square\in K_{\omega-1}} \frac{x - c_{\hat{\square}} - \varepsilon_2}{x - c_{\hat{\square}} - \varepsilon}. \tag{3.4.11}$$

In terms of these $\mathcal{Y}_\omega$'s we also have the fundamental refined qq-characters, which are obtained as the orbifolded crossed instanton partition functions [15],

$$\mathcal{X}_\omega(x) = \mathcal{Y}_{\omega+1}(x+\varepsilon) + \frac{\Lambda^2 z_\omega z_{\omega-1}^{-1}}{\mathcal{Y}_\omega(x)}, \tag{3.4.12}$$

whose expectation value in the gauge theory in the presence of the surface defect,

$$\langle \mathcal{X}_\omega(x)\rangle_s \equiv \frac{1}{\boldsymbol{\Psi}_s^{\text{inst}}} \sum_{\boldsymbol{\lambda}} \mathcal{X}_\omega(\mathcal{Y}[\boldsymbol{\lambda}])\mathfrak{q}^{|\boldsymbol{\lambda}|}\boldsymbol{\mu}_{\boldsymbol{\lambda}}^{\mathbb{Z}_N,s}(\mathbf{a},\boldsymbol{\varepsilon}) = T_{s,\omega}(x), \tag{3.4.13}$$

is a polynomial in x by the compactness theorem proven in [14]. In particular, we have the vanishing equations,

$$[x^{-n}]\langle \mathcal{X}_\omega(x)\rangle_s = 0, \quad n \in \mathbb{Z}_{>0}. \tag{3.4.14}$$

We study the coefficients of x^{-n} of the fundamental refined qq-character in the large x limit. The lowest order coefficients are given by:

$$
\begin{aligned}
[x^{-1}]\mathfrak{X}_\omega =& \frac{\varepsilon_1^2}{2}\left(k_\omega - k_{\omega+1} - \frac{a_{s^{-1}(\omega+1)}}{\varepsilon_1}\right)^2 - \frac{1}{2}a_{s^{-1}(\omega+1)}^2 + \varepsilon_1\varepsilon_2 k_\omega + \Lambda^2 z_\omega z_{\omega-1}^{-1} \\
&+ \frac{\varepsilon_1^2}{2}(k_\omega - k_{\omega+1}) + \varepsilon_1\left(\sum_{\square\in K_\omega} c_{\hat{\square}} - \sum_{\square\in K_{\omega+1}} c_{\hat{\square}}\right), \qquad (3.4.15a) \\
[x^{-2}]\mathfrak{X}_\omega =& \frac{\varepsilon_1^3}{6}(k_\omega - k_{\omega+1})^3 - \frac{\varepsilon_1^3}{2}(k_\omega - k_{\omega+1})^2 + \varepsilon_1^2\varepsilon_2 k_{\omega+1}(k_\omega - k_{\omega+1}) \\
&+ (\varepsilon - a_{s^{-1}(\omega+1)})\left(\frac{\varepsilon_1^2}{2}(k_\omega - k_{\omega+1})^2 - \frac{\varepsilon_1^2}{2}(k_\omega - k_{\omega+1}) + \varepsilon_1\varepsilon_2 k_{\omega+1} + \varepsilon_1\left(\sum_{\square\in K_\omega} c_{\hat{\square}} - \sum_{\square\in K_{\omega+1}} c_{\hat{\square}}\right)\right) \\
&+ \Lambda^2 z_\omega z_{\omega-1}^{-1}(a_{s^{-1}(\omega)} + \varepsilon_1(k_\omega - k_{\omega-1})) + \varepsilon_1^2(k_\omega - k_{\omega+1})\left(\sum_{\square\in K_\omega} c_{\hat{\square}} - \sum_{\square\in K_{\omega+1}} c_{\hat{\square}}\right) \\
&+ \frac{\varepsilon_1^3}{3}(k_\omega - k_{\omega+1}) - \varepsilon_1^2\left(\sum_{\square\in K_\omega} c_{\hat{\square}} - \sum_{\square\in K_{\omega+1}} c_{\hat{\square}}\right) + \varepsilon_1\left(\sum_{\square\in K_\omega} c_{\hat{\square}}^2 - \sum_{\square\in K_{\omega+1}} c_{\hat{\square}}^2\right) \\
&- \varepsilon_1\varepsilon_2\varepsilon k_{\omega+1} + 2\varepsilon_1\varepsilon_2\sum_{\square\in K_{\omega+1}} c_{\hat{\square}}. \qquad (3.4.15b)
\end{aligned}
$$

The expectation values of (3.4.15) yield the vanishing equations. We take the sum over $\omega \in \mathbb{Z}_N$, while simplifying (3.4.15b) using (3.4.15a), to get the following differential equations,

$$
0 = \left[\frac{\varepsilon_1^2}{2}\sum_\omega\left(z_\omega\frac{\partial}{\partial z_\omega} - \frac{a_{s^{-1}(\omega+1)}}{\varepsilon_1}\right)^2 + \Lambda^2\sum_\omega z_\omega z_{\omega-1}^{-1} - \frac{1}{2}\sum_\omega a_{s^{-1}(\omega+1)}^2 + \frac{1}{2}\varepsilon_1\varepsilon_2\Lambda\frac{\partial}{\partial\Lambda}\right]\mathbf{\Psi}_s^{\mathrm{inst}}(\mathbf{a},\boldsymbol{\varepsilon},\mathfrak{q},\mathbf{z}), \qquad (3.4.16a)
$$

$$
\begin{aligned}
0 = \Bigg[&-\frac{\varepsilon_1^3}{3}\sum_\omega\left(z_\omega\frac{\partial}{\partial z_\omega} - \frac{a_{s^{-1}(\omega+1)}}{\varepsilon_1}\right)^3 \\
&+ \Lambda^2\sum_\omega z_\omega z_{\omega-1}^{-1}\left(-\varepsilon_1\left(z_\omega\frac{\partial}{\partial z_\omega} + z_{\omega-1}\frac{\partial}{\partial z_{\omega-1}} - \frac{a_{s^{-1}(\omega+1)}}{\varepsilon_1} - \frac{a_{s^{-1}(\omega)}}{\varepsilon_1}\right) + \varepsilon_2\right) \\
&-\frac{1}{3}\sum_\omega a_{s^{-1}(\omega+1)}^3 + \frac{1}{2}\varepsilon_1\varepsilon_2\varepsilon\Lambda\frac{\partial}{\partial\Lambda} + 2\varepsilon_1\varepsilon_2\left\langle\sum_{\square\in K} c_{\hat{\square}}\right\rangle_s\Bigg]\mathbf{\Psi}_s^{\mathrm{inst}}(\mathbf{a},\boldsymbol{\varepsilon},\mathfrak{q},\mathbf{z}). \qquad (3.4.16b)
\end{aligned}
$$

Note that (3.4.16a) is the one-line rederivation of the results of [37, 38]. In the Nekrasov-Shatashvili limit ($\varepsilon_2 \to 0$), these differential equations produce the spectral equations for the Hamiltonians $\mathcal{O}_2$ and $\mathcal{O}_3$ of the periodic Toda system, as we shall see shortly in section 3.5.

3.5 Splitting of the surface defect partition function

Finally we study the splitting behavior of the regular orbifold surface defect partition functions and its relation with integrable systems. A crucial remark is that the differential equations (3.4.16) are still valid even at the special locus of the Coulomb moduli, $\left\{ \frac{a_{\alpha\beta}}{\varepsilon_1} \in \mathbb{Z} \setminus \{0\} \right\}$. Thus the surface defect partition function can be used as a probe for the special locus, where the bulk partition function does not provide a simple picture for the correspondence. Meanwhile, on the integrable system side the special locus still gives the well-defined spectral problem of mutually commuting Hamiltonians, except that the spectra become degenerate at the 0-th order due to the specially tuned Bloch angles. In particular, the differential equations that define the spectral problem are still the same. Therefore the surface defect partition function is expected to detect such a splitting behavior of the corresponding integrable system. In particular, we will observe that, while the surface defect partition function still has the additional singularities in the limit $\varepsilon_2 \to 0$, it splits into parts in such a way that those extra singularities are resolved in each split part.

First note that for generic values of Coulomb moduli the surface defect partition function exhibits the typical asymptotic behavior in $\varepsilon_2 \to 0$,

$$\widetilde{\mathbf{\Psi}}_s(\mathbf{a}, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z}) \equiv \prod_\omega z_\omega^{-\frac{a_{s-1(\omega+1)}}{\varepsilon_1}} \mathbf{\Psi}_s^{\text{inst}}(\mathbf{a}, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z}) = e^{\frac{\widetilde{\mathcal{W}}(\mathbf{a},\varepsilon_1,\Lambda)}{\varepsilon_2}} \left(\psi_s(\mathbf{a}, \varepsilon_1, \Lambda, \mathbf{z}) + \mathcal{O}(\varepsilon_2)\right), \quad (3.5.1)$$

up to some prefactor. Therefore the differential equations (3.4.16) realize the Schrödinger equations for the periodic Toda system

$$\left[\frac{\varepsilon_1^2}{2} \sum_\omega \left(z_\omega \frac{\partial}{\partial z_\omega} \right)^2 + \Lambda^2 \sum_\omega z_\omega z_{\omega-1}^{-1} - E_2(\mathbf{a}, \varepsilon_1, \Lambda) \right] \psi_s(\mathbf{a}, \varepsilon_1, \Lambda, \mathbf{z}) = 0, \quad (3.5.2a)$$

$$\left[-\frac{\varepsilon_1^3}{3} \sum_\omega \left(z_\omega \frac{\partial}{\partial z_\omega} \right)^3 - \varepsilon_1 \Lambda^2 \sum_\omega z_\omega z_{\omega-1}^{-1} \left(z_\omega \frac{\partial}{\partial z_\omega} + z_{\omega-1} \frac{\partial}{\partial z_{\omega-1}} \right) - E_3(\mathbf{a}, \varepsilon_1, \Lambda) \right] \psi_s(\mathbf{a}, \varepsilon_1, \Lambda, \mathbf{z}) = 0, \quad (3.5.2b)$$

where

$$E_2(\mathbf{a},\varepsilon_1,\Lambda) = \frac{1}{2}\sum_{\omega} a_\omega^2 - \frac{1}{2}\varepsilon_1\Lambda\frac{\partial\widetilde{\mathcal{W}}(\mathbf{a},\varepsilon_1,\Lambda)}{\partial\Lambda}, \tag{3.5.3a}$$

$$E_3(\mathbf{a},\varepsilon_1,\Lambda) = \frac{1}{3}\sum_{\omega} a_\omega^3 - \frac{1}{2}\varepsilon_1^2\Lambda\frac{\partial\widetilde{\mathcal{W}}(\mathbf{a},\varepsilon_1,\Lambda)}{\partial\Lambda} - 2\varepsilon_1 \lim_{\varepsilon_2\to 0}\varepsilon_2\left\langle \sum_{\square\in K} c_{\hat{\square}}\right\rangle_s \tag{3.5.3b}$$

are nothing but the eigenvalues of the Hamiltonians (3.2.10) we have derived in the theory without the surface defect.[5][6] Note that even though the meaning of the expectation values in (3.2.10b) and (3.5.3b) are different, the final results agree in the limit $\varepsilon_2 \to 0$. Thus the surface defect partition function provides a constructive way to obtain both the eigenfunctions and the eigenvalues of the Hamiltonians of the corresponding integrable system.

Now we attempt an analogous construction at the special locus of the Coulomb moduli. The investigation reveals the splitting behavior of the surface defect partition functions.

3.5.1 $N = 2$

Let us first consider the simplest case, $N = 2$, in which there are two choices for the regular orbifold surface defect corresponding to the elements of $S_2 = \{\mathrm{id}, (01)\}$. The Schrödinger equation (3.5.2a) is precisely the Mathieu equation up to some change of variables. At the special locus $\{a_{01} = m\varepsilon_1 \mid m \in \mathbb{Z}\setminus\{0\}\}$, we observe that the surface defect partition functions split into two parts,

$$\widetilde{\mathbf{\Psi}}_{\mathrm{id}}(a_{01} = m\varepsilon_1, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z}) \pm \widetilde{\mathbf{\Psi}}_{(01)}(a_{01} = m\varepsilon_1, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z}) = e^{\frac{\widetilde{\mathcal{W}}_m^{\pm}(\varepsilon_1,\Lambda)}{\varepsilon_2}}\left(\psi_m^{\pm}(\varepsilon_1,\Lambda,\mathbf{z}) + \mathcal{O}(\varepsilon_2)\right). \tag{3.5.4}$$

[5]The relative factor N in the second term is due to the map $(z_1, z_2) \mapsto (z_1, z_2^N)$ in the orbifold construction of the regular orbifold surface defect, which shifts the equivariant parameter as $\varepsilon_2 \to N\varepsilon_2$.

[6]Although the eigenvalue (3.5.3b) seems to depend on the choice $s \in S_N$ through the expectation value $\langle\cdots\rangle_s$, it turns out not to. This is consistent with the computation in the absence of the surface defect, (3.2.10b).

Note that (3.4.16a) guarantees the wavefunctions $\psi_m^\pm(\varepsilon_1, \Lambda, \mathbf{z})$ to be the split eigenfunctions of the Schrödinger equation (3.5.2a) with the split energy spectrum

$$E_{2,m}^\pm = \frac{m^2\varepsilon_1^2}{8} - \frac{1}{4}\varepsilon_1\Lambda\frac{\partial\widetilde{\mathcal{W}}_m^\pm(\varepsilon_1,\Lambda)}{\partial\Lambda}. \tag{3.5.5}$$

We decoupled the irrelevant center of mass contribution and rescaled by a factor of 2 for convenience. The splitting behavior exactly accounts for the broken degeneracy due to the quantum tunneling effects on the integrable system side. Note that (3.5.4) is not obvious in the sense that the *split twisted superpotential* $\widetilde{\mathcal{W}}_m^\pm$ is non-divergent and is independent of the fractional gauge coupling $\mathbf{z}$. Also, it should be emphasized that the split twisted superpotential $\widetilde{\mathcal{W}}_m^\pm$ shows the series expansion in Λ^2, as opposed to the Λ^4-expansion of the generic twisted superpotential.

We have checked that the split eigenfunctions $\psi_m^\pm$ and the split eigenvalues $E_{2,m}^\pm$ in (3.5.4) and (3.5.5) precisely match with the well-known results of the half-periodic and the periodic solutions for the Mathieu equation, for various $m \in \mathbb{Z}\setminus\{0\}$ to some order of Λ. Therefore the splitting of the surface defect partition functions accounts for the splitting of the degenerate levels in the integrable system, and the correspondence between the gauge theory and the integrable system is recovered for the special locus of the Coulomb moduli space. We present some specific examples of the computation in Appendix A.1.

3.5.2 $N = 3$

In the case $N = 3$, the Hamiltonians are no longer Hermitian and the eigenvalues are not necessarily real, yet the perturbative series is well-defined including the degenerate case. Therefore we can still compare the spectra and the wavefunctions obtained from the gauge theory with the quantum mechanical computations. As mentioned in section 3.2, for the non-degenerate cases the known dictionary of the correspondence works as stated. Let us turn to the degenerate cases. There are three types of degeneracy possible, which are 2-fold, 3-fold, and 6-fold respectively. Without loss of generality, those degeneracies occur at the loci

$$\begin{aligned}
&\text{2-fold}: \quad \{a_{01} = m\varepsilon_1, a_{02} \text{ is generic} \mid m \in \mathbb{Z} \setminus \{0\}\} \\
&\text{3-fold}: \quad \{a_{01} = a_{02} = m\varepsilon_1 \mid m \in \mathbb{Z} \setminus \{0\}\} \\
&\text{6-fold}: \quad \{a_{01} = m\varepsilon_1, a_{02} = l\varepsilon_1 \mid m, l \in \mathbb{Z} \setminus \{0\}, m \neq l\}.
\end{aligned}$$

There are some subtle issues for the 2-fold and 6-fold degeneracies that obstruct our understanding of the splitting of the surface defect partition function, so we leave them to future work. Here we discuss the splitting of the surface defect partition function for the 3-fold degeneracy.

We have 6 different regular surface defects corresponding to the elements $s \in S_3$. Due to the residual symmetry, only 3 out of 6 are independent of each another in the case of $a_{12} = 0$. We form the split surface defect partition functions as

$$\begin{aligned}
\left[\widetilde{\boldsymbol{\Psi}}_{(012)}(\boldsymbol{a}, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z}) + \zeta \widetilde{\boldsymbol{\Psi}}_{(021)}(\boldsymbol{a}, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z}) + \zeta^2 \widetilde{\boldsymbol{\Psi}}_{\mathrm{id}}(\boldsymbol{a}, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z})\right]\Bigg|_{a_{01}=a_{02}=m\varepsilon_1} \\
= e^{\frac{\widetilde{\mathcal{W}}^{\zeta}_m(\varepsilon_1, \Lambda)}{\varepsilon_2}} \left(\psi^{\zeta}_m(\varepsilon_1, \Lambda, \mathbf{z}) + \mathcal{O}(\varepsilon_2)\right), \quad (3.5.6)
\end{aligned}$$

where ζ is any third root of unity, $\zeta^3 = 1$. Therefore each surface defect partition function splits into three parts, accounting for the level splitting of the 3-fold degeneracy. The wave-functions $\psi^{\zeta}_m(\varepsilon_1, \Lambda, \mathbf{z})$ are the common split eigenfunctions of $\mathcal{O}_2$ and $\mathcal{O}_3$ by (3.4.16) with the split eigenvalues

$$E^{\zeta}_{2,m} = \frac{m^2\varepsilon_1^2}{3} - \frac{1}{2}\varepsilon_1 \Lambda \frac{\partial \widetilde{\mathcal{W}}^{\zeta}_m(\varepsilon_1, \Lambda)}{\partial \Lambda}, \quad (3.5.7a)$$

$$E^{\zeta}_{3,m} = \frac{2m^3\varepsilon_1^3}{27} - \frac{\varepsilon_1^2}{2} \Lambda \frac{\partial \widetilde{\mathcal{W}}^{\zeta}_m(\varepsilon_1, \Lambda)}{\partial \Lambda} - 2\varepsilon_1 c^{\zeta}_1(\varepsilon_1, \Lambda), \quad (3.5.7b)$$

where we have decoupled the irrelevant center of mass contribution and defned the *split* *expectation value*

$$
\begin{aligned}
&c_1^\zeta(\varepsilon_1,\Lambda) \\
&= \lim_{\varepsilon_2\to 0} \varepsilon_2 \frac{\left\langle \sum_{\square\in K} c_{\hat\square} \right\rangle_{(012)} \widetilde\Psi_{(012)} + \zeta \left\langle \sum_{\square\in K} c_{\hat\square} \right\rangle_{(021)} \widetilde\Psi_{(021)} + \zeta^2 \left\langle \sum_{\square\in K} c_{\hat\square} \right\rangle_{\mathrm{id}} \widetilde\Psi_{\mathrm{id}}}{\widetilde\Psi_{(012)} + \zeta\widetilde\Psi_{(021)} + \zeta^2\widetilde\Psi_{\mathrm{id}}} \Bigg|_{a_{01}=a_{02}=m\varepsilon_1} .
\end{aligned}
\tag{3.5.8}
$$

It is not obvious that $c_1^\zeta(\varepsilon_1,\Lambda)$ neither diverges nor depends on the fractional coupling $\mathbf{z}$; in those cases the split eigenvalue (3.5.7b) would not be well-defined. The computation shows that $c_1^\zeta(\varepsilon_1,\Lambda)$ indeed behaves as desired. Note that the split twisted superpotential $\widetilde{\mathcal{W}}_m^\zeta$ and the split expectation value c_1^ζ have the series expansions in Λ^2, as opposed to the Λ^6-expansion of the generic twisted superpotential and expectation value. We present some examples of computation in Appendix A.2.

3.6 Discussion

In this chapter we have studied the Bethe/gauge correspondence for the special locus of the Coulomb moduli of the gauge theory, where the integrable system becomes degenerate in the non-interacting (free) limit. The analysis on the gauge theory with partial noncommutativity and partial Ω-deformation revealed the emergence of extra massless modes of matter multiplet at the speical locus, which makes the generic effective description without matter multiplet inapplicable. We used half-BPS surface defects, which are constructed out of orbifolds, to investigate the problem. The orbifold surface defect provided a constructive approach for the common eigenfunctions as well as the spectra of the Hamiltonians of the integrable system. Namely, the non-perturbative Dyson-Schwinger equations can be used to show that the surface defect partition function satisfies the Schrödinger-type equations, which indeed reduce to the spectral equations for the Hamiltonians in the Nekrasov-Shatashvili limit. We have seen that at the special locus of the Coulomb moduli the orbifold surface defect partition functions split into parts. Each split part assumes the desired asymptotic behavior in the Nekrasov-Shatashvili limit so that the degenerate perturbative series for the eigenfunctions and the eigenvalues could be presicely reproduced from the gauge theory perspective. We have presented some examples of the splitting.

There is a natural generalization of the investigation, i.e. adding various flavors to the theory. It is manifest from the instanton counting procedure that the theories with various types of flavor share the same denomenator in the effective twisted superpotential. Thus $U(N)$ gauge theories with flavors show the same divergent phenomena at the special locus of the Coulomb moduli space, which are expected to correspond to the splitting of degeneracies in the integrable system side. We may introduce the regular surface defect in those theories, with some proper assignment of the colorings for the flavors, and investigate the splitting behavior at the special locus. Some theories with fundamental hypermultiplets have non-Hermitian Hamiltonians even in the simplest case $N = 2$. It would be a nontrivial check to see how the splitting works for those theories.

Another interesting issue to be considered is the 5d uplift. While $d = 4$, $\mathcal{N} = 2$ gauge theories correspond to the non-relativistic integrable systems realized on the Seiberg-Witten geometry, the $d = 5$, $\mathcal{N} = 1$ gauge theories compactified on a circle correspond to their relativistic cousins [49]. The main difference is that the spectral equations become difference equations instead of differential equations. It was checked in [50] at some low instanton numbers that the codimension-two surface defect partition function satisfies those difference equations, for the example of $\mathcal{N} = 1^*$ theory. It would be nice to construct a rigorous analytic proof of those relations as done in this work for the four-dimensional case, using the 5d version of the qq-characters [10]. The *algebraic engineering* of codimension-two defect partition functions *à la* [51] can be useful for this study. The splitting of degeneracies would persist in those relativistic integrable systems, and the insertion of codimension-two defects is expected to detect this splitting through their partition functions.

The study of resurgence in integrable systems can have a connection with our story. For example, let us consider the Mathieu system which corresponds to the pure $\mathcal{N}=2$ $SU(2)$ gauge theory. The exact spectrum of the Mathieu system around a minimum of the Mathieu potential $V(x)=\Lambda^2\cos x$ exhibits the trans-series expansion, which can be computed by the exact quantization condition [52, 53]. In [54], it was argued that this exact quantization condition can be regarded as the Nekrasov-Shatashvili quantization condition in the strong coupling regime. The analysis showed that the prepotential at the strong coupling regime gets non-perturbative corrections (in the sense of quantum mechanics). Using the connection between the weak and the strong coupling regimes described in [55, 56], we may look for the gauge theoretical understanding of a nontrivial relation between the aformentioned non-perturbative effect in the strong coupling regime and the non-perturbative effect in the weak coupling regime, i.e. the splitting of the degenerate levels studied in this chapter. The topological string point of view on the exact quantization in [57, 58, 59] can also be related along these lines.

It would also be interesting to clarify the implication of the other eigenfunctions for the split eigenvalues. For example, it is well-known that for the Mathieu system the second solution for the split eigenvalue includes a $\log z$ term. Actually the second solution for $a_{01}=0$ (where the splitting does not occur) can be obtained by taking a derivative of the surface defect partition function with respect to the Coulomb moduli. When $a_{01}=m\varepsilon_1$ this procedure is not available since the surface defect partition function has discontinuity at the special locus. However, we may insert a 't Hooft line operator on top of the surface operator to get a ε_2-shift of the Coulomb moduli [60, 61, 62], which becomes infinitesimal in the Nekrasov-Shatashvili limit. Since the configuration is expected to have a well-defined effective twisted superpotential in the Nekrasov-Shatashvili limit, its partition function may produce the second solution with log. Unfortunately, the supersymmetric localization for such configuration of non-local observables is not available as of yet.

Chapter 4

BPZ equations and non-perturbative Dyson-Schwinger equations

4.1 Introduction

The paradigm of BPS/CFT correspondence [10] is to establish exact connections between the correlation functions of half-BPS (local and non-local) observables in four-dimensional $\mathcal{N} = 2$ supersymmetric field theories and the correlation functions of primary and descendant fields in two-dimensional CFTs. One of the remarkable manifestations of the BPS/CFT correspondence is the AGT correspondence [63], where the S^4 partition functions of $\mathcal{N} = 2$ gauge theories are identified with the correlation functions of Liouvlle/Toda primary fields.

The AGT correspondence emerges most naturally in the six-dimensional point of view. As briefly discussed in section 3.2, the $\mathcal{N} = 2$ theory of class $\mathcal{S}$ can be obtained by compactifying the six-dimensional $\mathcal{N} = (0, 2)$ theory on a Riemann surface $\mathcal{C}$ [33]. In turn, the S^4 partition function of the class $\mathcal{S}$ theory is identical to the partition function of $\mathcal{N} = (0, 2)$ on $S^4 \times \mathcal{C}$. Now by compactifying along S^4 instead of $\mathcal{C}$, we expect the same partition function is equivalent to the partition function of a CFT on $\mathcal{C}$. Consequently, we expect

$$\mathcal{Z}_{\text{class } \mathcal{S}}(S^4) = \mathcal{Z}_{\text{CFT}}(\mathcal{C}). \tag{4.1.1}$$

The non-conformal $\mathcal{N} = 2$ gauge theories can also be obtained by taking various limits of the gauge theory parameters. Hence the six-dimensional point of view provides a physical intuition on the AGT correspondence.

The identity (4.1.1) was firstly conjectured for superconformal $SU(2)$ quiver gauge theories and the Liouville theory [63], and was soon extended to more general relationships between quiver gauge theories with other gauge groups and Toda field theories [64]. As described above, the four-dimensional $\mathcal{N} = 2$ superconformal field theories considered in the AGT correspondence are obtained by the compactification of the six-dimensional $\mathcal{N} = (0, 2)$ superconformal theory on a punctured Riemann surface $\mathcal{C}$. When there is a weakly-coupled Lagrangian description of the theory, we can compute its partition function in the Ω-background [6]. It was discovered that the instanton part of the partition function $\mathcal{Z}^{\text{instanton}}$ can be identified with certain conformal blocks in the Liouville/Toda field theory, and the partition function on a (squashed) sphere S^4_b [65, 66], which is equal to the integral of the absolute value squared of the full partition function, can be identified with correlation functions in the Liouville/Toda field theory on the Riemann surface $\mathcal{C}$.

It is interesting to drop the genericity assumption for the parameters of the theory. In the two-dimensional conformal field theory, we can make one of the fields in the correlation function degenerate. Belavin, Polyakov, and Zamolodchikov showed that the correlation function in the Liouville field theory that involves a degenerate field satisfies a linear partial differential equation as a result of the decoupling of the null descendant field [67, 68]. The order of the differential equation is the level of the null field in the corresponding degenerate representation. In the case of Toda field theories, similar differential equations has been derived for certain four-point correlation functions in [69, 70]. On the other hand, the gauge field configurations of the corresponding four-dimensional $\mathcal{N} = 2$ superconformal quiver gauge theories are constrained, leading to a differential equation on the instanton partition function. To confirm the BPS/CFT correspondence, we should be able to identify the differential equations derived from both the conformal field theory side and the gauge theory side. This program has been investigated carefully in the Nekrasov-Shatashvili limit [28] of the Ω-background, which corresponds to the classical limit $c \to \infty$ of two-dimensional conformal field theories [71, 72, 73, 74, 75, 76, 77, 9, 78, 79]. However, previous methods become less powerful when we would like to go beyond such limits.

In this chapter, we shall follow the idea of [32] to provide a derivation of the differential equation using the non-perturbative Dyson-Schwinger equations, which result from the fact that the path integral of the instanton partition function is invariant with respect to the transformations changing topological sectors of the field space. We review the result of [32] and study the case of $U(2)$ superconformal linear quiver gauge theories with the next-to-simplest constraint in this chapter. The natural generalization to $U(N)$ superconformal linear quiver gauge theories will be discussed in a follow-up work. Similar method has also been applied to the study of Bethe/gauge correspondence [30, 29, 28] in [2].

The rest of the chapter is organized as follows. In Section 4.2, we recall some basic facts about two-dimensional Liouville field theory and review the derivation of BPZ equations on the degenerate correlation functions. In Section 4.3, we review the relevant details of the AGT correspondence and discuss the restrictions on gauge theory parameters. In Section 4.4, we study the superconformal gauge theory with gauge group $U(N)$. We show that the instanton partition function at the simplest nontrivial degenerate point in the parameter space is a (generalized) hypergeometric function. After working out this simple warm-up example, we consider the $U(2)$ superconformal linear quiver gauge theory in Section 4.5. We review the second order differential equation on the instanton partition function derived in [32] and derive the third order differential equation for the next-to-simplest case. We also identify the differential equations derived from both sides using the AGT dictionary. Finally, we conclude in Section 4.6 and discuss possible directions for future work. In the Appendix B we review some standard material on the (generalized) hypergeometric function. In the Appendix C we derive the partition function of the $U(1)$ factor using the non-perturbative Dyson-Schwinger equations.

4.2 Degenerate correlation functions in the Liouville field theory

In this section, we recall some basic facts about two-dimensional Liouville field theory and present the derivation of the BPZ equations on the degenerate correlation functions.

4.2.1 Degenerate fields in the Liouville field theory

The two-dimensional Liouville conformal field theory is defined by the action

$$S_{\text{Liouville}} = \int d^2\sigma\sqrt{g}\left(\frac{1}{4\pi}\partial_a\phi\partial^a\phi + \mu e^{2b\phi} + \frac{Q}{4\pi}R\phi\right), \tag{4.2.1}$$

where the background charge $Q = b + b^{-1}$, and R is the Ricci scalar of the Riemann surface. The symmetry algebra of the theory is two independent copies of the Virasoro algebra, with the central charge $c = 1+6Q^2$. In the following, we focus on the chiral part, which is spanned by generators L_n for $n \in \mathbb{Z}$ and the central charge c, satisfying

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}. \tag{4.2.2}$$

For the Virasoro algebra, a conformal primary field V_Δ with the conformal dimension Δ is defined to be

$$L_{n>0}V_\Delta = 0, \quad L_0V_\Delta = \Delta V_\Delta. \tag{4.2.3}$$

The descendant fields are obtained by taking the linear combinations of the basis vectors

$L_{-\vec{n}}V_\Delta = L_{-n_1}L_{-n_2}\cdots L_{-n_l}V_\Delta$, where $\vec{n} = \{1 \le n_1 \le n_2 \le \cdots \le n_l\}$. The conformal dimension of the basis vector $L_{-\vec{n}}V_\Delta$ is $\Delta + |\vec{n}|$, where the number $|\vec{n}| = \sum_{i=1}^{l} n_i$ is called the level of $L_{-\vec{n}}V_\alpha$.

A primary field V_Δ is called degenerate if it has a null descendant field $\tilde{V} = \sum_{\vec{n}} C_{\vec{n}} L_{-\vec{n}} V_\Delta \neq V_\Delta$, such that $L_n\tilde{V} = 0$ for $n > 0$. If the null field is at the level one, then

$$L_n\left(L_{-1}V_\Delta\right) = 0, \quad n > 0. \tag{4.2.4}$$

This is automatically true for $n \ge 2$, and for $n = 1$ we have

$$0 = L_1 L_{-1} V_\Delta = 2L_0 V_\Delta = 2\Delta V_\Delta. \tag{4.2.5}$$

Thus the field $V_\Delta = 1$ with zero conformal dimension. If the level-two descendant field $\tilde{V} = \left(C_{1,1}L_{-1}^2 + C_2 L_{-2}\right) V_\Delta$ is null, then $L_n\tilde{V} = 0$ for $n \ge 1$. The nontrivial constraints are

$$\begin{aligned} 0 &= L_1\tilde{V} = \left(\left(4\Delta + 2\right)C_{1,1} + 3C_2\right)L_{-1}V_\Delta, \\ 0 &= L_2\tilde{V} = \left(6\Delta C_{1,1} + \left(4\Delta + \frac{c}{2}\right)C_2\right)V_\Delta. \end{aligned} \tag{4.2.6}$$

Therefore, we have

$$\begin{vmatrix} 4\Delta + 2 & 3 \\ 6\Delta & 4\Delta + \frac{c}{2} \end{vmatrix} = 0, \tag{4.2.7}$$

which gives two solutions

$$\Delta_{(2,1)} = -\frac{1}{2} - \frac{3}{4}b^2, \quad \tilde{V}_{(2,1)} = \left(\frac{1}{b^2}L_{-1}^2 + L_{-2}\right)V_{\Delta_{(2,1)}}, \tag{4.2.8}$$

$$\Delta_{(1,2)} = -\frac{1}{2} - \frac{3}{4b^2}, \quad \tilde{V}_{(1,2)} = \left(b^2 L_{-1}^2 + L_{-2}\right)V_{\Delta_{(1,2)}}. \tag{4.2.9}$$

If the level-three descendant field $\tilde{V} = \left(C_{1,1,1}L_{-1}^3 + C_{1,2}L_{-1}L_{-2} + C_3 L_{-3}\right)V_\Delta$ is null, then

$L_n \tilde{V} = 0$ for $n \geq 1$. Since $L_3 = [L_2, L_1]$, we only need

$$
\begin{aligned}
0 &= L_1 \tilde{V} = \left(\left(2\Delta + 4\right) C_{1,2} + 4 C_3\right) L_{-2} V_\Delta + \left(\left(6\Delta + 6\right) C_{1,1,1} + 3 C_{1,2}\right) L_{-1}^2 V_\Delta, \\
0 &= L_2 \tilde{V} = \left(\left(6 + 18\Delta\right) C_{1,1,1} + \left(4\Delta + 9 + \frac{c}{2}\right) C_{1,2} + 5 C_3\right) L_{-1} V_\Delta. \qquad (4.2.10)
\end{aligned}
$$

Therefore, we have

$$
\begin{vmatrix} 0 & 2\Delta + 4 & 4 \\ 6\Delta + 6 & 3 & 0 \\ 6 + 18\Delta & 4\Delta + 9 + \frac{c}{2} & 5 \end{vmatrix} = 0, \qquad (4.2.11)
$$

which gives two solutions

$$
\begin{aligned}
\Delta_{(3,1)} &= -1 - 2b^2, \quad \tilde{V}_{(3,1)} = \left(\frac{1}{4b^2} L_{-1}^3 + L_{-1} L_{-2} + \left(b^2 - \frac{1}{2}\right) L_{-3}\right) V_{\Delta_{(3,1)}}, \qquad (4.2.12) \\
\Delta_{(1,3)} &= -1 - \frac{2}{b^2}, \quad \tilde{V}_{(1,3)} = \left(\frac{b^2}{4} L_{-1}^3 + L_{-1} L_{-2} + \left(\frac{1}{b^2} - \frac{1}{2}\right) L_{-3}\right) V_{\Delta_{(1,3)}}. \qquad (4.2.13)
\end{aligned}
$$

Generally, the conformal dimension of a degenerate field can be read from the Kac determinant formula, and is given by

$$
\Delta_{(m,n)} = \frac{Q^2 - \left(mb + nb^{-1}\right)^2}{4}, \quad m, n \in \mathbb{Z}^+, \qquad (4.2.14)
$$

with the null vector being at the level mn.

4.2.2 BPZ equations

Now we are ready to derive the BPZ equations on the $(r+3)$-point correlation function of the conformal primary fields, with one of the primary fields being degenerate. In order to relate a correlation function involving Virasoro generators acting on a primary field with a correlation function of purely primary fields, we use the conformal Ward identities, which state that inserting the holomorphic energy-momentum tensor in a correlation function of primary fields yields

$$\left\langle T(z)\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle=\sum_{i=-1}^{r+1}\left(\frac{1}{z-z_i}\frac{\partial}{\partial z_i}+\frac{\Delta_i}{(z-z_i)^2}\right)\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle. \tag{4.2.15}$$

The simplest nontrivial example is the second order BPZ equation. We assume that $\Delta_0=\Delta_{(2,1)}$. The decoupling of the null descendant field (4.2.8) implies that the $(r+3)$-point correlation function satisfies

$$\left[\frac{1}{b^2}\frac{\partial^2}{\partial z_0^2}+\sum_{i\neq 0}\left(\frac{1}{z_0-z_i}\frac{\partial}{\partial z_i}+\frac{\Delta_i}{(z_0-z_i)^2}\right)\right]\left\langle V_{\Delta_{(2,1)}}(z_0)\prod_{i\neq 0}V_{\Delta_i}(z_i)\right\rangle=0. \tag{4.2.16}$$

Similarly, the third order BPZ equation with $\Delta_0=\Delta_{(3,1)}$ can be derived from the decoupling of the null vector (4.2.12),

$$\begin{aligned}0 \;=\; &\left[\frac{1}{4b^2}\frac{\partial^3}{\partial z_0^3}+\frac{\partial}{\partial z_0}\sum_{i\neq 0}\left(\frac{1}{z_0-z_i}\frac{\partial}{\partial z_i}+\frac{\Delta_i}{(z_0-z_i)^2}\right)\right.\\ &\left.-\left(b^2-\frac{1}{2}\right)\sum_{i\neq 0}\left(\frac{1}{(z_0-z_i)^2}\frac{\partial}{\partial z_i}+\frac{2\Delta_i}{(z_0-z_i)^3}\right)\right]\left\langle V_{\Delta_{(3,1)}}(z_0)\prod_{i\neq 0}V_{\Delta_i}(z_i)\right\rangle.\end{aligned} \tag{4.2.17}$$

There are additional constraints on the correlation functions due to the global conformal symmetry. Using the holomorphy of the energy-momentum tensor at infinity, $T(z)=\mathcal{O}\left(z^{-4}\right)$ as $z\to\infty$, we deduce the global conformal Ward identities

$$\left[\sum_{i=-1}^{r+1}\frac{\partial}{\partial z_i}\right]\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle = 0, \tag{4.2.18}$$

$$\left[\sum_{i=-1}^{r+1}\left(z_i\frac{\partial}{\partial z_i}+\Delta_i\right)\right]\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle = 0, \tag{4.2.19}$$

$$\left[\sum_{i=-1}^{r+1}\left(z_i^2\frac{\partial}{\partial z_i}+2z_i\Delta_i\right)\right]\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle = 0. \tag{4.2.20}$$

For our purpose, it is convenient to get rid of all the ∂_{-1} and ∂_{r+1} terms using (4.2.18) and

(4.2.20),

$$\begin{aligned}
\frac{\partial}{\partial z_{-1}}\left\langle \prod_{i=-1}^{r+1} V_{\alpha_i}(z_i)\right\rangle &= \left[\sum_{i=0}^{r} \frac{z_i^2 - z_{r+1}^2}{z_{r+1}^2 - z_{-1}^2}\frac{\partial}{\partial z_i} + \sum_{i=-1}^{r+1} \frac{2z_i\Delta_i}{z_{r+1}^2 - z_{-1}^2}\right]\left\langle \prod_{i=-1}^{r+1} V_{\Delta_i}(z_i)\right\rangle, \\
\frac{\partial}{\partial z_{r+1}}\left\langle \prod_{i=-1}^{r+1} V_{\alpha_i}(z_i)\right\rangle &= \left[\sum_{i=0}^{r} \frac{z_i^2 - z_{-1}^2}{z_{-1}^2 - z_{r+1}^2}\frac{\partial}{\partial z_i} + \sum_{i=-1}^{r+1} \frac{2z_i\Delta_i}{z_{-1}^2 - z_{r+1}^2}\right]\left\langle \prod_{i=-1}^{r+1} V_{\Delta_i}(z_i)\right\rangle
\end{aligned} \tag{4.2.21}$$

We then fix $z_{-1} = \infty$ and $z_{r+1} = 0$, and the remaining global conformal Ward identity (4.2.19) gives

$$\left[\sum_{i=0}^{r} (\nabla_i + \Delta_i) - \Delta_{-1} + \Delta_{r+1}\right]\left\langle V_{\Delta_{-1}}(\infty)\prod_{i=0}^{r} V_{\Delta_i}(z_i) V_{\Delta_{r+1}}(0)\right\rangle = 0. \tag{4.2.22}$$

Let us decouple a prefactor from the correlation function

$$\left\langle V_{\Delta_{-1}}(\infty)V_{\Delta_{(m,n)}}(z_0)\prod_{i=1}^{r} V_{\Delta_i}(z_i)V_{\Delta_{r+1}}(0)\right\rangle = \left[\left(\prod_{i=0}^{r} z_i^{L_i}\right)\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{T_{ij}}\right]\chi_{r+3}^{(m,n)}(\boldsymbol{z}), \tag{4.2.23}$$

where $\chi_{r+3}^{(m,n)}(\boldsymbol{z})$ only depends on the ratios of z_i, $i = 0, \cdots, r$. The identity (4.2.22) is satisfied if

$$\sum_{i=0}^{r} (L_i + \Delta_i) - \Delta_{-1} + \Delta_{r+1} = 0. \tag{4.2.24}$$

Using

$$\begin{aligned}
\left[z_i\frac{\partial}{\partial z_i}, \left(\prod_{i=0}^{r} z_i^{L_i}\right)\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{T_{ij}}\right] &= \left(L_i + \sum_{j=i+1}^{r} T_{ij}\frac{z_j}{z_i - z_j} + \sum_{j=0}^{i-1} T_{ji}\frac{z_i}{z_i - z_j}\right) \\
&\quad \times \left(\prod_{i=0}^{r} z_i^{L_i}\right)\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{T_{ij}},
\end{aligned} \tag{4.2.25}$$

the second order BPZ equation (4.2.16) can be express in terms of $\chi_{r+3}^{(m,n)}(\boldsymbol{z})$ as

$$\begin{aligned} 0 \;=\; & \Bigg[\frac{1}{b^2}\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)^2 - \left(1+\frac{1}{b^2}\right)\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right) \\ & + \sum_{i=1}^{r}\frac{z_0}{z_0 - z_i}\left(\nabla_i + L_i + \sum_{j=i+1}^{r} T_{ij}\frac{z_j}{z_i - z_j} + \sum_{j=0}^{i-1} T_{ji}\frac{z_i}{z_i - z_j}\right) \\ & + \sum_{i=1}^{r}\frac{z_0^2\Delta_i}{(z_0 - z_i)^2} + \Delta_{r+1}\Bigg]\chi_{r+3}^{(2,1)}(\boldsymbol{z}), \end{aligned} \tag{4.2.26}$$

and the third order BPZ equation (4.2.17) becomes

$$\begin{aligned} 0 \;=\; & \Bigg[\frac{1}{4b^2}\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)^3 - \left(\frac{3}{4b^2}+1\right)\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)^2 \\ & + \left(\frac{1}{b^2} + b^2 + \frac{3}{2} + \sum_{i=1}^{r}\frac{z_0^2\Delta_i}{(z_0 - z_i)^2} + \Delta_{r+1}\right)\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right) \\ & + \sum_{i=1}^{r}\frac{z_0}{z_0 - z_i}\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)\left(\nabla_i + L_i + \sum_{j=i+1}^{r} T_{ij}\frac{z_j}{z_i - z_j} + \sum_{j=0}^{i-1} T_{ji}\frac{z_i}{z_i - z_j}\right) \\ & - \left(b^2 + \frac{1}{2}\right)\sum_{i=1}^{r}\frac{z_0\,(2z_0 - z_i)}{(z_0 - z_i)^2}\left(\nabla_i + L_i + \sum_{j=i+1}^{r} T_{ij}\frac{z_j}{z_i - z_j} + \sum_{j=0}^{i-1} T_{ji}\frac{z_i}{z_i - z_j}\right) \\ & - \left(2b^2 + 1\right)\left(\sum_{i=1}^{r}\frac{z_0^3\Delta_i}{(z_0 - z_i)^3} + \Delta_{r+1}\right)\Bigg]\chi_{r+3}^{(3,1)}(\boldsymbol{z}), \end{aligned} \tag{4.2.27}$$

where we denote

$$\nabla_i = z_i\frac{\partial}{\partial z_i}. \tag{4.2.28}$$

We should determine L_i and T_{ij} when we identify the BPZ equations with the differential equations derived in the corresponding gauge theories.

4.3 AGT corresponence with surface defects

In this section, we review some results regarding the AGT correspondence of four-dimensional $\mathcal{N} = 2$ quiver gauge theories in the Ω-background [63]. We also discuss imposing constraints on the gauge theory parameters, which in special cases generate surface defects in four-dimensional theory as discussed in section 2.3. The insertion of surface defects correspond to the insertion of degenerate fields in the CFT side [60].

4.3.1 Dictionary of AGT correspondence

It is useful to summarize the dictionary of AGT correspondence in order to make this chapter self-contained. The main statement of the AGT correspondence is an identification between the $(r+3)$-point correlation function in the Liouville field theory with the partition function of superconformal quiver gauge theory with gauge group $SU(2)^r$.

Let us decompose the $U(2)$ gauge group into the $U(1)$ part and the $SU(2)$ part,

$$\bar{a}_i = \frac{1}{2}\sum_{\alpha=1}^{2} a_{i,\alpha}, \quad a'_{i,\alpha} = a_{i,\alpha} - \bar{a}_i. \tag{4.3.1}$$

From the point of view of an $SU(2)$ linear quiver gauge theory, the masses of the anti-fundamental, fundamental and bifundamental hypermultiplets are given by

$$\bar{\mu}_\alpha = a_{0,\alpha} - \bar{a}_1, \quad \mu_\alpha = a_{r+1,\alpha} - \bar{a}_r, \quad , \mu_{i,i+1} = \bar{a}_{i+1} - \bar{a}_i, \quad i = 1, \cdots, r-1. \tag{4.3.2}$$

If we identify the Liouville parameter b with the Ω-deformation parameters $\varepsilon_1, \varepsilon_2$ as

$$b^2 = \frac{\varepsilon_1}{\varepsilon_2}, \tag{4.3.3}$$

and relate the conformal dimensions Δ_i with the Coulomb parameters $\boldsymbol{a}$ in the following way,

$$\begin{aligned} \Delta_{-1} &= \frac{\varepsilon^2 - (a_{0,1} - a_{0,2})^2}{4\varepsilon_1\varepsilon_2}, \quad \Delta_{r+1} = \frac{\varepsilon^2 - (a_{r+1,1} - a_{r+1,2})^2}{4\varepsilon_1\varepsilon_2}, \\ \Delta_i &= \frac{(\bar{a}_{i+1} - \bar{a}_i)(\bar{a}_i - \bar{a}_{i+1} + \varepsilon)}{4\varepsilon_1\varepsilon_2}, \quad i = 0, \cdots, r, \end{aligned} \tag{4.3.4}$$

then we have

$$\begin{aligned}
&\left\langle V_{\Delta_{-1}}(\infty)\prod_{i=0}^{r}V_{\Delta_i}(z_i)V_{\Delta_{r+1}}(0)\right\rangle \\
= \;& f\left(\Delta_{-1},\cdots,\Delta_{r+1}\right)\left|\left(z_0^{\Delta_{-1}-\Delta_0-\frac{\varepsilon^2}{4\varepsilon_1\varepsilon_2}}\right)\left(\prod_{i=1}^{r-1}z_i^{-\Delta_i}\right)\left(z_r^{\frac{\varepsilon^2}{4\varepsilon_1\varepsilon_2}-\Delta_r-\Delta_{r+1}}\right)\right|^2 \\
&\times\int\prod_{i=1}^{r}[da_i']\left|\frac{\mathcal{Z}\left(\boldsymbol{a};\boldsymbol{z};\varepsilon_1,\varepsilon_2\right)}{\mathcal{Z}^{U(1)}\left(\boldsymbol{a};\boldsymbol{z};\varepsilon_1,\varepsilon_2\right)}\right|^2, \qquad (4.3.5)
\end{aligned}$$

where the prefactor $f\left(\Delta_{-1},\cdots,\Delta_{r+1}\right)$ is independent of $\boldsymbol{z}$, and $\mathcal{Z}^{U(1)}\left(\boldsymbol{a};\boldsymbol{z};\varepsilon_1,\varepsilon_2\right)$ is the $U(1)$ part of the partition function.

4.3.2 Degenerate partition function

In section 2.1 we assumed that the Coulomb moduli $\boldsymbol{a}$ are generic. Then the instanton partition function (2.1.31) contains an infinite sum over collections of Young diagrams $\boldsymbol{Y}$. However, we can tune some of the parameters to special values so as to force some of $Y^{(i,\alpha)}$ to have a constrained shape [32]. For example, we can adjust

$$a_{0,\alpha}=\begin{cases}a_{1,1}+(m-1)\varepsilon_1+(n-1)\varepsilon_2, & \alpha=1,\\ a_{1,\alpha}, & \alpha\neq 1,\end{cases} \qquad (4.3.6)$$

where $m,n\in\mathbb{Z}^+$. Since the measure of the instanton partition function contains a factor

$$\prod_{\square=(u,v)\in Y^{(1,\alpha)}}\left(a_{0,\alpha}-a_{1,\alpha}-\varepsilon_1\left(u-1\right)-\varepsilon_2\left(v-1\right)\right), \qquad (4.3.7)$$

the contribution to the instanton partition function vanishes unless the Young diagrams $Y^{(1,\alpha)}=\emptyset$ for $\alpha\neq 1$, and $\square=(m,n)\notin Y^{(1,1)}$. Hence the number of Young diagrams we need to sum over reduces drastically. In particular, when $m>1$ and $n=1$, the Young diagram $Y^{(1,1)}$ can have at most $m-1$ rows. According to the AGT dictionary, (4.3.6) corresponds to a degenerate field with the conformal dimension $\Delta_{(m,n)}$.

4.4 Superconformal theory with gauge group $U(N)$

In this section, we take a simple example to illustrate the basic idea of deriving the differential equation on the instanton partition function at a special point in the parameter space. We consider the $U(N)$ gauge theory with N fundamental hypermultiplets and N anti-fundamental hypermultiplets for general $N \geq 2$. At the degenerate point of parameter space,

$$a_{0,\alpha} = \begin{cases} a_{1,1} + \varepsilon_1, & \alpha = 1, \\ a_{1,\alpha}, & \alpha \neq 1, \end{cases} \tag{4.4.1}$$

the instanton partition function is only summed over the Young diagram $Y^{(1,1)}$ which has only one row,

$$Y^{(1)} = \left(\boxed{(1,1)}\boxed{(1,2)} \; \cdots \; \boxed{(1,k_1)}, \emptyset, \cdots, \emptyset \right). \tag{4.4.2}$$

Therefore, we can label the Young diagram $Y^{(1,1)}$ by the instanton charge k_1.

In this case, we face no obstruction in proving directly that the instanton partition function is a (generalized) hypergeometric function from the instanton partition function. The instanton partition function is

$$\begin{aligned} \mathcal{Z}^{\text{instanton}} &= \sum_{k_1=0}^{\infty} \frac{q_1^{k_1}}{k_1!} \frac{\prod_{\alpha=1}^{N} \left(\frac{a_{0,1}-a_{2,\alpha}+\varepsilon_2}{\varepsilon_2}\right)^{\overline{k_1}}}{\prod_{\alpha=2}^{N} \left(\frac{a_{0,1}-a_{0,\alpha}+\varepsilon_2}{\varepsilon_2}\right)^{\overline{k_1}}} && (4.4.3) \\ &= {}_N F_{N-1} \left(\left(\frac{a_{0,1} - a_{2,\alpha} + \varepsilon_2}{\varepsilon_2} \right)_{\alpha=1}^{N} ; \left(\frac{a_{0,1} - a_{0,\alpha} + \varepsilon_2}{\varepsilon_2} \right)_{\alpha=2}^{N} ; q_1 \right), && (4.4.4) \end{aligned}$$

which is a (generalized) hypergeometric function, and satisfies the (generalized) hypergeometric differential equation (see the Appendix B for details),

$$
\begin{aligned}
0 \quad = \quad & \left[q_1 \prod_{\alpha=1}^{N} \left(q_1 \frac{\partial}{\partial q_1} + \frac{a_{0,1} - a_{2,\alpha} + \varepsilon_2}{\varepsilon_2} \right) \right. \\
& \left. - q_1 \frac{\partial}{\partial q_1} \prod_{\alpha=2}^{N} \left(q_1 \frac{\partial}{\partial q_1} + \frac{a_{0,1} - a_{0,\alpha} + \varepsilon_2}{\varepsilon_2} - 1 \right) \right] \mathcal{Z}^{\text{instanton}}. \qquad (4.4.5)
\end{aligned}
$$

Now we would like to derive the above differential equation using the non-perturbative Dyson-Schwinger equations. There is only one fundamental qq-character in this theory,

$$
\mathcal{X}_1(x) = \mathcal{Y}_1(x+\varepsilon) + q_1 \frac{\mathcal{Y}_0(x)\mathcal{Y}_2(x+\varepsilon)}{\mathcal{Y}_1(x)}. \qquad (4.4.6)
$$

At the degenerate point (4.4.1), the value $\mathcal{Y}_1(x)[\boldsymbol{Y}]$ simplifies

$$
\begin{aligned}
\mathcal{Y}_1(x)[\boldsymbol{Y}] \quad &= \quad \left[\prod_{\alpha=1}^{N} (x - a_{1,\alpha}) \right] \left[\prod_{v=1}^{k_1} \frac{(x - a_{1,1} - \varepsilon_2 (v-1) - \varepsilon_1)(x - a_{1,1} - \varepsilon_2 v)}{(x - a_{1,1} - \varepsilon_2 (v-1))(x - a_{1,1} - \varepsilon_2 v - \varepsilon_1)} \right] \\
&= \quad \left[\prod_{\alpha=1}^{N} (x - a_{1,\alpha}) \right] \left[\frac{(x - a_{1,1} - \varepsilon_1)(x - a_{1,1} - \varepsilon_2 k_1)}{(x - a_{1,1})(x - a_{1,1} - \varepsilon_2 k_1 - \varepsilon_1)} \right] \\
&= \quad \left[(x - a_{1,1} - \varepsilon_1) \prod_{\alpha=2}^{N} (x - a_{1,\alpha}) \right] \left[\frac{x - a_{1,1} - \varepsilon_2 k_1}{x - a_{1,1} - \varepsilon_2 k_1 - \varepsilon_1} \right] \\
&= \quad \mathcal{Y}_0(x) \frac{x - a_{0,1} + \varepsilon_1 - \varepsilon_2 k_1}{x - a_{0,1} - \varepsilon_2 k_1}. \qquad (4.4.7)
\end{aligned}
$$

Accordingly, $\mathcal{X}_1(x)[\boldsymbol{Y}]$ becomes

$$
\begin{aligned}
\mathcal{X}_1(x)[\boldsymbol{Y}] \quad = \quad & \mathcal{Y}_0(x+\varepsilon) \left(1 + \frac{\varepsilon_1}{x + \varepsilon - a_{0,1} - \varepsilon_2 k_1} \right) \\
& + q_1 \mathcal{Y}_2(x+\varepsilon) \left(1 - \frac{\varepsilon_1}{x - a_{0,1} + \varepsilon_1 - \varepsilon_2 k_1} \right). \qquad (4.4.8)
\end{aligned}
$$

The x^{-1} coefficient of the large x expansion of $\mathcal{X}_1(x)[\boldsymbol{Y}]$ is given by

$$
\mathcal{X}_1^{(-1)}[\boldsymbol{Y}] = \varepsilon_1 \mathcal{Y}_0(a_{0,1} + \varepsilon_2 k_1) - q_1 \varepsilon_1 \mathcal{Y}_2(a_{0,1} + \varepsilon_2 k_1 + \varepsilon_2). \qquad (4.4.9)
$$

Using the relation

$$\begin{aligned}\langle k_1^p \rangle &= \mathcal{Z}^{\text{instanton}}\left(\boldsymbol{a};\vec{Y};\varepsilon_1,\varepsilon_2\right)^{-1}\sum_{\vec{Y}=\left\{Y^{(\alpha)}\right\}} q_1^{k_1}\mathcal{Z}^{\text{instanton}}\left(\boldsymbol{a};\vec{Y};\varepsilon_1,\varepsilon_2\right)k_1^p \\ &= \mathcal{Z}^{\text{instanton}}\left(\boldsymbol{a};\vec{Y};\varepsilon_1,\varepsilon_2\right)^{-1}\left(q_1\frac{\partial}{\partial q_1}\right)^p \mathcal{Z}^{\text{instanton}}\left(\boldsymbol{a};q_1;\varepsilon_1,\varepsilon_2\right), \end{aligned} \tag{4.4.10}$$

the equation $\langle \mathcal{X}_1^{(-1)} \rangle = 0$ becomes

$$0 = \left[\prod_{\alpha=1}^{N}\left(a_{0,1}+\varepsilon_2 q_1\frac{\partial}{\partial q_1}-a_{0,\alpha}\right) - q_1\prod_{\alpha=1}^{N}\left(a_{0,1}+\varepsilon_2 q_1\frac{\partial}{\partial q_1}+\varepsilon_2-a_{2,\alpha}\right)\right]\mathcal{Z}^{\text{instanton}}, \tag{4.4.11}$$

which coincides with the differential equation (4.4.5).

4.5 Superconformal linear quiver gauge theories

In this section, we would like to derive the differential equation on the instanton partition function of the superconformal linear quiver gauge theory using the non-perturbative Dyson-Schwinger equations. Recall that the fundamental qq-characters of the A_r-linear quiver gauge theory are given as (2.2.7). From the non-perturbative Dyson-Schwinger equation for these qq-characters,

$$[x^{-n}]\left\langle \mathcal{X}_l(x)\right\rangle = 0, \quad n \geq 1, \;\; l = 1, \cdots, r, \tag{4.5.1}$$

we derive the differential equations that the degenerate partition functions satisfy.

4.5.1 Large x expansion of fundamental $\mathcal{Y}$-observables

The first step is to compute the large x expansion of the $\mathcal{Y}$-observables,

$$\begin{aligned}\mathcal{Y}_i(x)[\boldsymbol{Y}] &= x^N \exp\left[\sum_{\alpha=1}^{N} \log\left(1-\frac{a_{i,\alpha}}{x}\right) + \sum_{\alpha=1}^{N}\sum_{\square\in Y^{(i,\alpha)}} \frac{\left(1-\frac{\widehat{c_\square}+\varepsilon_1}{x}\right)\left(1-\frac{\widehat{c_\square}+\varepsilon_2}{x}\right)}{\left(1-\frac{\widehat{c_\square}}{x}\right)\left(1-\frac{\widehat{c_\square}+\varepsilon}{x}\right)}\right] \\ &= x^N \exp\left(-\sum_{n=1}^{\infty}\frac{C_{i,n}[\boldsymbol{Y}]}{nx^n}\right),\end{aligned} \tag{4.5.2}$$

where

$$\begin{aligned}C_{i,n}[\boldsymbol{Y}] &= \mathrm{Tr}\Phi_i^n(0)[\boldsymbol{Y}] \\ &= \sum_{\alpha=1}^{N}\left\{a_{i,\alpha}^n + \sum_{\square\in Y^{(i,\alpha)}}\left[(\widehat{c_\square}+\varepsilon_1)^n + (\widehat{c_\square}+\varepsilon_2)^n - \widehat{c_\square}^{\,n} - (\widehat{c_\square}+\varepsilon)^n\right]\right\}.\end{aligned} \tag{4.5.3}$$

In particular, we have

$$\begin{aligned}C_{i,1}[\boldsymbol{Y}] &= \sum_{\alpha=1}^{N} a_{i,\alpha}, \\ C_{i,2}[\boldsymbol{Y}] &= \left(\sum_{\alpha=1}^{N} a_{i,\alpha}^2\right) - 2\varepsilon_1\varepsilon_2 k_i,\end{aligned} \tag{4.5.4}$$

We also have the similar expression for $\mathcal{Y}_i(x+\varepsilon)[\boldsymbol{Y}]$,

$$\begin{aligned}\mathcal{Y}_i(x+\varepsilon)[\boldsymbol{Y}] &= \prod_{\alpha=1}^{N}\left(x-(a_{i,\alpha}-\varepsilon)\right)\prod_{\square\in Y^{(i,\alpha)}}\frac{(x-\widehat{c_\square}+\varepsilon_1)(x-\widehat{c_\square}+\varepsilon_2)}{(x-\widehat{c_\square})(x-\widehat{c_\square}+\varepsilon)} \\ &= x^N\exp\left(-\sum_{n=1}^{\infty}\frac{C'_{i,n}[\boldsymbol{Y}]}{nx^n}\right),\end{aligned} \tag{4.5.5}$$

where

$$C'_{i,n}[\boldsymbol{Y}] = \sum_{\alpha=1}^{N}\left\{(a_{i,\alpha}-\varepsilon)^n + \sum_{\square\in Y^{(i,\alpha)}}\left[(\widehat{c_\square}-\varepsilon_1)^n + (\widehat{c_\square}-\varepsilon_2)^n - \widehat{c_\square}^{\,n} - (\widehat{c_\square}-\varepsilon)^n\right]\right\}. \tag{4.5.6}$$

Therefore, we obtain the large x expansion of $\Xi_i(x)[\boldsymbol{Y}]$,

$$\Xi_i(x)[\boldsymbol{Y}] = \frac{\mathcal{Y}_{i+1}(x+\varepsilon)[\boldsymbol{Y}]}{\mathcal{Y}_i(x)[\boldsymbol{Y}]} = 1 + \sum_{n=1}^{\infty} \frac{\zeta_{i,n}[\boldsymbol{Y}]}{x^n}. \tag{4.5.7}$$

The first two terms of $\zeta_{i,n}$ are given explicitly as

$$\begin{aligned} \zeta_{i,1} &= \mathcal{A}_i^{(1)}, \\ \zeta_{i,2} &= \mathcal{A}_i^{(2)} - \varepsilon_1\varepsilon_2 \left(k_i - k_{i+1}\right), \end{aligned} \tag{4.5.8}$$

where

$$\begin{aligned} \mathcal{A}_i^{(1)} &= \sum_{\alpha=1}^{N} \left(a_{i,\alpha} - a_{i+1,\alpha} + \varepsilon\right), \\ \mathcal{A}_i^{(2)} &= \frac{1}{2}\sum_{\alpha=1}^{N} \left[a_{i,\alpha}^2 - \left(a_{i+1,\alpha} - \varepsilon\right)^2\right] + \frac{1}{2}\left(\mathcal{A}_i^{(1)}\right)^2 \end{aligned} \tag{4.5.9}$$

4.5.2 Generating function of the fundamental qq-characters

After expanding the $\mathcal{Y}$-observables, we would like to calculate the large x expansion of the qq-characters. In order to deal with all of the fundamental qq-characters at the same time, we introduce the generating function

$$\begin{aligned} \mathcal{G}_r(x;t) &= \mathcal{Y}_0(x)^{-1}\Delta_r^{-1}\sum_{\ell=0}^{r+1} z_0 z_1 \cdots z_{\ell-1} t^{\ell} \mathcal{X}_{\ell}\left(x - \varepsilon(1-\ell)\right) \\ &= \Delta_r^{-1} \sum_{I\subset[0,r]} \left[\left(\prod_{i\in I} t z_i\right) \prod_{i\in I} \Xi_i\left(x + \varepsilon h_I(i)\right)\right], \end{aligned} \tag{4.5.10}$$

where

$$\Delta_r = \sum_{I\subset[0,r]} \left(\prod_{i\in I} t z_i\right) = \prod_{i=0}^{r} \left(1 + t z_i\right). \tag{4.5.11}$$

In the following, we would like to sum over $I \subset [0,r]$ to obtain the large x expansion of $\mathcal{G}_r(x;t)$,

$$\mathcal{G}_r(x;t) = \sum_{n=0}^{\infty} \frac{\mathcal{G}_r^{(-n)}(t)}{x^n}. \tag{4.5.12}$$

Let us define

$$u_i = \frac{tz_i}{1+tz_i}. \tag{4.5.13}$$

When $r=0$, $\mathcal{G}_0(t)$ is given by a sum over $I=\emptyset$ and $I=\{0\}$,

$$\mathcal{G}_0(x;t) = \frac{1}{1+tz_0} + \frac{tz_0}{1+tz_0}\Xi_0(x) = 1 + \sum_{n=1}^{\infty} \frac{u_0 \zeta_{0,n}}{x^n}. \tag{4.5.14}$$

Hence,

$$\mathcal{G}_0^{(0)}(t) = 1, \quad \mathcal{G}_0^{(-n)}(t) = u_0 \zeta_{0,n}, \quad n \in \mathbb{Z}^+. \tag{4.5.15}$$

For general $r \geq 1$, we can compute the value of the generating function (4.5.10) using the recurrence relation between $\mathcal{G}_r(x;t)$ and $\mathcal{G}_{r-1}(x;t)$. We divide the sum over $I \subset [0,r]$ into two classes: $r \notin I$ and $r \in I$,

$$\begin{aligned}
\mathcal{G}_r(x;t) &= \frac{1}{1+tz_r}\Delta_{r-1}^{-1} \sum_{I' \subset [0,r-1]} \left[\left(\prod_{i \in I'} tz_i \right) \prod_{i \in I'} \Xi_i \left(x + \varepsilon h_{I'}(i) \right) \right] \\
&\quad + \frac{tz_r}{1+tz_r}\Delta_{r-1}^{-1} \sum_{I' \subset [0,r-1]} \left[\left(\prod_{i \in I'} tz_i \right) \Xi_r \left(x + \varepsilon |I'| \right) \prod_{i \in I'} \Xi_i \left(x + \varepsilon h_{I'}(i) \right) \right] \\
&= \mathcal{G}_{r-1}(x;t) + u_r \Delta_{r-1}^{-1} \left(\Xi_r \left(x + \varepsilon t \frac{\partial}{\partial t} \right) - 1 \right) \left(\Delta_{r-1} \mathcal{G}_{r-1}(x;t) \right) \\
&= \mathcal{G}_{r-1}(x;t) + u_r \Delta_{r-1}^{-1} \sum_{n=1}^{\infty} \frac{\zeta_{r,n}}{\left(x + \varepsilon t \frac{\partial}{\partial t} \right)^n} \left(\Delta_{r-1} \mathcal{G}_{r-1}(x;t) \right) \\
&= \mathcal{G}_0(x;t) + \sum_{j=1}^{r} u_j \Delta_{j-1}^{-1} \sum_{n=1}^{\infty} \frac{\zeta_{j,n}}{\left(x + \varepsilon t \frac{\partial}{\partial t} \right)^n} \left(\Delta_{j-1} \mathcal{G}_{j-1}(x;t) \right).
\end{aligned} \tag{4.5.16}$$

Hence, we obtain the recursive relations,

$$
\begin{aligned}
\mathcal{G}_r^{(0)}(t) &= \mathcal{G}_0^{(0)}(t) = 1, && (4.5.17)\\
\mathcal{G}_r^{(-1)}(t) &= \sum_{j=0}^{r} u_j \zeta_{j,1}, && (4.5.18)\\
\mathcal{G}_r^{(-2)}(t) &= \sum_{j=0}^{r} u_j \zeta_{j,2} + \sum_{j=1}^{r} u_j \zeta_{j,1} \mathcal{G}_{j-1}^{(-1)}(t) \\
&\quad - \varepsilon \sum_{j=1}^{r} u_j \zeta_{j,1} \Delta_{j-1}^{-1} t \frac{\partial}{\partial t} \Delta_{j-1}, && (4.5.19)\\
\mathcal{G}_r^{(-3)}(t) &= \sum_{j=0}^{r} u_j \zeta_{j,3} + \sum_{j=1}^{r} u_j \left[\zeta_{j,1} \mathcal{G}_{j-1}^{(-2)}(t) + \zeta_{j,2} \mathcal{G}_{j-1}^{(-1)}(t) \right] \\
&\quad - \varepsilon \sum_{j=1}^{r} u_j \Delta_{j-1}^{-1} \left[\zeta_{j,1} t \frac{\partial}{\partial t} \left(\Delta_{j-1} \mathcal{G}_{j-1}^{(-1)}(t) \right) + 2 \zeta_{j,2} t \frac{\partial}{\partial t} \Delta_{j-1} \right] \\
&\quad + \varepsilon^2 \sum_{j=1}^{r} u_j \zeta_{j,1} \Delta_{j-1}^{-1} \left(t \frac{\partial}{\partial t} \right)^2 \Delta_{j-1}, && (4.5.20)\\
\mathcal{G}_r^{(-4)}(t) &= \sum_{j=0}^{r} u_j \zeta_{j,4} + \sum_{j=1}^{r} u_j \left[\zeta_{j,1} \mathcal{G}_{j-1}^{(-3)}(t) + \zeta_{j,2} \mathcal{G}_{j-1}^{(-2)}(t) + \zeta_{j,3} \mathcal{G}_{j-1}^{(-1)}(t) \right] \\
&\quad - \varepsilon \sum_{j=1}^{r} u_j \Delta_{j-1}^{-1} \left[\zeta_{j,1} t \frac{\partial}{\partial t} \left(\Delta_{j-1} \mathcal{G}_{j-1}^{(-2)}(t) \right) + 2 \zeta_{j,2} t \frac{\partial}{\partial t} \left(\Delta_{j-1} \mathcal{G}_{j-1}^{(-1)}(t) \right) + 3 \zeta_{j,3} t \frac{\partial}{\partial t} \Delta_{j-1} \right] \\
&\quad + \varepsilon^2 \sum_{j=1}^{r} u_j \Delta_{j-1}^{-1} \left[\zeta_{j,1} \left(t \frac{\partial}{\partial t} \right)^2 \left(\Delta_{j-1} \mathcal{G}_{j-1}^{(-1)}(t) \right) + 3 \zeta_{j,2} \left(t \frac{\partial}{\partial t} \right)^2 \Delta_{j-1} \right] \\
&\quad - \varepsilon^3 \sum_{j=1}^{r} u_j \zeta_{j,1} \Delta_{j-1}^{-1} \left(t \frac{\partial}{\partial t} \right)^3 \Delta_{j-1}. && (4.5.21)
\end{aligned}
$$

We further introduce the notation

$$
U_r \left[s_1, s_2, \cdots, s_\ell \right] \equiv \sum_{0 \le i_1 < \cdots < i_\ell \le r} \prod_{n=1}^{\ell} \left(u_{i_n} \zeta_{i_n, s_n} \right), \tag{4.5.22}
$$

where $[s_1, \cdots, s_\ell]$ is a sequence of non-negative integers, and we adopt the convention that

$\zeta_{i,0} = 1$. We have the following useful relations from the definition

$$\begin{aligned}
t\frac{\partial}{\partial t}\Delta_r &= \Delta_r U_r[0], && (4.5.23)\\
t\frac{\partial}{\partial t}\left(\Delta_r U_r[s_1, \cdots, s_\ell]\right) &= \Delta_r \left(\ell U_r[s_1, \cdots, s_\ell] + U_r^{\oplus}[s_1, \cdots, s_\ell]\right), && (4.5.24)\\
\sum_{j=1}^{r} u_j \zeta_{j,m} U_{j-1}[s_1, \cdots, s_\ell] &= U_r\left[s_1, s_2, \cdots, s_\ell, m\right], && (4.5.25)
\end{aligned}$$

where

$$U_r^{\oplus}[s_1, \cdots, s_\ell] \equiv U_r[0, s_1, \cdots, s_\ell] + U_r[s_1, 0, \cdots, s_\ell] + \cdots + U_r[s_1, \cdots, s_\ell, 0]. \qquad (4.5.26)$$

We also have

$$\begin{aligned}
\Delta_r^{-1} t\frac{\partial}{\partial t}\Delta_r &= U_r[0], && (4.5.27)\\
\Delta_r^{-1}\left(t\frac{\partial}{\partial t}\right)^2 \Delta_r &= U_r[0] + 2U_r[0, 0], && (4.5.28)\\
\Delta_r^{-1}\left(t\frac{\partial}{\partial t}\right)^3 \Delta_r &= U_r[0] + 6U_r[0, 0] + 6U_r[0, 0, 0], && (4.5.29)
\end{aligned}$$

After solving the recurrence relations, the first few terms of $\mathcal{G}_r^{(-n)}(t)$ can be written as

$$
\begin{aligned}
\mathcal{G}_r^{(0)}(t) &= 1, && (4.5.30)\\
\mathcal{G}_r^{(-1)}(t) &= U_r[1], && (4.5.31)\\
\mathcal{G}_r^{(-2)}(t) &= U_r[2] + U_r[1,1] - \varepsilon U_r[0,1], && (4.5.32)\\
\mathcal{G}_r^{(-3)}(t) &= U_r[3] + U_r[2,1] + U_r[1,2] - \varepsilon\left(U_r[1,1] + 2U_r[0,2]\right) + \varepsilon^2 U_r[0,1] \\
&\quad + U_r[1,1,1] - \varepsilon\left(2U_r[0,1,1] + U_r[1,0,1]\right) + 2\varepsilon^2 U_r[0,0,1], && (4.5.33)\\
\mathcal{G}_r^{(-4)}(t) &= U_r[4] + U_r[1,3] + U_r[2,2] + U_r[3,1] \\
&\quad - \varepsilon\left(U_r[2,1] + 2U_r[1,2] + 3U_r[0,3]\right) + \varepsilon^2\left(U_r[1,1] + U_r[0,2]\right) - \varepsilon^3 U_r[0,1] \\
&\quad + U_r[2,1,1] + U_r[1,2,1] + U_r[1,1,2] \\
&\quad - \varepsilon\left(3U_r[1,1,1] + 3U_r[0,2,1] + 3U_r[0,1,2] + 2U_r[1,0,2] + U_r[2,0,1]\right) \\
&\quad + \varepsilon^2\left(6U_r[0,1,1] + 3U_r[1,0,1] + 6U_r[0,0,2]\right) - 6\varepsilon^3 U_r[0,0,1] \\
&\quad + U_r[1,1,1,1] - \varepsilon\left(3U_r[0,1,1,1] + 2U_r[1,0,1,1] + U_r[1,1,0,1]\right) \\
&\quad + \varepsilon^2\left(6U_r[0,0,1,1] + 3U_r[0,1,0,1] + 2U_r[1,0,0,1]\right) - 6\varepsilon^3 U_r[0,0,0,1] && (4.5.34)
\end{aligned}
$$

In this chapter, we are interested in the special case $N = 2$, with $\mathcal{Y}_0(x) = x^2 - (a_{0,1} + a_{0,2})\,x + a_{0,1}a_{0,2}$. We can deduce from the non-perturbative Dyson-Schwinger equations (4.5.1) that $\langle \mathcal{Y}_0(x)\mathcal{G}_r(x;t)\rangle$ is a polynomial in x for arbitrary t. In particular, we have

$$
\begin{aligned}
0 &= \langle \mathcal{G}_r^{(-3)}(t)\rangle - (a_{0,1} + a_{0,2})\,\langle \mathcal{G}_r^{(-2)}(t)\rangle + a_{0,1}a_{0,2}\langle \mathcal{G}_r^{(-1)}(t)\rangle \\
&= \langle U_r[3]\rangle - (a_{0,1} + a_{0,2})\,\langle U_r[2]\rangle + a_{0,1}a_{0,2}\langle U_r[1]\rangle \\
&\quad + \langle U_r[2,1]\rangle + \langle U_r[1,2]\rangle - 2\varepsilon\langle U_r[0,2]\rangle - (a_{0,1} + a_{0,2} + \varepsilon)\,\langle U_r[1,1]\rangle + \varepsilon\,(a_{0,1} + a_{0,2} + \varepsilon)\,\langle U_r[0,1]\rangle \\
&\quad + \langle U_r[1,1,1]\rangle - \varepsilon\langle U_r[1,0,1]\rangle - 2\varepsilon\langle U_r[0,1,1]\rangle + 2\varepsilon^2\langle U_r[0,0,1]\rangle, && (4.5.35)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \langle \mathcal{G}_r^{(-4)}(t)\rangle - (a_{0,1}+a_{0,2})\,\langle \mathcal{G}_r^{(-3)}(t)\rangle + a_{0,1}a_{0,2}\langle \mathcal{G}_r^{(-2)}(t)\rangle \\
&= \langle U_r[4]\rangle - (a_{0,1}+a_{0,2})\,\langle U_r[3]\rangle + a_{0,1}a_{0,2}\langle U_r[2]\rangle \\
&\quad +\langle U_r[1,3]\rangle + \langle U_r[3,1]\rangle - 3\varepsilon\langle U_r[0,3]\rangle + \langle U_r[2,2]\rangle \\
&\quad -(a_{0,1}+a_{0,2}+\varepsilon)\,\langle U_r[2,1]\rangle - (a_{0,1}+a_{0,2}+2\varepsilon)\,\langle U_r[1,2]\rangle + \varepsilon\,(\varepsilon+2a_{0,1}+2a_{0,2})\,\langle U_r[0,2]\rangle \\
&\quad +(a_{0,1}+\varepsilon)\,(a_{0,2}+\varepsilon)\,\langle U_r[1,1]\rangle - \varepsilon\,(a_{0,1}+\varepsilon)\,(a_{0,2}+\varepsilon)\,\langle U_r[0,1]\rangle \\
&\quad +\langle U_r[2,1,1]\rangle + \langle U_r[1,2,1]\rangle + \langle U_r[1,1,2]\rangle \\
&\quad -\varepsilon\langle U_r[2,0,1]\rangle - 2\varepsilon\langle U_r[1,0,2]\rangle - 3\varepsilon\langle U_r[0,2,1]\rangle - 3\varepsilon\langle U_r[0,1,2]\rangle + 6\varepsilon^2\langle U_r[0,0,2]\rangle \\
&\quad -(a_{0,1}+a_{0,2}+3\varepsilon)\,\langle U_r[1,1,1]\rangle + \varepsilon\,(a_{0,1}+a_{0,2}+3\varepsilon)\,\langle U_r[1,0,1]\rangle \\
&\quad +2\varepsilon\,(a_{0,1}+a_{0,2}+3\varepsilon)\,\langle U_r[0,1,1]\rangle - 2\varepsilon^2\,(a_{0,1}+a_{0,2}+3\varepsilon)\,\langle U_r[0,0,1]\rangle \\
&\quad +\langle U_r[1,1,1,1]\rangle - \varepsilon\,(3\langle U_r[0,1,1,1]\rangle + 2\langle U_r[1,0,1,1]\rangle + \langle U_r[1,1,0,1]\rangle) \\
&\quad +\varepsilon^2\,(6\langle U_r[0,0,1,1]\rangle + 3\langle U_r[0,1,0,1]\rangle + 2\langle U_r[1,0,0,1]\rangle) - 6\varepsilon^3\langle U_r[0,0,0,1]\rangle. \qquad (4.5.36)
\end{aligned}
$$

By taking the residue of (4.5.35) at $t=-z_i^{-1}$, we have

$$
\begin{aligned}
0 &= \langle \zeta_{i,3}\rangle + \left[-a_{0,1}-a_{0,2}+\sum_{j=0}^{i-1}\frac{z_j}{z_j-z_i}\left(\mathcal{A}_j^{(1)}-2\varepsilon\right)+\sum_{j=i+1}^{r}\frac{z_j}{z_j-z_i}\mathcal{A}_j^{(1)}\right]\langle\zeta_{i,2}\rangle + a_{0,1}a_{0,2}\mathcal{A}_i^{(1)} \\
&\quad +\sum_{j=0}^{i-1}\frac{z_j}{z_j-z_i}\left[\mathcal{A}_i^{(1)}\langle\zeta_{j,2}\rangle - (a_{0,1}+a_{0,2}+\varepsilon)\left(\mathcal{A}_j^{(1)}-\varepsilon\right)\mathcal{A}_i^{(1)}\right] \\
&\quad +\sum_{j=i+1}^{r}\frac{z_j}{z_j-z_i}\left[\left(\mathcal{A}_i^{(1)}-2\varepsilon\right)\langle\zeta_{j,2}\rangle - (a_{0,1}+a_{0,2}+\varepsilon)\left(\mathcal{A}_i^{(1)}-\varepsilon\right)\mathcal{A}_j^{(1)}\right] \\
&\quad +\sum_{0\le i_1<i_2<i}\frac{z_{i_1}z_{i_2}}{(z_{i_1}-z_i)\,(z_{i_2}-z_i)}\left(\mathcal{A}_{i_1}^{(1)}-2\varepsilon\right)\left(\mathcal{A}_{i_2}^{(1)}-\varepsilon\right)\mathcal{A}_i^{(1)} \\
&\quad +\sum_{i_1=0}^{i-1}\sum_{i_2=i+1}^{r}\frac{z_{i_1}z_{i_2}}{(z_{i_1}-z_i)\,(z_{i_2}-z_i)}\left(\mathcal{A}_{i_1}^{(1)}-2\varepsilon\right)\left(\mathcal{A}_i^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)} \\
&\quad +\sum_{i<i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_{i_1}-z_i)\,(z_{i_2}-z_i)}\left(\mathcal{A}_i^{(1)}-2\varepsilon\right)\left(\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)}. \qquad (4.5.37)
\end{aligned}
$$

In particular, when $j=0$, we have

$$
\begin{aligned}
0 \;=\;& \langle\zeta_{0,3}\rangle - \left(a_{0,1}+a_{0,2}+\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\mathcal{A}_i^{(1)}\right)\langle\zeta_{0,2}\rangle + a_{0,1}a_{0,2}\mathcal{A}_0^{(1)} \\
& -\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\left[\left(\mathcal{A}_0^{(1)}-2\varepsilon\right)\langle\zeta_{i,2}\rangle - \left(a_{0,1}+a_{0,2}+\varepsilon\right)\left(\mathcal{A}_0^{(1)}-\varepsilon\right)\mathcal{A}_i^{(1)}\right] \\
& +\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{\left(z_0-z_{i_1}\right)\left(z_0-z_{i_2}\right)}\left(\mathcal{A}_0^{(1)}-2\varepsilon\right)\left(\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)}. \qquad (4.5.38)
\end{aligned}
$$

We also need the equation obtained by taking residue of (4.5.36) at $t=-z_0^{-1}$,

$$
\begin{aligned}
0 \;=\;& \langle\zeta_{0,4}\rangle - \left(a_{0,1}+a_{0,2}+\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\mathcal{A}_i^{(1)}\right)\langle\zeta_{0,3}\rangle + a_{0,1}a_{0,2}\langle\zeta_{0,2}\rangle \\
& -\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\Bigg[\left(\mathcal{A}_0^{(1)}-3\varepsilon\right)\langle\zeta_{i,3}\rangle + \langle\zeta_{0,2}\zeta_{i,2}\rangle - \left(a_{0,1}+a_{0,2}+\varepsilon\right)\mathcal{A}_i^{(1)}\langle\zeta_{0,2}\rangle \\
& +\left(\varepsilon\left(\varepsilon+2a_{0,1}+2a_{0,2}\right)-\left(a_{0,1}+a_{0,2}+2\varepsilon\right)\mathcal{A}_0^{(1)}\right)\langle\zeta_{i,2}\rangle + \left(a_{0,1}+\varepsilon\right)\left(a_{0,2}+\varepsilon\right)\left(\mathcal{A}_0^{(1)}-\varepsilon\right)\mathcal{A}_i^{(1)}\Bigg] \\
& +\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{\left(z_0-z_{i_1}\right)\left(z_0-z_{i_2}\right)}\Bigg[\left(\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)}\langle\zeta_{0,2}\rangle + \left(\mathcal{A}_0^{(1)}-3\varepsilon\right)\mathcal{A}_{i_2}^{(1)}\langle\zeta_{i_1,2}\rangle \\
& +\left(\mathcal{A}_0^{(1)}-3\varepsilon\right)\left(\mathcal{A}_{i_1}^{(1)}-2\varepsilon\right)\langle\zeta_{i_2,2}\rangle - \left(a_{0,1}+a_{0,2}+3\varepsilon\right)\left(\mathcal{A}_0^{(1)}-2\varepsilon\right)\left(\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)}\Bigg] \\
& -\sum_{1\le i_1<i_2<i_3\le r}\frac{z_{i_1}z_{i_2}z_{i_3}}{\left(z_0-z_{i_1}\right)\left(z_0-z_{i_2}\right)\left(z_0-z_{i_3}\right)}\left(\mathcal{A}_0^{(1)}-3\varepsilon\right)\left(\mathcal{A}_{i_1}^{(1)}-2\varepsilon\right)\left(\mathcal{A}_{i_2}^{(1)}-\varepsilon\right)\mathcal{A}_{i_3}^{(1)}. \qquad (4.5.39)
\end{aligned}
$$

4.5.3 Derivation of the differential equations

Now we are ready to derive the differential equations satisfied by the instanton partition function using the non-perturbative Dyson-Schwinger equations. The key point is that $\zeta_{0,n}$ take special values at a degenerate point in the parameter space.

4.5.3.1 Second order differential equation

In order to derive a second order differential equation, we should tune the parameters in the following way,

$$a_{0,1}=a_{1,1}+\varepsilon_1, \quad a_{0,2}=a_{1,2}. \qquad (4.5.40)$$

The configuration of the gauge fields are constrained so that the Young diagram $Y^{(1,1)}$ has only one row and $Y^{(1,2)}=\emptyset$,

$$Y^{(1)}=\left(\begin{array}{|c|c|c|c|}\hline (1,1) & (1,2) & \ldots & (1,k_1) \\ \hline \end{array}, \emptyset\right). \tag{4.5.41}$$

Hence, the Young diagram $Y^{(1,1)}$ is completely determined by the instanton charge k_1, and

$$\begin{aligned}
\Xi_0(x)[k_1] &= \frac{\mathcal{Y}_1(x+\varepsilon)[k_1]}{\mathcal{Y}_0(x)} \\
&= \frac{(x+\varepsilon-a_{0,1})(x+\varepsilon-a_{0,2})}{(x-a_{0,1})(x-a_{0,2})}\frac{x-a_{0,1}+2\varepsilon_1-\varepsilon_2(k_1-1)}{x-a_{0,1}+\varepsilon_1-\varepsilon_2(k_1-1)} \\
&= 1+\sum_{n=1}^{\infty}\frac{\zeta_{0,n}[k_1]}{x^n},
\end{aligned} \tag{4.5.42}$$

Which gives

$$\begin{aligned}
\zeta_{0,1}[k_1] &= 3\varepsilon_1+2\varepsilon_2, \\
\zeta_{0,2}[k_1] &= (2\varepsilon_1+\varepsilon_2)\,a_{0,1}+\varepsilon a_{0,2}+\varepsilon\,(2\varepsilon_1+\varepsilon_2)+\varepsilon_1\varepsilon_2 k_1, \\
\zeta_{0,3}[k_1] &= (2\varepsilon_1+\varepsilon_2)\,a_{0,1}^2+\varepsilon\,(2\varepsilon_1+\varepsilon_2)\,a_{0,1}+\varepsilon a_{0,2}^2+\varepsilon\,(2\varepsilon_1+\varepsilon_2)\,a_{0,2} \\
&\quad +2\varepsilon_1\varepsilon_2 a_{0,1}k_1+\varepsilon_1\varepsilon_2^2 k_1^2.
\end{aligned} \tag{4.5.43}$$

Hence, from (4.5.38), we have

$$\begin{aligned}
0 &= \varepsilon_1\varepsilon_2\,(a_{0,1}-a_{0,2})\,\langle k_1\rangle+\varepsilon_1\varepsilon_2^2\langle k_1^2\rangle \\
&\quad -\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\varepsilon_1\left[-a_{0,2}\mathcal{A}_i^{(1)}+\varepsilon_2\mathcal{A}_i^{(1)}\langle k_1\rangle+\mathcal{A}_i^{(2)}-\varepsilon_1\varepsilon_2\langle k_i-k_{i+1}\rangle\right] \\
&\quad +\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_0-z_{i_1})(z_0-z_{i_2})}\varepsilon_1\left(\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)}.
\end{aligned} \tag{4.5.44}$$

Using

$$\mathcal{Z}^{\text{instanton}}\langle k_1\rangle=-\nabla_0\mathcal{Z}^{\text{instanton}},\quad \mathcal{Z}^{\text{instanton}}\langle k_i-k_{i+1}\rangle=\nabla_i\mathcal{Z}^{\text{instanton}}, \tag{4.5.45}$$

we obtain a differential equation on the instanton partition function,

$$
\begin{aligned}
0 \quad = \quad & \Bigg\{ \varepsilon_2^2 \nabla_0^2 - \varepsilon_2 \left(a_{0,1} - a_{0,2} - \sum_{i=1}^{r} \frac{z_i}{z_0 - z_i} \mathcal{A}_i^{(1)} \right) \nabla_0 \\
& + \sum_{i=1}^{r} \frac{z_i}{z_0 - z_i} \left[a_{0,2} \mathcal{A}_i^{(1)} - \mathcal{A}_i^{(2)} + \varepsilon_1 \varepsilon_2 \nabla_i \right] \\
& + \sum_{1 \le i_1 < i_2 \le r} \frac{z_{i_1} z_{i_2}}{\left(z_0 - z_{i_1}\right)\left(z_0 - z_{i_2}\right)} \left(\mathcal{A}_{i_1}^{(1)} - \varepsilon \right) \mathcal{A}_{i_2}^{(1)} \Bigg\} \mathcal{Z}^{\text{instanton}}.
\end{aligned} \tag{4.5.46}
$$

This is the equation that was derived in [32] to confirm the BPS/CFT correspondence for this particular case.

4.5.3.2 Third order differential equation

The derivation can be extended to the next-to-simplest case, as we now explain. To obtain a third order differential equation, we tune the parameters

$$
a_{0,1} = a_{1,1} + 2\varepsilon_1, \quad a_{0,2} = a_{1,2}. \tag{4.5.47}
$$

In this case, the configurations of the gauge field are required to satisfy that the Young diagram $Y^{(1,1)}$ has at most two rows and $Y^{(1,2)} = \emptyset$,

$$
Y^{(1)} = \left(\begin{array}{|c|c|ccc|c|ccc|c|}
\hline
(1,1) & (1,2) & & \dots & & (1,y_2) & & \dots & & (1,y_1) \\
\hline
(2,1) & (2,2) & & \dots & & (2,y_2) & & & & \\
\hline
\end{array}, \emptyset \right), \tag{4.5.48}
$$

where we denote the number of boxes in the first and the second row of the Young diagram $Y^{(1,1)}$ as y_1 and y_2, respectively. The instanton charge $k_1 = y_1 + y_2$. Then, we have

$$
\begin{aligned}
\Xi_0(x)[\boldsymbol{Y}] &= \frac{\mathcal{Y}_1(x+\varepsilon)[\boldsymbol{Y}]}{\mathcal{Y}_0(x)} \\
&= \frac{(x+\varepsilon-a_{0,1})(x+\varepsilon-a_{0,2})}{(x-a_{0,1})(x-a_{0,2})} \\
&\quad \times \frac{x+\varepsilon-a_{0,1}+2\varepsilon_1-\varepsilon_2 y_1}{x+\varepsilon-a_{0,1}+\varepsilon_1-\varepsilon_2 y_1} \frac{x+\varepsilon-a_{0,1}+\varepsilon_1-\varepsilon_2 y_2}{x+\varepsilon-a_{0,1}-\varepsilon_2 y_2} \\
&= 1+\sum_{n=1}^{\infty} \frac{\zeta_{0,n}[\boldsymbol{Y}]}{x^n}. \qquad (4.5.49)
\end{aligned}
$$

We have

$$
\begin{aligned}
\zeta_{0,1} &= 4\varepsilon_1 + 2\varepsilon_2, \\
\zeta_{0,2} &= (3\varepsilon_1+\varepsilon_2)\, a_{0,1} + \varepsilon a_{0,2} + \varepsilon\,(3\varepsilon_1+\varepsilon_2) + \varepsilon_1\varepsilon_2 k_1, \qquad (4.5.50)
\end{aligned}
$$

while $\zeta_{0,4}$ are related to $\zeta_{0,3}$ as

$$
\begin{aligned}
\zeta_{0,4} &= \left(3a_{0,1} - 2\varepsilon_1 - \varepsilon_2 + \frac{3}{2}\varepsilon_2 k_1\right)\zeta_{0,3} - \frac{1}{2}\varepsilon_1\varepsilon_2^3 k_1^3 - \frac{1}{2}\varepsilon_1\varepsilon_2^2\,(6a_{0,1}-\varepsilon_1)\, k_1^2 \\
&\quad -\varepsilon_2\left(\frac{3}{2}(5\varepsilon_1+\varepsilon_2)\, a_{0,1}^2 + \frac{1}{2}(\varepsilon_1+3\varepsilon_2)(3\varepsilon_1+\varepsilon_2)\, a_{0,1} + \frac{3}{2}\varepsilon a_{0,2}^2 + \frac{1}{2}\varepsilon\,(7\varepsilon_1+3\varepsilon_2)\, a_{0,2}\right) k_1 \\
&\quad -2\,(3\varepsilon_1+\varepsilon_2)\, a_{0,1}^3 - \varepsilon_2\,(3\varepsilon_1+\varepsilon_2)\, a_{0,1}^2 + \varepsilon\,(2\varepsilon_1+\varepsilon_2)(3\varepsilon_1+\varepsilon_2)\, a_{0,1} - 3\varepsilon a_{0,1} a_{0,2}^2 \\
&\quad -2\varepsilon\,(3\varepsilon_1+\varepsilon_2)\, a_{0,1}a_{0,2} + \varepsilon a_{0,2}^3 + \varepsilon\,(5\varepsilon_1+2\varepsilon_2)\, a_{0,2} + \varepsilon\,(2\varepsilon_1+\varepsilon_2)(3\varepsilon_1+\varepsilon_2)\, a_{0,2} \qquad (4.5.51)
\end{aligned}
$$

Using 4.5.38, (4.5.51) and (4.5.37), we can get rid of all terms with $\langle\zeta_{0,3}\rangle$, $\langle\zeta_{0,4}\rangle$ and $\langle\zeta_{i,3}\rangle$ in (4.5.39), and we obtain a differential equation on the instanton partition funtion

$$\begin{aligned}
0 \;=\; & \Bigg\{\frac{\varepsilon_2^3}{2}\nabla_0^3+\frac{\varepsilon_2^2}{2}\left(-3a_{0,1}+3a_{0,2}+\varepsilon_1+3\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\mathcal{A}_i^{(1)}\right)\nabla_0^2 \\
&+\varepsilon_2\Bigg[\left(a_{0,1}-a_{0,2}\right)\left(a_{0,1}-a_{0,2}-\varepsilon_1\right) \\
&-\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\left(2\mathcal{A}_i^{(2)}-2\varepsilon_1\varepsilon_2\nabla_i+\mathcal{A}_i^{(1)}\left(2a_{0,1}-4a_{0,2}+\frac{\varepsilon_2}{2}\right)\right) \\
&+\sum_{i=1}^{r}\frac{z_i^2}{\left(z_0-z_i\right)^2}\mathcal{A}_i^{(1)}\left(\mathcal{A}_i^{(1)}-\varepsilon_1-\frac{\varepsilon_2}{2}\right) \\
&+2\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{\left(z_0-z_{i_1}\right)\left(z_0-z_{i_2}\right)}\left(2\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)}\Bigg]\nabla_0 \\
&-\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\left(2a_{0,1}-2a_{0,2}+\varepsilon_2\right)\left(a_{0,2}\mathcal{A}_i^{(1)}-\mathcal{A}_i^{(2)}+\varepsilon_1\varepsilon_2\nabla_i\right) \\
&+\sum_{i=1}^{r}\frac{z_i^2}{\left(z_0-z_i\right)^2}\left(2\mathcal{A}_i^{(1)}-2\varepsilon_1-\varepsilon_2\right)\left(a_{0,2}\mathcal{A}_i^{(1)}-\mathcal{A}_i^{(2)}+\varepsilon_1\varepsilon_2\nabla_i\right) \\
&+2\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{\left(z_0-z_{i_1}\right)\left(z_0-z_{i_2}\right)}\Bigg[\left(\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)}\left(-a_{0,1}+a_{0,2}-\varepsilon\right) \\
&+\mathcal{A}_{i_1}^{(1)}\left(-\mathcal{A}_{i_2}^{(2)}+a_{0,2}\mathcal{A}_{i_2}^{(1)}+\varepsilon_1\varepsilon_2\nabla_i\right) \\
&+\mathcal{A}_{i_2}^{(1)}\left(-\mathcal{A}_{i_1}^{(2)}+a_{0,2}\mathcal{A}_{i_1}^{(1)}+\varepsilon_1\varepsilon_2\nabla_i\right)\Bigg] \\
&+2\sum_{1\le i_1<i_2<i_3\le r}\frac{z_{i_1}z_{i_2}z_{i_3}}{\left(z_0-z_{i_1}\right)\left(z_0-z_{i_2}\right)\left(z_0-z_{i_3}\right)}\left(\mathcal{A}_{i_1}^{(1)}\left(3\mathcal{A}_{i_2}^{(1)}-\varepsilon\right)\mathcal{A}_{i_3}^{(1)}-2\varepsilon\mathcal{A}_{i_2}^{(1)}\mathcal{A}_{i_3}^{(1)}\right) \\
&+\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}^2z_{i_2}}{\left(z_0-z_{i_1}\right)^2\left(z_0-z_{i_2}\right)}\left(\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\left(2\mathcal{A}_{i_1}^{(1)}-2\varepsilon_1-\varepsilon_2\right)\mathcal{A}_{i_2}^{(1)} \\
&+\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}^2}{\left(z_0-z_{i_1}\right)\left(z_0-z_{i_2}\right)^2}\left(\mathcal{A}_{i_1}^{(1)}-\varepsilon\right)\mathcal{A}_{i_2}^{(1)}\left(2\mathcal{A}_{i_2}^{(1)}-2\varepsilon_1-\varepsilon_2\right)\Bigg\}\mathcal{Z}^{\text{instanton}} \qquad (4.5.52)
\end{aligned}$$

4.5.4 Identification with the BPZ equations

The final step is to identify the differential equations we derived in the gauge theory side with the BPZ equations by solving the undetermined parameters L_i and T_{ij}. We find that there exists the following solution

$$
\begin{aligned}
L_0 &= \Delta_{-1} - \Delta_0 - \frac{\varepsilon^2 - (a_{1,1} - a_{1,2})^2}{4\varepsilon_1\varepsilon_2}, \\
L_i &= -\Delta_i - \frac{\left(a_{i,1} - a_{i,2}\right)^2}{4\varepsilon_1\varepsilon_2} + \frac{\left(a_{i+1,1} - a_{i+1,2}\right)^2}{4\varepsilon_1\varepsilon_2}, \quad i = 1, \cdots, r-1, \\
L_r &= -\Delta_r - \Delta_{r+1} + \frac{\varepsilon^2 - \left(a_{r,1} - a_{r,2}\right)^2}{4\varepsilon_1\varepsilon_2}, \\
T_{ij} &= \frac{\left(\mathcal{A}_i^{(1)} - 2\varepsilon\right)\mathcal{A}_j^{(1)}}{2\varepsilon_1\varepsilon_2}.
\end{aligned} \tag{4.5.53}
$$

With this identification of parameters, we observe the precise agreement between (4.2.26), (4.2.27) and (4.5.46), (4.5.52). It is easy to check that (4.2.24) is also satisfied. Notice that the prefactor can also be written as

$$
\begin{aligned}
\left(\prod_{i=0}^{r} z_i^{L_i}\right) \prod_{0\le i<j\le r} \left(1 - \frac{z_j}{z_i}\right)^{T_{ij}} &= \left(z_0^{\Delta_{-1}-\Delta_0-\frac{\varepsilon^2}{4\varepsilon_1\varepsilon_2}}\right)\left(\prod_{i=1}^{r-1} z_i^{-\Delta_i}\right)\left(z_r^{\frac{\varepsilon^2}{4\varepsilon_1\varepsilon_2}-\Delta_r-\Delta_{r+1}}\right) \\
&\quad \times \prod_{i=1}^{r} q_i^{-\frac{\left(a_{i,1}-a_{i,2}\right)^2}{4\varepsilon_1\varepsilon_2}} \prod_{0\le i<j\le r}\left(1 - \frac{z_j}{z_i}\right)^{\frac{2\left(\bar{a}_i - \bar{a}_{i+1}\right)\left(\bar{a}_j - \bar{a}_{j+1} + \varepsilon\right)}{\varepsilon_1\varepsilon_2}}
\end{aligned} \tag{4.5.54}
$$

which give the expected tree-level partition function and the $U(1)$ part of the partition function (see the Appendix C for details). Therefore, we confirm the BPS/CFT correspondence.

4.6 Discussion

In this chapter, we perform the derivation of the differential equation on the instanton partition function at a special point in the parameter space using the method of non-perturbative Dyson-Schwinger equations, and identify the differential equations with the BPZ equations in the Liouville field theory. Therefore, we confirm the main assertion of the BPS/CFT correspondence.

There are several obvious generalizations of the contents of this chapter. First of all, it is natural to consider the general degenerate fields with conformal dimension $\Delta_{(m,n)}$, and derive the differential equation of order mn on the instanton partition function. The computation will be unavoidably lengthy, but the basic idea is the same. To simplify the derivation, it is sometimes useful to consider the non-perturbative Dyson-Schwinger equations of both fundamental and non-fundamental qq-characters.

We can also generalize the discussion to the $U(N)$ superconformal linear quiver gauge theories. However, from the knowledge of corresponding Toda field theory, we do not expect to obtain a differential equation on the instanton partition function. Instead, the equations derived from the non-perturbative Dyson-Schwinger equations will generally relate the instanton partition function with expectation values of certain BPS observables. Only if we take the Nekrasov-Shatashvili limit can we get an differential equation on the instanton partition function. The non-conformal A_2-quiver $SU(3)$ gauge theory and the degenerate irregular conformal block in the A_2 Toda field theory were studied in [2] along this direction. The detailed discussion on general quiver will appear in a separate work.

In spite of the successful application of the non-perturbative Dyson-Schwinger equations to derive the BPZ equations, there are still some open problems. From the point of view of conformal field theory, it is equally good to choose any one of the fields to be degenerate, and we have the BPZ equation for every choice. In the corresponding four-dimensional theory, we need to tune the parameters in the following way for arbitrary $i = 0, \cdots, r$,

$$a_{i,\alpha} = \begin{cases} a_{i+1,1} + (m-1)\varepsilon_1 + (n-1)\varepsilon_2, & \alpha = 1, \\ a_{i+1,\alpha}, & \alpha \neq 1. \end{cases} \tag{4.6.1}$$

However, we do not get the expected constraints of the from (4.6.1). For example, the constraint is $Y^{(i+1,\alpha)} \subset Y^{(i,\alpha)}$ rather than $Y^{(i+1,\alpha)} = Y^{(i,\alpha)}$ for $\alpha \neq 1$. This problem is associated with the annoying $U(1)$ factor in the AGT dictionary. We may have to figure out how to factor out the $U(1)$ factor at the level of the measure $\mathcal{Z}^{\text{instanton}}(\boldsymbol{a}; \boldsymbol{Y}; \varepsilon_1, \varepsilon_2)$. A progress in this direction will also lead us immediately to a derivation of the BPZ equation for the conformal field theory on a torus.

Chapter 5

Opers, surface defects, and Yang-Yang functional

5.1 Introduction

The dynamics of supersymmetric gauge theories is a rewarding research subject. The exact low-energy description of the four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry was proposed in [24, 25] for the $SU(2)$ theories with various matter multiplets. The proposal has been generalized in the subsequent works, allowing for different gauge groups and matter representations. In many cases the Coulomb branch of the moduli space of vacua is a family of algebraic curves (called the Seiberg-Witten curves) equipped with meromorphic differential. The periods of the differential compute the central charges of the supersymmetry algebra determining the masses of the BPS particles at this vacuum. The microscopic study of these theories using direct quantum field theory methods and supersymmetric localization was initiated in [6], leading to the exact computation of the partition functions of a deformed version of the theory, the realization they coincide with the partition functions of some two dimensional chiral theory, and connecting that theory to the M- and string theory fivebranes [6, 80, 19, 33, 63].

The method of [6] reduces the computation of the path integral to a problem of counting fixed points under the action of the global symmetry group on a finite dimensional BPS field configurations. More specifically, the partition function can be written as a product of analytic functions,

$$\mathcal{Z}(\mathbf{a},\mathbf{m},\boldsymbol{\varepsilon},\mathfrak{q}) = \mathcal{Z}^{\text{classical}}(\mathbf{a},\boldsymbol{\varepsilon},\mathfrak{q})\ \mathcal{Z}^{\text{1-loop}}(\mathbf{a},\mathbf{m},\boldsymbol{\varepsilon})\ \mathcal{Z}^{\text{inst}}(\mathbf{a},\mathbf{m},\boldsymbol{\varepsilon},\mathfrak{q}). \tag{5.1.1}$$

Here $\mathfrak{q}$ schematically denotes the gauge couplings of the theory, while $\mathbf{a}$, $\mathbf{m}$, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ denote the equivariant parameters for the group of global gauge symmetry, the group of flavor symmetry, and the group of Lorentz symmetry, respectively. $\varepsilon_{1,2}$ are also called Ω-deformation parameters (See appendix 2.1 for a more detailed review of the $\mathcal{N}=2$ partition functions). The effective prepotential is then obtained by taking the limit (while keeping $\mathbf{a},\mathbf{m},\mathfrak{q}$ generic)

$$\mathcal{F}(\mathbf{a},\mathbf{m},\mathfrak{q}) = \lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\varepsilon_2 \log \mathcal{Z}(\mathbf{a},\mathbf{m},\boldsymbol{\varepsilon},\mathfrak{q}), \tag{5.1.2}$$

which provides the direct microscopic derivation of the results in [24, 25] (either using the limit shape approach [7], or the blowup equations [81]).

Meanwhile, it was observed in [26, 27, 36] that the Coulomb branch of vacua of a $\mathcal{N}=2$ theory canonically has a structure of a base $\mathcal{B}$ of an algebraic integrable system. The full structure is revealed when the theory is compactified [82] on a circle S^1_R [35]. The moduli space of the effective $\mathcal{N}=4, d=3$ theory is a hyper-Kähler manifold which metrically collapses to the Coulomb moduli space $\mathcal{B}$ of the four-dimensional theory in the limit $R\to\infty$ [82]. In this limit, one of the complex structures, say, I is singled out, with respect to which we have a holomorphic symplectic form Ω_I. For finite R, the moduli space is a Ω_I-Lagrangian fibration over $\mathcal{B}$ by abelian varieties. More specifically, the Coulomb branch $\mathcal{B}$ is parametrized by the expectation values $u_k = \langle \mathcal{O}_k \rangle$ of chiral observables (these are local operators anticommuting with the four nilpotent supercharges of one Lorentz chirality).

These observables carry over to the theory with finite R. We define the Hamiltonians to be the I-holomorphic functions on the moduli space of the compactified theory by

$$H_k = \langle \mathcal{O}_k \rangle \ , \quad k = 1, \cdots, \dim \mathcal{B}, \tag{5.1.3}$$

and it is not difficult to show that these functions Poisson-commute with respect to the Ω_I^{-1}.

5.1.1 Quantization via gauge theory: Effective twisted superpotential as Yang-Yang functional

The remarkable correspondence between the gauge theory and integrable system was promoted to the quantum level in [28], by placing the gauge theory into the realm of Bethe/gauge correspondence [29, 30]. We consider the theory in the Ω-background affecting two out of four dimensions of spacetime. Equivalently, we take the *Nekrasov-Shatashvili limit* ($\varepsilon_1 = \hbar \neq 0, \varepsilon_2 \to 0$) of the general Ω-background, so that the theory retains the two-dimensional $\mathcal{N} = (2,2)$ supersymmetry. The effective action includes the twisted F-term given by the effective twisted superpotential,

$$\widetilde{\mathcal{W}}(\mathbf{a}, \mathbf{m}, \hbar, \mathfrak{q}) = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \mathcal{Z}(\mathbf{a}, \mathbf{m}, \varepsilon_1 = \hbar, \varepsilon_2, \mathfrak{q}). \tag{5.1.4}$$

Typically, the theories with four supercharges have isolated vacua. In this way the Ω-deformation of the four dimensional theory lifts the continuous moduli of vacua. The discrete set of vacua is in one-to-one correspondence with the solutions to the following equation,

$$\exp \frac{\partial \widetilde{\mathcal{W}}(\mathbf{a}, \mathbf{m}, \hbar, \mathfrak{q})}{\partial a_\alpha} = 1, \quad \alpha = 1, \cdots, \dim \mathcal{B} \tag{5.1.5}$$

In the context of Bethe/gauge correspondence, this equation is identified with the Bethe equation which determines the set of joint eigenvalues of the mutually commuting Hamiltonians. The Coulomb moduli $\mathbf{a}$ in (5.1.5) map to the quasi-momenta, or Bethe roots, of the integrable system. The spectrum of the Hamiltonians for a given solution $\mathbf{a}_*$ of (5.1.5) is computed as

$$u_k(\mathbf{a}_*, \mathbf{m}, \hbar, \mathfrak{q}) = \langle \mathcal{O}_k \rangle^{\varepsilon_1=\hbar, \varepsilon_2=0;\, \mathbf{m}, \mathfrak{q}}_{\mathbf{a}=\mathbf{a}_*}, \tag{5.1.6}$$

The Ω-deformation parameter $\hbar$ plays the role of the Planck constant of the quantum integrable system. The potential $\widetilde{\mathcal{W}}$ of the Eqs. (5.1.5) determining the Bethe roots is identified with *the Yang-Yang functional* [83] in the context of the integrable system. The effective twisted superpotential, or the Yang-Yang functional, can be written in the following form according to the decomposition of (5.1.1),

$$\widetilde{\mathcal{W}}(\mathbf{a}, \mathbf{m}, \hbar, \mathfrak{q}) = \widetilde{\mathcal{W}}^{\text{classical}}(\mathbf{a}, \mathbf{m}, \hbar) \log \mathfrak{q} + \widetilde{\mathcal{W}}^{\text{1-loop}}(\mathbf{a}, \mathbf{m}, \hbar) + \widetilde{\mathcal{W}}^{\text{inst}}(\mathbf{a}, \mathbf{m}, \hbar, \mathfrak{q}). \tag{5.1.7}$$

The 1-loop part depends on the regularization scheme but is independent of the gauge coupling $\mathfrak{q}$, while the instanton part is expanded as a series in $\mathfrak{q}$. The series can be exactly computed by taking the Nekrasov-Shatashvili limit of the Young diagram expansion of the instanton partition function. See appendix 2.1 for more background on the localization computation of the effective twisted superpotential.

5.1.2 Hitchin systems, flat connections, and opers

In this chapter, we study a specific subclass of the four-dimensional $\mathcal{N} = 2$ theories, which is called the class $\mathcal{S}$ theories [33]. The class $\mathcal{S}$ theory $\mathcal{T}[\mathfrak{g}, \mathcal{C}]$ ($\mathfrak{g} = ADE$) is the four-dimensional $\mathcal{N} = 2$ superconformal theory engineered by compactifying the 6-dimensional $\mathcal{N} = (0, 2)$ superconformal theory of type $\mathfrak{g}$ on the Riemann surface $\mathcal{C}$, with a partial topological twist. As we discussed earlier, the further compactification of $\mathcal{T}[\mathfrak{g}, \mathcal{C}]$ on a circle S^1 yields a three-dimensional $\mathcal{N} = 4$ gauge theory whose Coulomb moduli space is the phase space of the Seiberg-Witten integrable system. By changing the order of compactification on $\mathcal{C} \times S^1$[84], it can be verified that the moduli space is equivalent to the moduli space $\mathcal{M}_H(G, \mathcal{C})$ of the

Hitchin pairs $(\mathcal{P}, \varphi)$, that is, the locus of the Hitchin equations on $\mathcal{C}$ [34],

$$\begin{aligned} &F_A + [\varphi, \bar{\varphi}] = 0 \\ &\bar{\partial}_A \varphi = 0, \quad \partial_A \bar{\varphi} = 0. \end{aligned} \tag{5.1.8}$$

modulo the G-gauge transformations. Here, G is the simple Lie group corresponding to $\mathfrak{g}$, A is a G-connection on the principal G-bundle $\mathcal{P} \to \mathcal{C}$, and $\varphi \in \Gamma(\mathcal{C}, K_{\mathcal{C}} \otimes \mathrm{ad}_{\mathcal{P}})$ is the $\mathfrak{g}_{\mathbb{C}}$-valued (1,0)-form called the Higgs field. Note that $\mathcal{C}$ may have punctures, and the Higgs field is prescribed to have specific singular behaviors at those punctures. Therefore, the Seiberg-Witten integrable system for the class $\mathcal{S}$ theory $\mathcal{T}[\mathfrak{g}, \mathcal{C}]$ is the Hitchin integrable system with the phase space $\mathcal{M}_H(G, \mathcal{C})$.

As discussed in [85], we can view the Hitchin moduli space $\mathcal{M}_H$ as a hyper-Kähler quotient of the affine space $\mathcal{W}$ of all the field configurations of (A, φ). $\mathcal{W}$ is hyper-Kähler with a natural $\mathbb{P}^1$-family of complex structures,

$$\mathcal{I} = aI + bJ + cK, \quad \mathcal{I}^2 = -1, \quad \text{for} \quad a^2 + b^2 + c^2 = 1, \tag{5.1.9}$$

where we may choose the convention that I, J, and K are the complex structures with the holomorphic coordinates $(A_{\bar{z}}, \varphi_z)$, $(\mathcal{A}_z \equiv A_z + i\varphi_z, \mathcal{A}_{\bar{z}} \equiv A_{\bar{z}} + i\varphi_{\bar{z}})$, and $(A_z + \varphi_z, A_{\bar{z}} - \varphi_{\bar{z}})$, respectively. The corresponding Kähler forms are

$$\begin{aligned} \omega_I &= -\frac{1}{4\pi} \int_{\mathcal{C}} \mathrm{Tr}\, (\delta A \wedge \delta A - \delta\varphi \wedge \delta\varphi), \\ \omega_J &= \frac{1}{2\pi} \int_{\mathcal{C}} |d^2 z|\ \mathrm{Tr}\, (\delta\varphi_{\bar{z}} \wedge \delta A_z + \delta\varphi_z \wedge \delta A_{\bar{z}}), \\ \omega_K &= \frac{1}{2\pi} \int_{\mathcal{C}} \mathrm{Tr}\, (\delta A \wedge \delta\varphi). \end{aligned} \tag{5.1.10}$$

Then the Hitchin equations (5.1.8) are just the moment map equations for these Kähler forms. Therefore $\mathcal{M}_H(G, \mathcal{C})$ is also hyper-Kähler with the same complex structures and

Kähler forms. We also define $\Omega_I = \omega_J + i\omega_k$ and its cyclic permutations,

$$\begin{aligned}
\Omega_I &= \frac{1}{\pi}\int_{\mathcal{C}} |d^2z| \ \mathrm{Tr}\ (\delta\varphi_z \wedge \delta A_{\bar z}), \\
\Omega_J &= -\frac{i}{4\pi}\int_{\mathcal{C}} \mathrm{Tr}\ (\delta\mathcal{A} \wedge \delta\mathcal{A}), \\
\Omega_K &= -\frac{i}{2\pi}\int_{\mathcal{C}} |d^2z| \ \mathrm{Tr}(\delta A_{\bar z} \wedge \delta A_z - \delta\varphi_{\bar z} \wedge \delta\varphi_z - \delta\varphi_{\bar z} \wedge \delta A_z - \delta\varphi_z \wedge \delta A_{\bar z}),
\end{aligned} \tag{5.1.11}$$

each of which is a holomorphic symplectic (2,0)-form with respect to the complex structure I, J, and K, respectively.

The complete integrability of $\mathcal{M}_H(G, \mathcal{C})$ is manifest when we work in the complex structure I. We restrict our attention to the case $\mathfrak{g} = A_{N-1}$ from now on. Let us define the Hitchin fibration by the map,

$$\begin{gathered}
\pi : \mathcal{M}_H(A_{N-1}, \mathcal{C}) \longrightarrow \mathcal{B} \equiv \bigoplus_{k=2}^{N} H^0(\mathcal{C}, K_{\mathcal{C}}^k), \\
(\mathcal{P}, \varphi) \longmapsto \left(\mathrm{Tr}\varphi^k\right)_{k=2}^{N}.
\end{gathered} \tag{5.1.12}$$

It is possible to show that under the partial topological twist, the vacuum expectation values of the chiral observables of $U(1)$ R-charge k exactly span $H^0(\mathcal{C}, K_{\mathcal{C}}^k)$. Therefore, we observe that the base $\mathcal{B}$ of the Hitchin fibration is precisely the Coulomb moduli space of $\mathcal{T}[A_{N-1}, \mathcal{C}]$. It is clear from the expression for Ω_I in (5.1.11) that all the base elements are mutually Poisson-commuting under Ω_I. A dimension counting also shows that $\dim \mathcal{B} = \frac{1}{2}\dim \mathcal{M}_H(A_{N-1}, \mathcal{C})$. Finally, the preimage of $u = (u_k(z))_{k=2}^N \in \mathcal{B}$ can be shown to be an abelian variety, the Jacobian $\mathrm{Jac}(\Sigma_u)$ of the spectral curve

$$\Sigma_u = \{x \in T^*\mathcal{C} \mid x^N + \sum_{k=2}^{N} u_k(z) x^{N-k} = 0\} \subset T^*\mathcal{C}, \tag{5.1.13}$$

establishing the algebraic integrable structure of $\mathcal{M}_H(A_{N-1}, \mathcal{C})$. The spectral curve Σ_u is identified with the Seiberg-Witten curve of the theory $\mathcal{T}[A_{N-1}, \mathcal{C}]$.

On the other hand, we can alternatively view $\mathcal{M}_H(A_{N-1}, \mathcal{C})$ through the complex structure J. Up to some stability issue that we do not discuss here, the hyper-Kähler quotient can be equivalently performed by imposing only the moment map equation for Ω_J,

$$\mathcal{F} \equiv d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0, \tag{5.1.14}$$

and moding out the $G_{\mathbb{C}}(= SL(N))$-gauge transformations. Thus, the Hitchin moduli space $\mathcal{M}_H(A_{N-1}, \mathcal{C})$ is identified with the moduli space of flat $SL(N)$-connections on $\mathcal{C}$, $\mathcal{M}_{\text{flat}}(SL(N), \mathcal{C})$. It is convenient to use the holonomy map to express $\mathcal{M}_{\text{flat}}(SL(N), \mathcal{C})$ as the character variety, i.e., the representations of the fundamental group of $\mathcal{C}$,

$$\mathcal{M}_{\text{flat}}(SL(N), \mathcal{C}) = \{\rho \in \text{Hom}(\pi_1(\mathcal{C}), SL(N)) \mid [\rho(\gamma_i)] \text{ fixed}\} / SL(N), \tag{5.1.15}$$

where $\{i\}$ enumerates all the punctures in $\mathcal{C}$, γ_i is the loop encircling the i-th puncture only, and the bracket $[\cdots]$ denotes the conjugacy class. The Poisson structure induced by Ω_J on $\mathcal{M}_{\text{flat}}(SL(N), \mathcal{C})$ can be explicitly written as the skein-relations on the Wilson loops [86, 87].

To see the quantization at work, the class $\mathcal{S}$ theory $\mathcal{T}[A_{N-1}, \mathcal{C}]$ is subject to the Ω-deformation in the Nekrasov-Shatashvili limit. This is most effectively implemented by deforming the underlying geometry into the product of a cylinder and a cigar-like geometry, $X^4 = \mathbb{R} \times S^1 \times \mathcal{D}^2$ [31]. The following metric on $\mathcal{D}^2$ is taken,

$$\begin{aligned} & ds^2 = dr^2 + f(r) d\theta^2, \quad r \in \mathbf{I} = [0, \infty], \quad \theta \in [0, 2\pi), \\ & \text{with } f(r) \sim r^2 \qquad \text{for } r \sim 0, \\ & \qquad f(r) \sim \text{const} \quad \text{for sufficiently large } r \end{aligned} \tag{5.1.16}$$

Note that this metric asymptotes to $X^4 \sim \mathbb{R} \times S^1 \times \mathbf{I} \times \widetilde{S^1}$. One recalls that the Ω-deformation with respect to the isometries of the two-torus can be undone by a redefinition of the fields of the theory [31]. In the limit where both circles S^1 and $\widetilde{S^1}$ are small we can approximate the theory by its reduction. The dependence of the theory on the radii of the circles S^1 and

$\widetilde{S^1}$ is Q-exact, where Q is the supercharge preserved by the Ω-deformation. The dimensional reduction along the two-torus $S^1 \times \widetilde{S^1}$ results in a two-dimensional $\mathcal{N} = (4,4)$ sigma model, with the worldsheet $\mathbb{R} \times \mathbf{I}$ and the target space $\mathcal{M}_H(A_{N-1}, \mathcal{C})$. The quantization of the Hitchin integrable system arises by correctly specifying the boundary conditions at $0, \infty \in \mathbf{I}$ [31]. The boundary condition at $\infty \in \mathbf{I}$ determines the space of states in the integrable system, implemented by a ω_K-Lagrangian brane. It is also argued in [31] that the effect of the Ω-deformation is correctly accounted by the boundary condition at $0 \in \mathbf{I}$ corresponding to the canonical coisotropic brane of $\mathcal{M}_H(A_{N-1}, \mathcal{C})$ [88]. Surprisingly, this brane could be T-dualized along the fibers of the Hitchin fibration to produce a brane supported on a distinguished $J_\hbar$-holomorphic $\Omega_{J_\hbar}$-Lagrangian submanifold of $\mathcal{M}_H(A_{N-1}, \mathcal{C})$: conjecturally, the *variety of opers* [89]. Here, $J_\hbar$ differs from $I, -I$, and is determined by the Ω-deformation parameter $\hbar$. In the absence of punctures on $\mathcal{C}$ all complex structures different from $I, -I$ are diffeomorphic. When punctures are present the diffeomorphism rotating $J_\hbar$ to J changes the masses of the matter hypermultiplets, and, accordingly, the eigenvalues of the monodromy around the punctures. With this subtlety understood, we shall skip the subscript $\hbar$ in the notation for the complex structure J in what follows.

The variety $\mathcal{O}_N[\mathcal{C}] = \{\widehat{\mathfrak{D}}\}$ of opers can be represented as a set of N-th order meromorphic differential operators

$$\widehat{\mathfrak{D}} = \partial_z^N + t_2(z)\partial_z^{N-2} + \cdots + t_N(z) : K_{\mathcal{C}}^{-\frac{N-1}{2}} \longrightarrow K_{\mathcal{C}}^{\frac{N+1}{2}} \otimes \mathcal{O}(N \cdot D), \tag{5.1.17}$$

where D is the divisor of punctures. Here we view $\widehat{\mathfrak{D}}$ as an element of $\mathcal{M}_{\text{flat}}(SL(N), \mathcal{C})$ by associating it to the representation

$$\begin{aligned} \rho_{\widehat{\mathfrak{D}}} : \pi_1(\mathcal{C}) &\longrightarrow SL(N) \\ \gamma &\longmapsto M_\gamma(\widehat{\mathfrak{D}}), \end{aligned} \tag{5.1.18}$$

where $M_\gamma(\widehat{\mathfrak{D}})$ is the $SL(N)$-valued monodromy of the solutions of $\widehat{\mathfrak{D}}$ along the loop γ. More specifically, the conjugacy class of the monodromy around each puncture is fixed, so that

$$\mathcal{O}_N[\mathcal{C}] = \left\{ \widehat{\mathfrak{D}} \;\middle|\; \left[M_{\gamma_i}(\widehat{\mathfrak{D}})\right] \text{ fixed} \right\}, \tag{5.1.19}$$

leaving only $\dim \mathcal{O}_N[\mathcal{C}] = \dim \mathcal{B}$ degrees of freedom for the meromorphic functions $(t_k(z))_{k=2}^N$ which is equal to the half of the dimension of the full moduli space $\mathcal{M}_{\text{flat}}(SL(N), \mathcal{C})$. In fact, as an oper (5.1.17) can be regarded as a quantization of the Seiberg-Witten curve (5.1.13), the variety of opers $\mathcal{O}_N[\mathcal{C}]$ provides a quantization of the Coulomb moduli space $\mathcal{B}$, and the holomorphic functions on $\mathcal{O}_N[\mathcal{C}]$ precisely correspond to the *off-shell* spectra of the mutually commuting quantum Hitchin Hamiltonians [90].

The ω_K-Lagrangian brane at infinity $\infty \in \mathbf{I}$ is T-dualized to another ω_K-Lagrangian brane L. The ground states of open strings with two ends on $\mathcal{O}_N[\mathcal{C}]$ and L, respectively, define the space of morphisms in Fukaya category

$$\mathcal{H} = \text{Hom}(\mathcal{O}_N[\mathcal{C}], L). \tag{5.1.20}$$

The space of morphisms between two Lagrangian branes in Fukaya category is the symplectic Floer homology $HF^\bullet_{\text{symp}}(\mathcal{O}_N[\mathcal{C}], L)$, which can be obtained as a cohomology of a complex spanned by the intersection points with the differential obtained by studying pseudo-holomorphic disks with boundaries on $\mathcal{O}_N[\mathcal{C}]$ and L. For hyper-Kähler manifolds, such as the Hitchin space in our case, there is no contribution coming from the disks of non-zero relative degree, thus the space of states are determined by the classical intersection points.[1] In other words, the problem of quantization reduces to enumeration of the intersection of the variety of opers and a ω_K-Lagrangian brane. The isolated intersection point defines a common eigenstate of the quantum Hamiltonians. The spectra of quantum Hamiltonians are the holomorphic functions on the variety of opers restricted to this locus.

[1] There is a subtlety when the Lagrangians are not transversal. It appears the lift of degeneracy of the ground states [2, 47] in quantum mechanics corresponds to such singularities.

5.1.3 Nekrasov-Rosly-Shatashvili conjecture

Since the variety of opers $\mathcal{O}_N[\mathcal{C}]$ is a complex Lagrangian submanifold of $\mathcal{M}_{\text{flat}}(SL(N), \mathcal{C})$, there exists the generating function $\mathcal{S}\left[\mathcal{O}_N[\mathcal{C}]\right]$ for $\mathcal{O}_N[\mathcal{C}]$,

$$\boldsymbol{\beta}_i = \frac{\partial \mathcal{S}\left[\mathcal{O}_N[\mathcal{C}]\right]}{\partial \boldsymbol{\alpha}_i}, \qquad i = 1, \cdots, \frac{1}{2}\dim \mathcal{M}_{\text{flat}}(SL(N), \mathcal{C}), \tag{5.1.21}$$

for any Darboux coordinate system $\{\boldsymbol{\alpha}_i, \boldsymbol{\beta}_j\} = \delta_{ij}$ on $\mathcal{M}_{\text{flat}}(SL(N), \mathcal{C})$. In [91], it was suggested that there exists a specific Darboux coordinate system (which we refer to as *the NRS coordinate system*), in which the generating function for the variety of opers is identified with the effective twisted superpotential, up to a contribution from the boundary at the infinity which is independent of the gauge coupling, namely,

$$\mathcal{S}\left[\mathcal{O}_N[\mathcal{C}]\right] = \frac{1}{\varepsilon_1}\left(\widetilde{\mathcal{W}}\left[\mathcal{T}[A_{N-1}, \mathcal{C}]\right] - \widetilde{\mathcal{W}}_\infty\right). \tag{5.1.22}$$

In the $N = 2$ case, the NRS coordinate system on the moduli space of $SL(2, \mathbb{C})$-flat connections essentially restricts to the coordinate systems proposed in [92, 93, 94] for the $SU(2)$ flat connections, Teichmüller space (which is a component of the moduli space of $SL(2, \mathbb{R})$-flat connections) and the $SO(1, 2)$-flat connections, respectively. The intuition behind the above equivalence is that as we vary the complex structure of $\mathcal{C}$, the corresponding variation of $\mathcal{O}_2[\mathcal{C}]$ is represented by a closed holomorphic one-form on $\mathcal{O}_2[\mathcal{C}]$, which is a derivative of a holomorphic function since $\mathcal{O}_2[\mathcal{C}]$ is simply-connected. As we noted earlier, the holomorphic functions on $\mathcal{O}_2[\mathcal{C}]$ are the spectra of the quantum Hamiltonians, which are, in the spirit of the Bethe/gauge correspondence,

$$u = \mathfrak{q}\frac{\partial \widetilde{\mathcal{W}}\left[\mathcal{T}[A_1, \mathcal{C}]\right]}{\partial \mathfrak{q}}. \tag{5.1.23}$$

Since the complex structure of $\mathcal{C}$ is controlled by the gauge coupling $\mathfrak{q}$, this motivated [91] to identify the generating function for the variety of opers with the effective twisted superpotential, and thereby with the Yang-Yang functional. As a result, the classical symplectic geometry (which operates with symplectic manifolds and thier Lagrangian subvarieties), the $\mathcal{N} = 2$ gauge theory, and quantum integrable system (which belongs to the domain of noncommutative algebras, their commutative subalgebras, and representation theory) are nicely interconnected through the equality (5.1.22). Note that this is a finite-dimensional version of the quantum/classical duality studied at some examples in [95], which connects the integrable quantum field theories to the classical nonlinear differential equations.

There were many questions that remain unanswered. Some of them are:

1. Can one precisely describe the variety of opers $\mathcal{O}_N[\mathcal{C}]$ as of a deformation of the Coulomb moduli space $\mathcal{B}$ (of course, the first order deformation is simply the WKB approximation)? In particular, how the meromorphic coefficients $(t_k(z))_{k=2}^N$ in (5.1.17) are related to the expectation values $(u_k)_{k=2}^N$ of the chiral observables in (5.1.6)?

2. How is the NRS coordinate system generalized to the higher rank case, at least for $\mathfrak{g} = A_{N-1}$?[2]

3. How should the equality (5.1.22) be understood? Specifically, the left hand side is written in the NRS coordinates, while the right hand side is written in the gauge theoretic terms. How do we match these parameters?[3]

4. Most importantly, derive the equality (5.1.22) from the first principles of the gauge theory (to all orders in the gauge coupling $\mathfrak{q}$)?

We address these questions below:

5.1.4 Outline

The key players of the work are the half-BPS codimension two (surface) defects in the four-dimensional $\mathcal{N} = 2$ gauge theories. The surface defects can be constructed in several ways

[2] In the genus one case it was done in [96, 97].
[3] Some of these questions are addressed in [98] from a geometric point of view.

[19, 18, 60]. The exact computation of their partition functions became accessible in part by [99, 100, 20], and in a more general setting in [15, 21]. In particular, the explicit forms of the surface defects as the observables in the underlying gauge theory were written down in [21].

Meanwhile, the analysis of the analytic properties of the $\mathcal{N} = 2$ partition functions became available since [10]. The qq-characters were introduced as gauge theory observables, which can be constructed out of the spiked instanton configurations [14, 15, 16]. The crucial property of these observables is the regularity of their expectation values [10], which follows from the compactness theorem [14]. From the regularity of qq-characters follows the vanishing theorem for the non-regular parts of the expectation values, thereby constraining the partition functions. We call these vanishing equations *the non-perturbative Dyson-Schwinger equations* [10].

In section 5.2, we recall two independent constructions of surface defects: the quiver and the orbifold. In section 5.3, we describe the fundamental qq-character for the surface defects, and derive the non-perturbative Dyson-Schiwnger equations for their partition functions. We show that the final equations satisfied by the surface defect partition functions can be regarded as a quantized version of the opers, in the sense that they reduce to the differential equations for the opers in the Nekrasov-Shatashvili limit $\varepsilon_2 \to 0$. The relations of the expectation values of the chiral observables to the holomorphic coordinates on the variety of opers are naturally revealed through this procedure, clarifying in what sense the variety of opers is a quantization of the Coulomb moduli space.

Being solutions to the non-perturbative Dyson-Schwinger equations, in the Nekrasov-Shatashvili limit the asymptotics χ of the appropriately normalized surface defect partition function becomes the oper solution $\widehat{\mathfrak{D}}\chi = 0$. Consequently, the monodromy of the solutions of the oper can be obtained by first computing the monodromy of the surface defect partition functions and then taking the Nekrasov-Shatashvili limit. However, each surface defect partition function has its own convergence domain, and to compute the monodromy we need *the connection matrix* which links the surface defect partition functions lying on different domains. This is the subject of the section 5.4. Namely, we present how the surface defect partition function is analytically continued to another convergence domain, and how they can be *glued* together. In fact, the analytically continued quiver surface defect partition function is shown to be identical to a specific orbifold surface defect partition function, suggesting the equivalence of the two distinct types of surface defects. It may be regarded as an independent nontrivial result in itself, realizing the duality between the surface defects [48] at the level of the partition functions.

In relating the gauge effective theory twisted superpotential to the generating function of the variety of opers, we need to specify the Darboux coordinate system on the moduli space of flat connections relevant to the correspondence. More precisely, we need at least the coordinates on the patch of the moduli space, in which the theory has a weak coupling description (the twisted superpotential is defined, of course, everywhere, however we can only compute it directly in quantum field theory in that region). It may appear that the coupling constant of the theory, being the complex moduli of the underlying Riemann surface, has nothing to do with the coordinate charts on the moduli space of flat connections in the J-complex structure, as the latter depends only on the topology of $\mathcal{C}$. The explanation is the following. The continuous dependence on the couplings $\mathfrak{q}$ is indeed absent. However, the universality classes of the Lagrangians describing the theory depend on the type of the degeneration of the Riemann surface $\mathcal{C}$, the so-called pair-of-pants decomposition. The latter is determined by the choice of a handlebody (together with an embedded graph) whose boundary is $\mathcal{C}$ (with the punctures being the end-points of the graph edges).

With this understood, in section 5.5, we propose Darboux coordinates on a particular patch of the moduli space of flat $SL(N)$-connections on the $r+3$-punctured sphere. Our coordinates agree (up to a simple shift) with the NRS coordinates [91] restricted to the corresponding patch of the $SL(2)$-moduli space. We verify the canonical Poisson relations for the proposed coordinate system by using the geometric representation of Poisson brackets between the Wilson loops in the classical Chern-Simons theory. We compute explicitly the invariants of the holonomies of flat connections in our main $r = 1$ example.[4]

Finally, the monodromy data of opers is computed in section 5.6. More precisely, we compute the analytic continuation of the surface defect partition functions, using the results of section 5.4. Then we take the Nekrasov-Shatashvili limit of the resulting transfer matrices to reduce them to the monodromies of the opers. Then we express those data in terms of the generalized NRS coordinates proposed in the section 5.5. This procedure reveals that the effective twisted superpotential is naturally identified with the generating function of the variety of opers. The conclusions and discussions are presented in the section 5.7. The appendices contain some computational details.

5.2 Surface defects

We start on the Hitchin system side. We will mainly consider the four-punctured Riemann sphere $\mathcal{C} = \mathbb{P}^1 \backslash \{0, \mathfrak{q}, 1, \infty\}$. All the punctures are assumed to be *regular*. That is, we only allow a simple pole for the Higgs field φ at each puncture. Moreover, we call a puncture *maximal* when the residue of φ at the puncture belongs to a generic semisimple conjugacy class of $\mathfrak{g} = A_{N-1}$, and *minimal* when the residue is in a maximally degenerate semisimple conjugacy class (as in [107, 108]). We assume the punctures at 0 and ∞ are maximal (this is the typical limit of a Hitchin system on a stably degenerate curve, see [109]), while the punctures at $\mathfrak{q}$ and 1 are minimal. In what follows in listing the punctures we underline the minimal ones, as in $\{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$. We shall also denote by $\underline{\mathcal{C}}$ the punctured Riemann surface together with the assignment of the minimal and maximal punctures, e.g. $\underline{\mathcal{C}} = \mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$. There is no distinction between the maximal and the minimal punctures in

[4] There are alternative approaches to the construction of Darboux coordinates from spectral networks [101], motivated by the work of A. Voros [102] on the exact WKB approximation, and from symplectic doubles [103], motivated by the work of W. Thurston on the measured laminations. The spectral networks were used in [101, 104, 103, 98] to generalize Fock-Goncharov [105], NRS [91], Goldman [86] and Fenchel-Nielsen [93] coordinates. We stress that we only work on an open subset of the moduli space, so the subtleties discussed in [105, 106, 103], forcing one to work on certain covers of the moduli space, are not visible at the level we are working.

the $N = 2$ case. For $N > 2$ the difference is significant. The corresponding class $\mathcal{S}$ theory $\mathcal{T}[A_{N-1}, \underline{\mathcal{C}}]$ is the superconformal $\mathcal{N} = 2$ gauge theory with the gauge group $SU(N)$ and the $2N$ hypermultiplets, whose gauge coupling is $\mathfrak{q}$ and the masses of the hypermultiplets are determined by specific combinations of the eigenvalues of the residue of φ [33].

A half-BPS surface defect on $\mathcal{T}[A_{N-1}, \underline{\mathcal{C}}]$ can be constructed in several ways. Here we present two constructions relevant to our study, which were reviewed in 2.3. It is convenient to treat the gauge group formally as $U(N)$, by making an overall shift in the masses of the hypermultiplets, as we do throughout the discussion. The $SU(N)$ gauge theory parameters can be easily recovered by shifting back the Coulomb moduli and the masses of hypermultiplets.

5.2.1 The quiver construction

The construction starts with the superconformal A_2-quiver $U(N)$ gauge theory. As reviewed in appendix 2.1 in detail, the equivariant localization reduces the instanton partition function of the theory to that of a grand canonical ensemble on the $2N$-tuples of Young diagrams $\boldsymbol{\lambda} = \{\lambda^{(\mathbf{i},\alpha)} \mid \mathbf{i} = 1, 2,\ \alpha = 1, \cdots, N\}$. It can be conveniently written as

$$\mathcal{Z}_{A_2}\left(\mathbf{a}_0; \mathbf{a}_1; \mathbf{a}_2; \mathbf{a}_3 | \varepsilon_1, \varepsilon_2 | \mathfrak{q}_1, \mathfrak{q}_2\right) = \sum_{\boldsymbol{\lambda}} \prod_{\mathbf{i}=1,2} \mathfrak{q}_{\mathbf{i}}^{|\boldsymbol{\lambda}^{(\mathbf{i})}|}\, \epsilon\left[\mathcal{T}_{A_2}[\boldsymbol{\lambda}]\right] \tag{5.2.1}$$

where the character $\mathcal{T}_{A_2}$ is

$$\begin{aligned}\mathcal{T}_{A_2} = \sum_{\mathbf{i}=1,2} \left(N_{\mathbf{i}} K_{\mathbf{i}}^* + q_{12} N_{\mathbf{i}}^* K_{\mathbf{i}} - P_{12} K_{\mathbf{i}} K_{\mathbf{i}}^*\right) - M_0 K_1^* - q_{12} M_3^* K_2 \\ - N_1 K_2^* - q_{12} N_2^* K_1 + P_{12} K_1 K_2^*,\end{aligned} \tag{5.2.2}$$

and the ϵ-operation[5], also known as the plethystic exponent, converts a character into the product of weights,

$$\epsilon(R) = \frac{\prod_{w\in R^-} w(\theta)}{\prod_{w\in R^+} w(\theta)} \quad \text{for} \quad \theta \in \mathrm{Lie}(T_H), \quad R = \sum_{w\in R^+} e^{w(\theta)} - \sum_{w\in R^-} e^{w(\theta)}. \tag{5.2.3}$$

Let us choose $\beta \in \{1, \cdots, N\}$, and tune the Coulomb moduli of the first gauge node as

$$\begin{cases} a_{1,\beta} = a_{0,\beta} - \varepsilon_2 \\ a_{1,\alpha} = a_{0,\alpha} & \text{for } \alpha \neq \beta \end{cases} \tag{5.2.4}$$

We define the defect partition function as $\mathcal{Z}_{A_2}$ with the constrained Coulomb parameters:

$$\mathcal{Z}^L_\beta \equiv \mathcal{Z}_{A_2}\left(\mathbf{a}_0;\, a_{1,\alpha} = a_{0,\alpha} - \varepsilon_2 \delta_{\alpha,\beta};\, \mathbf{a}_2;\, \mathbf{a}_3 \mid \varepsilon_1, \varepsilon_2 \mid \mathfrak{q}_1 = z^{-1}, \mathfrak{q}_2 = \mathfrak{q}\right). \tag{5.2.5}$$

The constraints can be succintly expressed as the relation between the characters

$$M_0 = N_1 - P_2 \mu, \tag{5.2.6}$$

where we have defined $\mu = e^{\beta(a_{0,\beta} - \varepsilon_2)}$. Note that due to the constraints, almost all the Young diagrams for the first gauge node have vanishing contributions to the partition function, except the ones of the form

$$\boldsymbol{\lambda}^{(1)} = \left(\varnothing, \cdots, \varnothing, \left.\begin{array}{c}\square\\\square\\\square\\\square\\\vdots\\\square\end{array}\right\} k\ , \varnothing, \cdots, \varnothing \right), \tag{5.2.7}$$

which is empty $\lambda^{(1,\alpha)} = \varnothing$ except the single-columned $\lambda^{(1,\beta)}$.

We can view the constraint (5.2.4) as adding an extra equation in the ADHM construction for the quiver instanton moduli space, as we now recall. First, the A_2-quiver $U(N)$ theory

[5] Not to be confused with the Ω-deformation parameters $\varepsilon_1, \varepsilon_2$.

can be obtained by the $\mathbb{Z}_4$-orbifold procedure from the $\mathcal{N} = 2^*$ $U(4N)$ theory. The ADHM data for the $\mathcal{N} = 2^*$ $U(4N)$ gauge theory is the following collection of linear maps between complex vector spaces:

$$\begin{aligned} &B_{1,2,3,4} : K \longrightarrow K \\ &I : N \longrightarrow K \\ &J : K \longrightarrow N, \end{aligned} \tag{5.2.8}$$

where $N = \mathbb{C}^{4N}$ and $K = \mathbb{C}^{k_1+k_2}$. The reason for strange dimensions of these spaces will become clear momentarily. The extended ADHM equations are written as [14]

$$\begin{aligned} &[B_1, B_2] + IJ + [B_3, B_4]^\dagger = 0 \\ &[B_1, B_3] + [B_4, B_2]^\dagger = 0 \\ &[B_1, B_4] + [B_2, B_3]^\dagger = 0 \\ &s^+ \equiv B_3 I + (JB_4)^\dagger = 0 \\ &s^- \equiv B_4 I - (JB_3)^\dagger = 0. \end{aligned} \tag{5.2.9}$$

We also impose the stability condition (cf. (2.1.25))

$$\mathbb{C}[B_1, B_2, B_3, B_4]I(N) = K. \tag{5.2.10}$$

Upon the $\mathbb{Z}_4$-orbifolding, the spaces N and K become $\mathbb{Z}_4$-modules, and therefore can be decomposed according to the $\mathbb{Z}_4$-representations

$$N = \bigoplus_{\omega \in \mathbb{Z}_4} N_\omega \otimes \mathcal{R}_\omega, \quad K = \bigoplus_{\omega \in \mathbb{Z}_4} K_\omega \otimes \mathcal{R}_\omega. \tag{5.2.11}$$

The coupling constant is also fractionalized accordingly, $\mathfrak{q}_\omega$ for $\omega \in \mathbb{Z}_4$. We manually set $\mathfrak{q}_0 = \mathfrak{q}_3 = 0$, then we are restricted to $K_0 = K_3 = 0$ due to the measure factor $\mathfrak{q}_\omega^{|K_\omega|}$. Let $|K_1| = k_1$ and $|K_2| = k_2$. Also, we impose the $\mathbb{Z}_4$-weights to the space N in such a way that

$N_\omega = \mathbb{C}^N$ for each $\omega \in \mathbb{Z}_4$. Let the maps

$$\Omega_N : N \longrightarrow N, \quad \Omega_K : K \longrightarrow K, \tag{5.2.12}$$

be defined by the diagonal action of i^ω to the elements in N_ω and K_ω. Then we impose the conditions for the ADHM data

$$\begin{aligned}
&\Omega_K^{-1}\, B_{1,2}\, \Omega_K = B_{1,2} \\
&\Omega_K^{-1}\, B_3\, \Omega_K = iB_3 \\
&\Omega_K^{-1}\, B_4\, \Omega_K = -iB_4 \\
&\Omega_K^{-1}\, I\, \Omega_N = I \\
&\Omega_N^{-1}\, J\, \Omega_K = J,
\end{aligned} \tag{5.2.13}$$

which fractionalize these matrices as

$$\begin{aligned}
&B_{\omega,1} : K_\omega \longrightarrow K_\omega \\
&B_{\omega,2} : K_\omega \longrightarrow K_\omega \\
&B_{\omega,3} : K_\omega \longrightarrow K_{\omega+1} \\
&B_{\omega,4} : K_\omega \longrightarrow K_{\omega-1} \\
&I_\omega : N_\omega \longrightarrow K_\omega \\
&J_\omega : K_\omega \longrightarrow N_\omega.
\end{aligned} \tag{5.2.14}$$

Note that many of these maps are identically zero due to the restriction $K_0 = K_3 = 0$. Hence only the following equations survive among the ADHM equations (5.2.9),

$$
\begin{aligned}
&[B_{1,1}, B_{1,2}] + I_1 J_1 - B_{1,3}^\dagger B_{2,4}^\dagger = 0 \\
&[B_{2,1}, B_{2,2}] + I_2 J_2 - B_{2,4}^\dagger B_{1,3}^\dagger = 0 \\
&B_{2,1} B_{1,3} - B_{1,3} B_{1,1} + B_{2,2}^\dagger B_{2,4}^\dagger - B_{2,4}^\dagger B_{1,2}^\dagger = 0 \\
&B_{1,1} B_{2,4} - B_{2,4} B_{2,1} + B_{1,3}^\dagger B_{2,2}^\dagger - B_{1,2}^\dagger B_{1,3}^\dagger = 0 \\
&s_1^+ \equiv B_{1,3} I_1 + B_{2,4}^\dagger J_1^\dagger = 0 \\
&s_2^- \equiv B_{2,4} I_2 - B_{1,3}^\dagger J_2^\dagger = 0.
\end{aligned}
\tag{5.2.15}
$$

The stability condition also becomes

$$
\mathbb{C}[B_{1,1}, B_{1,2}, B_{2,1}, B_{2,2}, B_{1,3}, B_{2,4}] I(N) = K. \tag{5.2.16}
$$

We find that the sum of the squares of the norms of the first two equations of (5.2.15) can be simplified, using the other four equations, into a sum of squares,

$$
\begin{aligned}
0 = &||[B_{1,1}, B_{1,2}] + I_1 J_1||^2 + ||[B_{2,1}, B_{2,2}] + I_2 J_2||^2 + ||B_{1,3} I_1||^2 + ||B_{2,4} I_2||^2 \\
&+ ||B_{2,1} B_{1,3} - B_{1,3} B_{1,1}||^2 + ||B_{1,1} B_{2,4} - B_{2,4} B_{2,1}||^2.
\end{aligned}
\tag{5.2.17}
$$

Applying the last two equations to the stability condition, we can commute $B_{1,3}$ and $B_{2,4}$ through all the way to hit $I_1(N_1)$ or $I_2(N_2)$, respectively. This vanishes as a result of the third and the fourth equations. Hence, the stability condition is reduced to

$$
\mathbb{C}[B_{\mathbf{i},1}, B_{\mathbf{i},2}] I_{\mathbf{i}}(N_{\mathbf{i}}) = K_{\mathbf{i}}, \quad \mathbf{i} = 1, 2. \tag{5.2.18}
$$

This implies $B_{1,3} = B_{2,4} = 0$. The first and the second equations of (5.2.17) provide the reduced ADHM equations

$$
[B_{\mathbf{i},1}, B_{\mathbf{i},2}] + I_{\mathbf{i}} J_{\mathbf{i}} = 0, \quad \mathbf{i} = 1, 2, \tag{5.2.19}
$$

which are precisely the ADHM equations for the instanton moduli space of the A_2-quiver $U(N)$ theory.

In this construction of the A_2-theory, the constraint (5.2.4) can be understood as adding an equation "s_0^+" : $N_0 \longrightarrow K_1$. Note that we neglected the equation

$$s_0^+ \equiv B_{0,3} I_0 + B_{1,4}^\dagger J_0^\dagger : N_0 \longrightarrow K_1, \tag{5.2.20}$$

since it is identically zero by $B_{0,3} = I_0 = B_{1,4} = J_0 = 0$ due to the restriction $K_0 = 0$. However, we can avoid this restriction if we first set

$$N_0 = \widetilde{N} \oplus L, \quad N_1 = \widetilde{N} \oplus q_2 L, \tag{5.2.21}$$

where we have chosen an one-dimensional subspace $L \subset N_0$, which corresponds the choice of $\beta \in \{1, \cdots, N\}$ in the constraint (5.2.4). Then we may define a non-vanishing map

$$s_0^+ = I_1\Big|_{\widetilde{N}} \oplus B_{1,2} I_1\Big|_L : N_0 \longrightarrow K_1. \tag{5.2.22}$$

Adding the equation $s_0^+ = 0$ to the ADHM construction, we find that the space K_1 is further restricted by the stability condition (5.2.18),

$$K_1 = \mathbb{C}[B_{1,1}, B_{1,2}]\, I_1(N_1) = \mathbb{C}[B_{1,1}]\, I_1(L). \tag{5.2.23}$$

In other words, the Young diagram that denotes the space K_1 only grows in one direction from the chosen basis vector $I_1(L)$. This exactly manifests the single-columnedness expressed in (5.2.7). The physics picture of what is happening is the following. The constraint (5.2.4) makes N hypermultiplets nearly massless (exactly massless in the absence of Ω-deformation). The theory can then go to the Higgs branch, where the gauge group is partially Higgsed to a subgroup, by the expectation values of the hypermultiplet scalars. Now, the theory allows

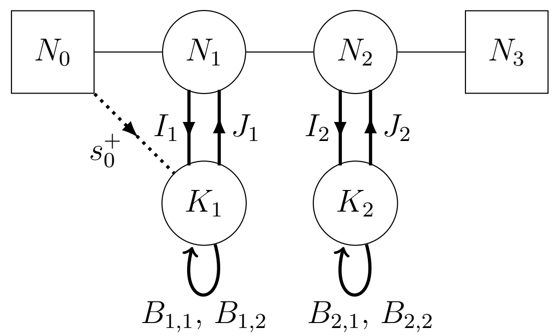

Figure 5.1: The ADHM data for the A_2-quiver gauge theory, with the extra map s_0^+.

for the half-BPS field configurations where the gauge group is restored along a codimension two defect, essentially a vortex string. Consequently, the gauge field configuration of the first gauge node is squeezed into a two-dimensional plane (ε_1-plane), effectively forming a vortex. The resulting two-dimensional supersymmetric sigma model couples to the remaining four-dimensional A_1-theory, generating a surface defect in the four-dimensional point of view.

We can confirm that the 2d-4d coupled system arises at the level of the partition function. First we have the simplified expression for

$$K_1 = \mu \frac{1-q_1^k}{1-q_1}. \tag{5.2.24}$$

Therefore, the character (5.2.2) can also be simplified into

$$\mathcal{T}_{A_2} = [N_2 K_2^* + q_{12} N_2^* K_2 - P_{12} K_2 K_2^* - N_1 K_2^* - q_{12} M_3^* K_2] + \left[P_2 \mu q_1^k K_1^* + q_{12} K_1 (N_1^* - S_2^*) \right]. \tag{5.2.25}$$

Accordingly, the partition function (5.2.1) of the A_2-quiver gauge theory is reduced to the expectation value of an observable in the A_1-quiver gauge theory

$$\mathcal{Z}_\beta^L = \sum_{\boldsymbol{\lambda}^{(2)}} \mathfrak{q}_2^{|\boldsymbol{\lambda}^{(2)}|} \mathcal{I}_\beta^L[\boldsymbol{\lambda}^{(2)}] \, \epsilon \left[\mathcal{T}_{A_1}[\boldsymbol{\lambda}^{(2)}] \right] = \langle \mathcal{I}_\beta^L \rangle \, \mathcal{Z}_{A_1}, \tag{5.2.26}$$

where we have defined the character for the A_1-theory

$$\mathcal{T}_{A_1} \equiv N_2 K_2^* + q_{12} N_2^* K_2 - P_{12} K_2 K_2^* - N_1 K_2^* - q_{12} M_3^* K_2, \tag{5.2.27}$$

which defines the instanton partition function of the A_1-theory by

$$\mathcal{Z}_{A_1} \equiv \sum_{\boldsymbol{\lambda}^{(2)}} \mathfrak{q}_2^{|\boldsymbol{\lambda}^{(2)}|} \epsilon \left[\mathcal{T}_{A_1}[\boldsymbol{\lambda}^{(2)}] \right], \tag{5.2.28}$$

and the surface defect as an element of the chiral ring

$$\begin{aligned} \mathcal{I}_\beta^L[\boldsymbol{\lambda}^{(2)}] &\equiv \sum_{k=0}^{\infty} \mathfrak{q}_1^k \, \epsilon \left[P_2 \mu q_1^k K_1^* + q_{12} K_1 (N_1^* - S_2^*) \right] \\ &= \sum_{k=0}^{\infty} \mathfrak{q}_1^k \, \epsilon \left[\sum_{l=1}^{k} q_1^l (P_2 + \mu q_2 (N_1^* - S_2^*)) \right] \\ &= \sum_{k=0}^{\infty} \mathfrak{q}_1^k \prod_{l=1}^{k} \frac{\mathcal{Y}_2(a_{0,\beta} + l\varepsilon_1)[\boldsymbol{\lambda}^{(2)}]}{P_0(a_{0,\beta} + l\varepsilon_1)}. \end{aligned} \tag{5.2.29}$$

Here, we have used the $\mathcal{Y}$-observable (2.1.42) for the second gauge node, and $P_0(x) \equiv \prod_{\alpha=1}^{N}(x - a_{0,\alpha})$ by definition. Let us focus on the zero bulk instanton sector, $|\boldsymbol{\lambda}^{(2)}| = 0$. The contribution of this sector is the vortex partition function of a two-dimensional gauged linear sigma model. This sigma model generates the surface defect, when coupled to the four-dimensional bulk [19, 99]. The $\mathcal{Y}$-observable in this sector simply reduces to a polynomial $\mathcal{Y}_2(x) \to A_2(x) \equiv \prod_{\alpha=1}^{N}(x - a_{2,\alpha})$. The partition function (5.2.29) is exactly that of the gauged linear sigma model on the $\mathrm{Hom}(\mathcal{O}(-1), \mathbb{C}^N)$-bundle over $\mathbb{P}^{N-1}$ whose Kähler modulus is $\mathfrak{q}_1$ [21]. For the non-trivial sectors of the four-dimension, the two-dimensional sigma model couples to the four-dimensional gauge theory through the non-perturbative corrections to the $\mathcal{Y}$-observable. Thus, the full partition function (5.2.26) represents the 2d-4d coupled system in this manner.

It is instructive to cast the surface defect partition function (5.2.29) into the form relevant to our study. Recall that the $\mathcal{Y}$-observable (2.1.42) can be written as a ratio

$$\mathcal{Y}_{\mathbf{i}}(x)[\boldsymbol{\lambda}] = \prod_{\alpha=1}^{N} \frac{\prod_{\square\in\partial_+\lambda^{(\mathbf{i},\alpha)}}(x-c_\square)}{\prod_{\square\in\partial_-\lambda^{(\mathbf{i},\alpha)}}(x-c_\square-\varepsilon)}. \tag{5.2.30}$$

This suggests to represent the $\mathcal{Y}$-observable as a ratio of two entire functions [10],

$$\mathcal{Y}_{\mathbf{i}}(x) = \frac{\mathcal{Q}_{\mathbf{i}}(x)}{\mathcal{Q}_{\mathbf{i}}(x-\varepsilon_1)}, \tag{5.2.31}$$

where we have defined the $\mathcal{Q}$-observable

$$\mathcal{Q}_{\mathbf{i}}(x)[\boldsymbol{\lambda}] \equiv \prod_{\alpha=1}^{N} \left(\frac{(-\varepsilon_1)^{\frac{x-a_{\mathbf{i},\alpha}}{\varepsilon_1}}}{\Gamma\left(-\frac{x-a_{\mathbf{i},\alpha}}{\varepsilon_1}\right)} \prod_{\square\in\lambda^{(\mathbf{i},\alpha)}} \frac{x-c_\square-\varepsilon_2}{x-c_\square} \right). \tag{5.2.32}$$

Therefore, the surface defect (5.2.29) can be understood as an infinite sum of $\mathcal{Q}$-observables,

$$\mathcal{I}^L_\beta[\boldsymbol{\lambda}^{(2)}] = \sum_{k=0}^{\infty} \mathfrak{q}_1^k \left(\prod_{\alpha=1}^{N} \frac{\Gamma\left(1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)}{\varepsilon_1^k\,\Gamma\left(k+1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)} \right) \frac{\mathcal{Q}_2(a_{0,\beta}+k\varepsilon_1)[\boldsymbol{\lambda}^{(2)}]}{\mathcal{Q}_2(a_{0,\beta})[\boldsymbol{\lambda}^{(2)}]}. \tag{5.2.33}$$

We will observe that the $\mathcal{Q}$-observable reduces to the so-called Baxter $\mathcal{Q}$-function in the Nekrasov-Shatashvili limit. It will be more apparent in section 5.3 that this representation is useful for our purpose.

Likewise, we can similarly impose the constraints for the Coulomb moduli in the second gauge node,

$$\begin{cases} a_{2,\beta} = a_{3,\beta} - \varepsilon - \varepsilon_2 \\ a_{2,\alpha} = a_{3,\alpha} - \varepsilon \qquad\qquad \alpha \neq \beta, \end{cases} \tag{5.2.34}$$

for some chosen $\beta \in \{1, \cdots, N\}$. Here, we are using the abbreviated notation $\varepsilon \equiv \varepsilon_1 + \varepsilon_2$.

The constraint can also be written as

$$q_{12}^{-1} M_3 = N_2 - P_2 \mu, \tag{5.2.35}$$

where now $\mu = e^{\beta(a_{3,\beta} - \varepsilon - \varepsilon_2)}$. For a reason that will be clarified in section 5.4.2, we make the following re-definition for the parameters after imposing the constraints (5.2.35),

$$\begin{aligned} a_{0,\alpha} &\longrightarrow -a_{0,\alpha} - \varepsilon \\ a_{1,\alpha} &\longrightarrow -a_{1,\alpha} \qquad\qquad\qquad \alpha = 1, \cdots, N \\ a_{3,\alpha} &\longrightarrow -a_{3,\alpha} + 2\varepsilon, \end{aligned} \tag{5.2.36}$$

The corresponding partition function,

$$\mathcal{Z}_\beta^R \equiv \mathcal{Z}_{A_2} \left(-a_{0,\alpha} - \varepsilon;\ -a_{1,\alpha};\ -a_{3,\alpha} + \varepsilon - \varepsilon_2 \delta_{\alpha,\beta};\ -a_{3,\alpha} + 2\varepsilon \mid \varepsilon_1, \varepsilon_2 \mid \mathfrak{q}_1 = \mathfrak{q}, \mathfrak{q}_2 = \mathfrak{q}^{-1} z \right), \tag{5.2.37}$$

can be likewise simplified to:

$$\mathcal{Z}_\beta^R = \sum_{\boldsymbol{\lambda}^{(1)}} \mathfrak{q}_1^{|\boldsymbol{\lambda}^{(1)}|} \mathcal{I}_\beta^R[\boldsymbol{\lambda}^{(1)}]\, \epsilon \left[\mathcal{T}_{A_1}[\boldsymbol{\lambda}^{(1)}] \right], \tag{5.2.38}$$

where the character for the A_1-theory is now

$$\mathcal{T}_{A_1} \equiv N_1 K_1^* + q_{12} N_1^* K_1 - P_{12} K_1 K_1^* - M_0 K_1^* - q_{12} N_2^* K_1, \tag{5.2.39}$$

and the surface defect is

$$\begin{aligned} \mathcal{I}_\beta^R[\boldsymbol{\lambda}^{(1)}] &\equiv \sum_{k=0}^{\infty} \mathfrak{q}_2^k \prod_{l=1}^{k} \frac{\mathcal{Y}_1(-a_{3,\beta} + l\varepsilon_1)}{P_3(-a_{3,\beta} + 2\varepsilon + l\varepsilon)} \\ &= \sum_{k=0}^{\infty} \mathfrak{q}_2^k \prod_{\alpha=1}^{N} \left(\frac{\Gamma\left(1 + \frac{a_{3,\alpha} - a_{3,\beta}}{\varepsilon_1}\right)}{\varepsilon_1^k\, \Gamma\left(k + 1 + \frac{a_{3,\alpha} - a_{3,\beta}}{\varepsilon_1}\right)} \right) \frac{\mathcal{Q}_1(-a_{3,\beta} + k\varepsilon_1)}{\mathcal{Q}_1(-a_{3,\beta})}, \end{aligned} \tag{5.2.40}$$

where we have used the $\mathcal{Y}$-observable for the first gauge node and $P_3(x) \equiv \prod_{\alpha=1}^{N}(x+a_{3,\alpha}-2\varepsilon)$ (Be cautious about the re-definition of the parameters). Also, the $\mathcal{Y}$-observable has been replaced by a ratio of $\mathcal{Q}$-observables in the second line. Note that the bulk coupling is now $\mathfrak{q}_1$, while the Kähler modulus for the two-dimensional sigma model is $\mathfrak{q}_2$. Thus it is natural to expect that the $\mathfrak{q}_2$ of $\mathcal{Z}^L_\beta$ would correspond to $\mathfrak{q}_1$ of $\mathcal{Z}^R_\beta$, when we try to connect these partition functions. The issue will be clarified in section 5.4.

5.2.2 The orbifold construction

We construct a surface defect by placing the gauge theory on an orbifold. We first form an orbifold $\mathbb{C}_{\varepsilon_1} \times (\mathbb{C}_{\varepsilon_2}/\mathbb{Z}_p)$ by the following $\mathbb{Z}_p$-action on $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\varepsilon_2}$

$$\zeta : (z_1, z_2) \longmapsto (z_1, \zeta z_2), \quad \zeta \equiv \exp\left(\frac{2\pi i}{p}\right) \in \mathbb{Z}_p. \tag{5.2.41}$$

Here, $\mathbb{C}_{\varepsilon_i}$ denotes the complex plane with the equivariant paramter ε_i for the $\mathbb{C}^\times$-action. Then the surface defect is constructed as a prescription of performing the path integral only over the $\mathbb{Z}_p$-invariant field configurations. Indeed, under the map $(z_1, z_2) \mapsto (\tilde{z}_1 \equiv z_1, \tilde{z}_2 \equiv z_2^p)$, the orbifold $\mathbb{C}_{\varepsilon_1} \times (\mathbb{C}_{\varepsilon_2}/\mathbb{Z}_p)$ is mapped to $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{p\varepsilon_2}$, and the field configurations are allowed to be singular along the surface $\tilde{z}_2 = 0$. Therefore, the resulting theory on $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{p\varepsilon_2}$ can be interpreted as a surface defect inserted upon the underlying gauge theory.

To fully characterize the surface defect, we have to specify how the field configurations are projected out by the $\mathbb{Z}_p$-action. We present here how this is done for the A_1-theory. Let us introduce the coloring function

$$c : [N] \longrightarrow \mathbb{Z}_p. \tag{5.2.42}$$

Then the space N is decomposed according to the $\mathbb{Z}_p$-representations,

$$N = \bigoplus_{\omega \in \mathbb{Z}_p} N_\omega \otimes \mathcal{R}_\omega, \quad N_\omega \equiv \sum_{\alpha \in c^{-1}(\omega)} e^{\beta a_\alpha}. \tag{5.2.43}$$

Also,

$$K = \bigoplus_{\omega \in \mathbb{Z}_p} K_\omega \otimes \mathcal{R}_\omega, \quad K_\omega \equiv \sum_{\alpha=1}^{N} e^{\beta a_\alpha} \sum_{i=1}^{l\left(\lambda^{(\alpha)}\right)} q_1^{i-1} \sum_{\substack{1 \leq j \leq \lambda_i^{(\alpha)} \\ c(\alpha)+j-1 \equiv \omega \bmod p}} q_2^{j-1}, \tag{5.2.44}$$

where $\mathcal{R}_\omega$ is the one-dimensional irreducible representation of $\mathbb{Z}_p$ with the weight ω, and $l\left(\lambda^{(\alpha)}\right) = \lambda_1^{(\alpha)\,t}$ is the number of rows in the Young diagram $\lambda^{(\alpha)}$. It is straightforward to include the fundamental matter fields, namely,

$$M = \bigoplus_{\omega \in \mathbb{Z}_p} M_\omega \otimes \mathcal{R}_\omega. \tag{5.2.45}$$

Now as explained, we perform the path integral only for the $\mathbb{Z}_p$-invariant field configurations, projecting out the non-invariant contributions. At the level of the character (see (2.1.32)), this is to pick up the $\mathbb{Z}_p$-invariant piece, namely,

$$\mathcal{T}^{\mathbb{Z}_p} \equiv \left[\frac{1}{P_{12}}\left(-SS^* + M^* S\right)\right]^{\mathbb{Z}_p}, \tag{5.2.46}$$

from which the partition function is given by

$$\mathcal{Z}^{\mathbb{Z}_p, c} \equiv \sum_{\boldsymbol{\lambda}} \mathfrak{q}_\omega^{|K_\omega|}\, \epsilon\left[\mathcal{T}^{\mathbb{Z}_p}[\boldsymbol{\lambda}]\right]. \tag{5.2.47}$$

Though providing a concrete formula, it is not so obvious from (5.2.47) that the partition function can be interpreted as an insersion of an observable in the A_1-theory. Thus it is important to properly construct the projection of the set of N-tuples of Young diagrams

$$\rho : \boldsymbol{\lambda} \longmapsto \boldsymbol{\Lambda}, \tag{5.2.48}$$

where $\boldsymbol{\Lambda}$ is supposed to enumerates the fixed points of the instanton moduli space of the A_1-theory on $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{p\varepsilon_2} = \mathbb{C}^2/\mathbb{Z}_p$.

The construction of the map ρ can be done as follows. Let us first re-define the Coulomb moduli by the shift

$$\widetilde{a}_\alpha \equiv a_\alpha - \varepsilon_2\, c(\alpha), \tag{5.2.49}$$

so that

$$\widetilde{N}_\omega \equiv \sum_{\alpha \in c^{-1}(\omega)} e^{\beta \widetilde{a}_\alpha}, \quad \widetilde{N} \equiv \sum_{\omega=0}^{p-1} \widetilde{N}_\omega. \tag{5.2.50}$$

and

$$\widetilde{K}_\omega \equiv K_\omega q_2^{-\omega} = \sum_{\alpha=1}^{N} e^{\beta \widetilde{a}_\alpha} \sum_{i=1}^{l\left(\lambda^{(\alpha)}\right)} q_1^{i-1} \sum_{j=1 \text{ or } 2}^{l_{i,\alpha,\omega}} \widetilde{q}_2^{\,j-1}, \tag{5.2.51}$$

where

$$l_{i,\alpha,\omega} = \left[\frac{\lambda_i^{(\alpha)} + c(\alpha) - \omega + p - 1}{p} \right] \tag{5.2.52}$$

$$\widetilde{q}_2 \equiv q_2^p \quad (\widetilde{\varepsilon}_2 \equiv p\varepsilon_2), \tag{5.2.53}$$

and the lower limit of the sum over j is equal to 1 for $c(\alpha) \leq \omega$ and 2 for $c(\alpha) > \omega$. In particular, for $\omega = p-1$,

$$\widetilde{K} \equiv \widetilde{K}_{p-1} = \sum_{\alpha=1}^{N} e^{\beta \widetilde{a}_\alpha} \sum_{i=1}^{l\left(\Lambda^{(\alpha)}\right)} q_1^{i-1} \sum_{j=1}^{\Lambda_i^{(\alpha)}} \widetilde{q}_2^{\,j-1}, \tag{5.2.54}$$

where we have defined a new N-tuple of Young diagrams $\mathbf{\Lambda} = \left(\Lambda^{(\alpha)}\right)_{\alpha=1}^{N}$ by

$$\Lambda_i^{(\alpha)} = l_{i,\alpha,p-1} \equiv \left[\frac{\lambda_i^{(\alpha)} + c(\alpha)}{p}\right]. \tag{5.2.55}$$

The map ρ is defined by this relation, $\rho(\boldsymbol{\lambda}) = \mathbf{\Lambda}$.

The partition function (5.2.47) is a weighted sum over $\boldsymbol{\lambda}$, which can be first summed over $\rho^{-1}(\mathbf{\Lambda})$ for fixed $\mathbf{\Lambda}$ and then summed over $\mathbf{\Lambda}$. We will show the sum over $\rho^{-1}(\mathbf{\Lambda})$ provides an observable insertion to the A_1-theory whose measure for the partition function is given by $\mathbf{\Lambda}$. First, the vector multiplet contribution in the measure (5.2.46) is

$$-\sum_{\omega,\omega',\omega''=0}^{p-1} \frac{S_\omega S_{\omega'}^*}{P_1(1-\widetilde{q}_2)} q_2^{\omega''} \delta^{\mathbb{Z}_p}_{\omega-\omega'+\omega''}, \tag{5.2.56}$$

where we have used the identity

$$\frac{1}{1-q_2\mathcal{R}_1} = \frac{1}{1-\widetilde{q}_2} \sum_{\omega=0}^{p-1} q_2^{\omega}\, \mathcal{R}_\omega. \tag{5.2.57}$$

After defining

$$\widetilde{S}_\omega \equiv S_\omega q_2^{-\omega}, \quad \widetilde{S} \equiv \sum_{\omega=0}^{p-1} \widetilde{S}_\omega = \widetilde{N} - P_1\widetilde{P}_2\widetilde{K}, \tag{5.2.58}$$

the character (5.2.56) can be written as

$$-\frac{\widetilde{S}\widetilde{S}^*}{\widetilde{P}_{12}} + \frac{1}{P_1} \sum_{0\le\omega<\omega'<p} \widetilde{S}_{\omega'}\widetilde{S}_{\omega}^*. \tag{5.2.59}$$

Note that the first term is precisely the vector multiplet contribution to the partition function of the A_1-theory on $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\widetilde{\varepsilon}_2}$ in the $\widetilde{k}$-instanton sector. The second term is interpreted as an observable insertion to this theory. We can further simplify it by introducing

$$\Sigma_\omega \equiv \widetilde{N}_0 + \cdots + \widetilde{N}_{\omega-1} - P_1 \widetilde{K}_{\omega-1} + \widetilde{q}_2 P_1 \widetilde{K}, \quad \omega = 1, \cdots, p. \tag{5.2.60}$$

The second term in (5.2.59) is now given by

$$\frac{1}{P_1} \sum_{\omega=1}^{p-1} (\Sigma_{\omega+1} - \Sigma_\omega) \Sigma_\omega^*. \tag{5.2.61}$$

Similarly, the matter contribution in the measure (5.2.46) can be written as

$$\sum_{\omega,\omega',\omega''=0}^{p-1} \frac{M_{\omega'}^* S_\omega}{P_1(1-\widetilde{q}_2)} q_2^{\omega''} \delta^{\mathbb{Z}_p}_{\omega-\omega'+\omega''}. \tag{5.2.62}$$

After defining $\widetilde{M}_\omega \equiv M_\omega q_2^{-\omega}$ and $\widetilde{M} \equiv \widetilde{q}_2^{-1} \sum_{\omega=0}^{p-1} \widetilde{M}_\omega$, it can be re-expressed as

$$\frac{\widetilde{M}^* \widetilde{S}}{\widetilde{P}_{12}} + \frac{1}{P_1} \sum_{\omega=0}^{p-1} \widetilde{M}_\omega^* \Sigma_{\omega+1} \tag{5.2.63}$$

Note that the first term is the usual fundamental matter contribution to the measure of the A_1-theory on $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\widetilde{\varepsilon}_2}$ in the $\widetilde{k}$-instanton sector. The second term is interpreted as an observable insertion to the theory.

We also introduce the auxiliary variables (z_ω) and $\mathfrak{q}$ to express the fractionalized couplings,

$$\begin{aligned} \mathfrak{q}_\omega &\equiv \frac{z_{\omega+1}}{z_\omega}, \quad \omega = 0, \cdots, p-2, \\ \mathfrak{q}_{p-1} &= \mathfrak{q} \frac{z_0}{z_{p-1}}, \end{aligned} \tag{5.2.64}$$

so that

$$\prod_{\omega=0}^{p-1} \mathfrak{q}_\omega = \mathfrak{q}. \tag{5.2.65}$$

Note that $\mathfrak{q}$ is weighted with the power $\widetilde{k}$ in the measure and therefore is the bulk coupling of the A_1-theory on $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\widetilde{\varepsilon}_2}$.

As a result, the full partition function (5.2.47) can be written as

$$\mathcal{Z}^{\mathbb{Z}_p,c} = \sum_{\mathbf{\Lambda}} \mathfrak{q}^{|\mathbf{\Lambda}|} \, \mathcal{I}_c[\mathbf{\Lambda}] \, \epsilon \left[\frac{1}{\widetilde{P}_{12}} \left(-\widetilde{S}\widetilde{S}^* + \widetilde{M}^* \widetilde{S} \right) \right] = \langle \mathcal{I}_c \rangle \, \mathcal{Z}_{A_1}, \tag{5.2.66}$$

where the surface defect is expressed as a chiral ring element

$$\mathcal{I}_c[\mathbf{\Lambda}] \equiv \sum_{\boldsymbol{\lambda} \in \rho^{-1}(\mathbf{\Lambda})} \prod_{\omega=0}^{p-1} z_\omega^{k_{\omega-1}-k_\omega} \, \epsilon \left[\frac{1}{P_1} \left(\sum_{\omega=1}^{p-1} \left((\widetilde{M}_{\omega-1} - \Sigma_\omega)^* \Sigma_\omega + \Sigma_{\omega+1} \Sigma_\omega^* \right) + \widetilde{M}_{p-1}^* \widetilde{S} \right) \right]. \tag{5.2.67}$$

Let us focus on the zero-instanton sector, $|\mathbf{\Lambda}| = \widetilde{k} = 0$. An element of the inverse image $\lambda = \left(\lambda^{(\alpha)} \right)_{\alpha=1}^N \in \rho^{-1}(\varnothing)$ is of the form

$$\lambda^{(\alpha)} = \begin{array}{|c|c|c|c|} \hline c(\alpha) & c(\alpha){+}1 & \cdots & p{-}2 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots & \\ \hline \vdots & \vdots & & \multicolumn{1}{c}{} \\ \cline{1-3} \vdots & & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \tag{5.2.68}$$

where the number in each box denotes its color. We may define the length of the column of color ω to be $d_{\omega+1,\alpha}$ ($c(\alpha) \leq \omega < p-1$). Note that $k_{\omega-1} = \sum_\alpha d_{\omega,\alpha}$. Consequently, we have

$$\widetilde{K}_\omega = \sum_{c(\alpha) \leq \omega} e^{\beta \widetilde{a}_\alpha} \sum_{i=1}^{d_{\omega+1,\alpha}} q_1^{i-1}, \tag{5.2.69}$$

from which we simplify (5.2.60) as

$$\Sigma_\omega = \sum_{c(\alpha)<\omega} e^{\beta \widetilde{a}_\alpha} q_1^{d_{\omega,\alpha}}. \tag{5.2.70}$$

Therefore, the partition function (5.2.67) is reduced to a sum over the non-negative integers

$$\left\{ \begin{array}{c|c} d_{\omega,\alpha} \geq 0 & \begin{array}{l} \omega = 1, \cdots, p-1, \quad c(\alpha) < \omega \\ d_{\omega,\alpha} \geq d_{\omega+1,\alpha} \end{array} \end{array} \right\}, \tag{5.2.71}$$

with the simplified Σ_ω given above. This is precisely the partition function of the gauged linear sigma model on the $\bigoplus_{\omega=1}^{p-1} \mathrm{Hom}(\mathcal{E}_\omega, \widetilde{M}_{\omega-1})$-bundle over the partial flag variety $\mathrm{Flag}(l_1, l_2, \cdots, N)$, with

$$l_\omega \equiv |\{\alpha \mid c(\alpha) < \omega\}|, \tag{5.2.72}$$

under certain stability condition [21]. Here, $\mathcal{E}_\omega$ is the ω-th tautological bundle with $\mathrm{rk}\mathcal{E}_\omega = l_\omega$. The Kähler moduli are precisely $\{\mathfrak{q}_{\omega-1} = z_\omega / z_{\omega-1} \mid \omega = 1, \cdots, p-1\}$.

In the non-zero instanton sector of the four-dimensional theory, the sigma model couples to the four-dimensional gauge theory through (5.2.67) in a non-trivial way, generating a surface defect. In this way, the full partition function (5.2.66) represents the 2d-4d coupled system.

The investigations in this chapter mainly utilize the special case, the $(N-1,1)$-type $\mathbb{Z}_2$-orbifold. That is, we set $p = 2$ and assign the coloring function

$$c_\beta(\alpha) \equiv \begin{cases} 1 & \text{for } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}, \tag{5.2.73}$$

for some chosen $\beta \in \{1, \cdots, N\}$. We also set $\widetilde{M}_0, \widetilde{M}_1 = \mathbb{C}^N$. For later use, it is instructive to separate out the instanton part in the partition function (5.2.66),

$$\mathcal{Z}_\beta^{\mathbb{Z}_2} = \sum_{\mathbf{\Lambda}} \mathfrak{q}^{|\mathbf{\Lambda}|}\, \mathcal{I}_\beta[\mathbf{\Lambda}]\, \epsilon \left[\widetilde{N}^* \widetilde{K} + q_{12}^{-1} \widetilde{N} \widetilde{K}^* - P_{12}^* \widetilde{K} \widetilde{K}^* - \widetilde{M}^* \widetilde{K} \right], \tag{5.2.74}$$

where the instanton part of the surface defect is

$$\begin{aligned} \mathcal{I}_\beta[\mathbf{\Lambda}] = \sum_{\boldsymbol{\lambda} \in \rho^{-1}(\mathbf{\Lambda})} z^{k_0 - k_1}\, \epsilon \Big[& (\widetilde{K}_0 - \widetilde{K}_1)(\widetilde{N}_0 - P_1 \widetilde{K}_0 + \tilde{q}_2 P_1 \widetilde{K}_1)^* \\ & + q_1 \widetilde{N}_1 (\widetilde{K}_0 - \tilde{q}_2 \widetilde{K}_1)^* - \widetilde{M}_0^* (\widetilde{K}_0 - \tilde{q}_2 \widetilde{K}_1) - \widetilde{P}_2 \widetilde{M}_1^* \widetilde{K}_1 \Big]. \end{aligned} \tag{5.2.75}$$

In this special case, the target space of the two-dimensional sigma model that generates the surface defect is the $\mathrm{Hom}(\mathcal{O}(-1), \mathbb{C}^N)$-bundle over $\mathbb{P}^{N-1}$, which is exactly the same with that of the quiver surface defect in section 5.2.1. Thus it is natural to expect the two distinct types of surface defects are actually related to each other. However, it is not so obvious from the explicit expressions for their partition functions, (5.2.26) and (5.2.66), how they can really be associated. In particular, the combinatorics that define these partition functions are quite different; one involves a simple sum over non-negative intergers while the other involves the non-trivial mapping ρ between N-tuples of Young diagrams. We come back to this problem in section 5.4.2.

5.3 Dyson-Schwinger equations and opers

We investigate the non-perturbative Dyson-Schwinger equations satisfied by the surface defect partition functions that we constructed in the previous section. The primary object of this investigation is the qq-character, which is a gauge theory observable formed as a certain Laurent polynomial of $\mathcal{Y}$-observables [10]. The most general qq-characters were constructed in [10, 15] from the spiked instanton configurations, by integratng out the degrees of freedom orthogonal to the four-dimensional gauge theory. The compactness theorem for the spiked instanton moduli space proven in [14] provided the crucial property of the qq-character, the holomorphicity of its expectation value. Schematically,

$$\left\langle \mathcal{X}(\mathcal{Y}(x)) \right\rangle = \frac{1}{\mathcal{Z}^{\text{inst}}} \sum_{\boldsymbol{\lambda}} \mathcal{X}(\mathcal{Y}(x)[\boldsymbol{\lambda}])\, \mathfrak{q}^{|\boldsymbol{\lambda}|}\, \boldsymbol{\mu}_{\boldsymbol{\lambda}} = T(x), \tag{5.3.1}$$

where $T(x)$ is a polynomial in x of certain degree. Therefore, the qq-character generates an infinite number of constraints that the partition function satisfies, from the expectation values of its non-regular parts

$$[x^{-n}] \left\langle \mathcal{X}(\mathcal{Y}(x)) \right\rangle = 0, \quad n \geq 1, \tag{5.3.2}$$

which we call *the non-perturbative Dyson-Schwinger equations.*

In this section, we present the fundamental qq-characters relevant to each surface defect, and study the consequences of their non-perturbative Dyson-Schwinger equations. For other analysis on the non-perturbative Dyson-Schwinger equations, see [32, 3] in the context of the BPS/CFT correspondence, and [2] in the context of the Bethe/gauge correspondence.

5.3.1 The quiver

As in section 5.2.1, we start with the A_2-quiver gauge theory with the $U(N)$ gauge group. The fundamental qq-characters for this theory is given by [10]

$$\mathcal{X}_1(x) = \mathcal{Y}_1(x+\varepsilon) + \mathfrak{q}_1 \frac{\mathcal{Y}_0(x)\mathcal{Y}_2(x+\varepsilon)}{\mathcal{Y}_1(x)} + \mathfrak{q}_1\mathfrak{q}_2 \frac{\mathcal{Y}_0(x)\mathcal{Y}_3(x+\varepsilon)}{\mathcal{Y}_2(x)}, \tag{5.3.3a}$$

$$\mathcal{X}_2(x) = \mathcal{Y}_2(x+\varepsilon) + \mathfrak{q}_2 \frac{\mathcal{Y}_1(x)\mathcal{Y}_3(x+\varepsilon)}{\mathcal{Y}_2(x)} + \mathfrak{q}_1\mathfrak{q}_2 \frac{\mathcal{Y}_0(x-\varepsilon)\mathcal{Y}_3(x+\varepsilon)}{\mathcal{Y}_1(x-\varepsilon)}, \tag{5.3.3b}$$

where $\mathcal{Y}_0(x) \equiv \prod_{\alpha=1}^{N}(x-a_{0,\alpha})$ and $\mathcal{Y}_3(x) \equiv \prod_{\alpha=1}^{N}(x-a_{3,\alpha})$ by definition.[6] We construct the surface defect by imposing the constraints (5.2.4) or (5.2.34) for the Coulomb moduli. In

[6] Be cautious about the re-definition of parameters (5.2.36) when we deal with the case (5.2.34). The expressions for $\mathcal{Y}_0$ and $\mathcal{Y}_3$ also change correspondingly.

each case, the $\mathscr{Y}$-observable of the first or the second gauge node is simplified to

$$\mathscr{Y}_1(x)[\boldsymbol{\lambda}^{(1)}(k_1)] = \mathscr{Y}_0(x)\frac{x - a_{0,\beta} + \varepsilon_2 - k_1\varepsilon_1}{x - a_{0,\beta} - k_1\varepsilon_1}, \quad \text{for (5.2.4)} \tag{5.3.4a}$$

$$\mathscr{Y}_2(x)[\boldsymbol{\lambda}^{(2)}(k_2)] = \mathscr{Y}_3(x+\varepsilon)\frac{x + a_{3,\beta} - \varepsilon_1 - k_2\varepsilon_1}{x + a_{3,\beta} - \varepsilon - k_2\varepsilon_1}, \quad \text{for (5.2.34)} \tag{5.3.4b}$$

It is now straighforward to plug (5.3.4) back into (5.3.3) and compute their expectation values of the non-regular parts. However, it is convenient to follow the systematic procedure establihshed in [3], which was reviewed in the Chapter 4. First let us define

$$\mathcal{G}(x;t) \equiv \frac{1}{\mathscr{Y}_0(x)\prod_{i=0}^{2}(1+tz_i)}\sum_{l=0}^{3} z_0 z_1 \cdots z_{l-1}\, t^l\, \mathcal{X}_l(x - \varepsilon(1-l)) = \sum_{n=0}^{\infty}\frac{\mathcal{G}^{(-n)}(t)}{x^n}, \tag{5.3.5}$$

where we have defined the parameters z_i by $\mathfrak{q}_i \equiv \frac{z_i}{z_{i-1}}$ ($z_{-1} = \infty$ and $z_3 = 0$ by definition), and t is an auxiliary parameter. The non-perturbative Dyson-Schwinger equations imply

$$[x^{-n}]\left\langle \mathscr{Y}_0(x)\mathcal{G}(x;t)\right\rangle = 0, \quad n \geq 1, \tag{5.3.6}$$

for any value of t. As we summarized the systematic approach for computing $\mathcal{G}^{(-n)}(t)$ in the Chapter 4, we do not reproduce it here. We focus on presenting the results below.

5.3.1.1 $N = 2$

We observe that $\mathscr{Y}_0(x) = \prod_{\alpha}^{N}(x - a_{0,\alpha})$ is a polynomial of degree N. Hence in the case of $N = 2$, the x^{-1}-term in (5.3.6) is

$$0 = \left\langle \mathcal{G}^{(-3)}(t)\right\rangle - (a_{0,1} + a_{0,2})\left\langle \mathcal{G}^{(-2)}(t)\right\rangle + a_{0,1}a_{0,2}\left\langle \mathcal{G}^{(-1)}(t)\right\rangle. \tag{5.3.7}$$

Recall that with the constraints (5.2.4) the Young diagram $\boldsymbol{\lambda}^{(1)}$ for the first gauge node is restricted to be single-columned. Thus we can simplify

$$\begin{aligned}\left\langle \sum_{\square\in\boldsymbol{\lambda}^{(1)}} c_\square \right\rangle &= (a_{0,\beta}-\varepsilon)\left\langle k_1\right\rangle + \frac{\varepsilon_1}{2}\left\langle k_1(k_1+1)\right\rangle \\ &= \frac{1}{\mathcal{Z}^L_\beta}\left[\left(a_{0,\beta}-\varepsilon-\frac{\varepsilon_1}{2}\right)\mathfrak{q}_1\frac{\partial}{\partial\mathfrak{q}_1}+\frac{\varepsilon_1}{2}\left(\mathfrak{q}_1\frac{\partial}{\partial\mathfrak{q}_1}\right)^2\right]\mathcal{Z}^L_\beta.\end{aligned} \tag{5.3.8}$$

Using this relation, the residue of (5.3.7) at $t=-z_0^{-1}$ can be written as the following second order differential equation

$$\begin{aligned}0=\Bigg[&\varepsilon_1^2\left(z_0\frac{\partial}{\partial z_0}\right)^2-\varepsilon_1\left(\sum_{i=1}^{2}\frac{z_i}{z_i-z_0}\mathcal{A}_i^{(1)}+2a_{0,\beta}-a_{0,1}-a_{0,2}\right)\left(z_0\frac{\partial}{\partial z_0}\right)\\ &+\sum_{i=1}^{2}\frac{z_i}{z_i-z_0}\left(\frac{1}{2}\left(\mathcal{A}_i^{(2)}+\left(\mathcal{A}_1^{(1)}\right)^2\right)-\varepsilon_1\varepsilon_2 z_i\frac{\partial}{\partial z_i}-(a_{0,1}+a_{0,2}-a_{0,\beta})\mathcal{A}_i^{(1)}\right)\\ &+\frac{z_1z_2}{(z_1-z_0)(z_2-z_0)}\left(\mathcal{A}_1^{(1)}-\varepsilon\right)\mathcal{A}_2^{(1)}\Bigg]\mathcal{Z}^L_\beta,\end{aligned} \tag{5.3.9}$$

where we have introduced

$$\mathcal{A}_i^{(n)}\equiv\sum_{\alpha=1}^{N}\left(a_{i,\alpha}^n-(a_{i+1,\alpha}-\varepsilon)^n\right),\qquad i=1,2. \tag{5.3.10}$$

Here $N=2$ but we will also extend to the higher N by the same expression. In particular, for $n=1$ we can write

$$\mathcal{A}_i^{(1)}=N(\bar{a}_i-\bar{a}_{i+1}+\varepsilon), \tag{5.3.11}$$

where we have defined $\bar{a}_i\equiv\frac{1}{N}\sum_{\alpha=1}^{N}a_{i,\alpha}$. For our purpose of investigating the relations with the opers, it is important to re-define the partition function as

$$\widetilde{\mathcal{Z}}_{A_2}\equiv\prod_{i=0}^{2}z_i^{L_i}\prod_{0\le i<j\le 2}\left(1-\frac{z_j}{z_i}\right)^{T_{ij}}\mathcal{Z}_{A_2}, \tag{5.3.12}$$

where we have multiplied the prefactors with the exponents,

$$
\begin{aligned}
L_i &\equiv \frac{(a_{i+1,1}-a_{i+1,2})^2-(a_{i,1}-a_{i,2})^2}{4\varepsilon_1\varepsilon_2}+\frac{(\bar{a}_i-\bar{a}_{i+1}+\varepsilon)(\bar{a}_i-\bar{a}_{i+1})}{\varepsilon_1\varepsilon_2}, \quad i=0,1,2, \\
T_{ij} &= \frac{2(\bar{a}_j-\bar{a}_{j+1}+\varepsilon)(\bar{a}_i-\bar{a}_{i+1})}{\varepsilon_1\varepsilon_2}, \quad i,j=0,1,2.
\end{aligned}
\tag{5.3.13}
$$

With the constraints (5.2.4) imposed on these prefactors, the modification for the surface defect partition function $\mathcal{Z}^L_\beta$ is simpler than the most generic case. Let us set $z_0 = z$, $z_1 = 1$, and $z_2 = \mathfrak{q}$ by using the redundancy of overall scaling of z_i's. Then we find the prefactors (with the overall constant that we choose at our convenience) for $\mathcal{Z}^L_\beta$ can be written as

$$
\begin{aligned}
&\left(-\frac{1}{z}\right)^{-r_{L,\beta}} \mathfrak{q}^{-\Delta_{\mathfrak{q}}-\Delta_0+\frac{\varepsilon^2-(a_{2,1}-a_{2,2})^2}{4\varepsilon_1\varepsilon_2}} \\
&\left(1-\frac{1}{z}\right)^{\frac{2\bar{a}_0-2\bar{a}_2+2\varepsilon_1+\varepsilon_2}{2\varepsilon_1}} \left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{\bar{a}_2-\bar{a}_3+\varepsilon}{\varepsilon_1}} (1-\mathfrak{q})^{\frac{2(\bar{a}_2-\bar{a}_3+\varepsilon)(2\bar{a}_0-2\bar{a}_2-\varepsilon_2)}{\varepsilon_1\varepsilon_2}},
\end{aligned}
\tag{5.3.14}
$$

where we have defined

$$
(r_{L,\beta})_{\beta=1,2} = \left(\frac{-a_{0,1}+a_{0,2}+\varepsilon+\varepsilon_2}{2\varepsilon_1},\ \frac{a_{0,1}-a_{0,2}+\varepsilon+\varepsilon_2}{2\varepsilon_1}\right), \tag{5.3.15}
$$

and

$$
\begin{aligned}
\Delta_0 &\equiv \frac{\varepsilon^2-(a_{3,1}-a_{3,2})^2}{4\varepsilon_1\varepsilon_2} \\
\Delta_{\mathfrak{q}} &\equiv -\frac{(\bar{a}_2-\bar{a}_3)(\bar{a}_2-\bar{a}_3+\varepsilon)}{\varepsilon_1\varepsilon_2} \\
\Delta_1 &\equiv -\frac{(2\bar{a}_0-2\bar{a}_2+2\varepsilon_1+\varepsilon_2)(2\bar{a}_0-2\bar{a}_2-\varepsilon_2)}{4\varepsilon_1\varepsilon_2} \\
\Delta_\infty &\equiv \frac{\varepsilon^2-(a_{0,1}-a_{0,2})^2}{4\varepsilon_1\varepsilon_2}.
\end{aligned}
\tag{5.3.16}
$$

For the re-defined partition function $\widetilde{\mathcal{Z}}^{L}_{\beta}$, the differential equation (5.3.9) becomes

$$\begin{aligned} 0 = \bigg[& \varepsilon_1^2 \partial^2 - \varepsilon_1\varepsilon_2 \frac{2z-1}{z(z-1)}\partial + \varepsilon_1\varepsilon_2 \frac{\mathfrak{q}-1}{z(z-1)(z-\mathfrak{q})}\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} \\ & +\varepsilon_1\varepsilon_2 \left(\frac{\Delta_0}{z^2} + \frac{\Delta_1}{(z-1)^2} + \frac{\Delta_{\mathfrak{q}}}{(z-\mathfrak{q})^2} - \frac{-\frac{2\varepsilon+\varepsilon_2}{4\varepsilon_1} + \Delta_1 + \Delta_{\mathfrak{q}} + \Delta_0 - \Delta_\infty}{z(z-1)} \right) \bigg] \widetilde{\mathcal{Z}}^{L}_{\beta}. \end{aligned} \tag{5.3.17}$$

We can view this differential equation as the second-order differential operator $\hat{\hat{\mathfrak{D}}}_2$ annihilating the modified partition function $\widetilde{\mathcal{Z}}^{L}_{\beta}$. Note that the operator $\hat{\hat{\mathfrak{D}}}_2$ is independent of β, so that each choice of $\beta \in \{1,2\}$ provides a solution to $\hat{\hat{\mathfrak{D}}}_2$. We may regard $\hat{\hat{\mathfrak{D}}}_2$ as the quantization of the $SL(2)$-oper $\widehat{\mathfrak{D}}_2$ for the four-punctured sphere $\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}$, as we now argue.

Under the Nekrasov-Shatashvili limit ($\varepsilon_1 \neq 0, \varepsilon_2 \to 0$), the surface defect partition function (5.2.26) is dominated by the limit shape [7]. Viewed as the expectation value of $\mathcal{I}^{L}_{\beta}$ in the A_1-theory, the surface defect partition function gets the singular contribution, or the effective twisted superpotential, from the bulk A_1-theory, while the observable $\mathcal{I}^{L}_{\beta}$ only contributes regular terms. Therefore, we arrive at the following asymptotics of the partition function

$$\widetilde{\mathcal{Z}}^{L}_{\beta}(\mathbf{a}_2 \equiv \mathbf{a}, z, \mathfrak{q}) = e^{\frac{\widetilde{\mathcal{W}}(\mathbf{a},\mathfrak{q})}{\varepsilon_2}} \left(\chi_\beta(\mathbf{a}, z, \mathfrak{q}) + \mathcal{O}(\varepsilon_2) \right), \tag{5.3.18}$$

where we have omitted the subscript for the Coulomb moduli $\mathbf{a}_2$ since it precisely becomes the Coulomb moduli $\mathbf{a}$ of the A_1-theory. $\widetilde{\mathcal{W}}$ is a part of the effective twisted superpotential of the underlying A_1-gauge theory,

$$\begin{aligned} \widetilde{\mathcal{W}} &\equiv \lim_{\varepsilon_2\to 0} \varepsilon_2 \log \widetilde{\mathcal{Z}}^{L}_{\beta} \\ &= \widetilde{\mathcal{W}}^{\text{classical}} + \widetilde{\mathcal{W}}^{\text{inst}} + \widetilde{\mathcal{W}}^{\text{extra}}, \end{aligned} \tag{5.3.19}$$

where we have defined

$$\widetilde{\mathcal{W}}^{\text{classical}} \equiv -\frac{(a_1-a_2)^2}{4\varepsilon_1}\log\mathfrak{q} \tag{5.3.20a}$$

$$\widetilde{\mathcal{W}}^{\text{inst}} \equiv \lim_{\varepsilon_2\to 0}\varepsilon_2\log\mathcal{Z}^{\text{inst}}_{A_1} \tag{5.3.20b}$$

$$\widetilde{\mathcal{W}}^{\text{extra}} \equiv \varepsilon_1\left(\frac{1}{4}-\delta_{\mathfrak{q}}-\delta_0\right)\log\mathfrak{q}+\frac{2(\bar{a}_0-\bar{a})(\bar{a}-\bar{a}_3+\varepsilon_1)}{\varepsilon_1}\log(1-\mathfrak{q}), \tag{5.3.20c}$$

where the instanton partition function $\mathcal{Z}^{\text{inst}}_{A_1}$ for the A_1-theory is given by (5.2.28). In particular, $\widetilde{\mathcal{W}}^{\text{inst}}$ is fully determined by the Young diagram expansions reviewed in appendix 2.1. Also, we have defined the limit,

$$\varepsilon_2\Delta_i \xrightarrow{\varepsilon_2\to 0} \varepsilon_1\delta_i, \quad i=0,\mathfrak{q},1,\infty. \tag{5.3.21}$$

We have emphasized that (5.3.19) is only a part of the full effective twisted superpotential, since we are missing the 1-loop term. This is because the 1-loop term is independent of the gauge coupling and therefore ignorant of the differential equation that the partition function satisfies. The missing 1-loop part will re-combine in section 5.6.

Thus, under the Nekrasov-Shatashvili limit the equation for the differential operator $\hat{\hat{\mathfrak{D}}}_2$ becomes

$$0=\left[\partial^2+\frac{\delta_0}{z^2}+\frac{\delta_1}{(z-1)^2}+\frac{\delta_{\mathfrak{q}}}{(z-\mathfrak{q})^2}-\frac{\delta_1+\delta_{\mathfrak{q}}+\delta_0-\delta_\infty}{z(z-1)}+\frac{H}{z(z-1)(z-\mathfrak{q})}\right]\chi_\beta, \tag{5.3.22}$$

which is exactly the equation for *the Heun's oper*, the Fuchsian differential operator $\hat{\mathfrak{D}}_2$ of degree 2 with fixed conjugacy class of monodromy at each puncture of $\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}$. The variety $\mathcal{O}_2[\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}]$ of these opers is spanned by *the accessory parameter*,

$$\begin{aligned} H &\equiv -\mathfrak{q}(1-\mathfrak{q})\frac{1}{\varepsilon_1}\frac{\partial\widetilde{\mathcal{W}}}{\partial\mathfrak{q}} \\ &= (1-\mathfrak{q})\left(\frac{1}{2\varepsilon_1^2}\lim_{\varepsilon_2\to 0}\left\langle \mathcal{O}_2\right\rangle_{A_1}-\frac{1}{4}+\delta_{\mathfrak{q}}+\delta_0\right)+\frac{2(\bar{a}_0-\bar{a})(\bar{a}-\bar{a}_3+\varepsilon_1)}{\varepsilon_1^2}\mathfrak{q}. \end{aligned} \tag{5.3.23}$$

All the terms are just some constants except the expectation value of chiral observable $\mathcal{O}_2 = \mathrm{Tr}\phi_2^2$. Thus, a holomorphic coordinate on the variety $\mathcal{O}_2[\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}]$ of opers is provided by the expectation value of the chiral observable $\mathcal{O}_2$ in the limit $\varepsilon_2 \to 0$. The variety $\mathcal{O}_2[\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}]$ of opers is a quantization of the Coulomb moduli space of $\mathcal{T}[A_1,\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}]$ in this sense. The expectation value $\lim_{\varepsilon_2\to 0}\left\langle \mathcal{O}_2 \right\rangle_{A_1}$ is also identified with the off-shell spectrum of the quantum Hitchin system on $\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}$ through the Bethe/gauge correspondence. Hence, we observe that the relation (5.3.23) establishes the connection between the accessory parameter H of $\widehat{\mathfrak{D}}_2$ and the off-shell spectrum of quantum Hitchin Hamiltonian. A proper on-shell condition is expected to be introduced by a ω_K-Lagrangian brane which intersects with $\mathcal{O}_2[\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}]$ at isolated points. As we argued earlier, the holomorphic coordinate, i.e., the expectation value, (5.3.23) evaluated at these points gives the on-shell spectrum of the quantum Hitchin system.

Remarks

- It was checked in [110, 111] that the series expansion (5.3.23) for the accessory parameter H matches with the direct computation in which H is determined by fixing the monodromy of the oper $\widehat{\mathfrak{D}}_2$ along the A-cycle (see Figure 5.5), up to some low orders in the gauge coupling $\mathfrak{q}$. The derivation above is purely gauge theoretical and therefore guarantees the validity to all orders in $\mathfrak{q}$.

- The series expansion for the instanton partition function is valid when $0 < |\mathfrak{q}_1|, |\mathfrak{q}_2| < 1$. This implies that the solutions $\tilde{\mathcal{Z}}_\beta^L$ for the operator $\hat{\widehat{\mathfrak{D}}}_2$ are in the convergence domain $0 < |\mathfrak{q}| < 1 < |z|$.

- The solution χ_β for the oper $\widehat{\mathfrak{D}}_2$ can be represented as a sum of the Baxter $\mathcal{Q}$-functions, by using (5.2.33) and taking the limit $\varepsilon_2 \to 0$. This expression reflects that the equation for the oper is the Fourier transform of the Baxter TQ-equation.

- The fact that (5.3.17) coincides with well-known null-vector decoupling equation in two-dimensional CFT [67], see also [112, 113, 114], confirms the paradigm of the BPS/CFT correspondence [80] at the example of the AGT correspondence [63].

Similarly, it is not too difficult to derive a closed differential equation for $\mathcal{Z}^R_\beta$. Again, we re-define partition function as in (5.3.12) with the prefactors (5.3.13), yet with the constraints (5.2.34). Also we need the re-definition of parameters (5.2.36) for the prefactors this time. By setting $z_0 = 1$, $z_1 = \mathfrak{q}$, and $z_2 = z$, the relevant prefactor for $\mathcal{Z}^R_\beta$ is

$$\begin{aligned}&\left(-\frac{\mathfrak{q}}{z}\right)^{-r_{R,\beta}} \mathfrak{q}^{\frac{\varepsilon^2-(a_{1,1}-a_{1,2})^2}{4\varepsilon_1\varepsilon_2}-\Delta_0-\Delta'_{\mathfrak{q}}+\frac{2\varepsilon+\varepsilon_2}{4\varepsilon_1}} \\ &(1-\mathfrak{q})^{\frac{(\bar{a}_0-\bar{a}_1+\varepsilon)(2\bar{a}_1-2\bar{a}_3-\varepsilon_2)}{\varepsilon_1\varepsilon_2}} (1-z)^{\frac{\bar{a}_0-\bar{a}_1+\varepsilon}{\varepsilon_1}} \left(1-\frac{z}{\mathfrak{q}}\right)^{\frac{2\bar{a}_1-2\bar{a}_3+2\varepsilon_1+\varepsilon_2}{2\varepsilon_1}}\end{aligned} \tag{5.3.24}$$

where we have defined

$$(r_{R,\beta})_{\beta=1,2} \equiv \left(\frac{-a_{3,1}+a_{3,2}+\varepsilon}{2\varepsilon_1}, \ \frac{a_{3,1}-a_{3,2}+\varepsilon}{2\varepsilon_1}\right), \tag{5.3.25}$$

and

$$\begin{aligned}\Delta'_{\mathfrak{q}} &\equiv -\frac{(2\bar{a}_1-2\bar{a}_3-\varepsilon_2)(2\bar{a}_1-2\bar{a}_3+2\varepsilon_1+\varepsilon_2)}{4\varepsilon_1\varepsilon_2} \\ \Delta'_1 &\equiv -\frac{(\bar{a}_0-\bar{a}_1)(\bar{a}_0-\bar{a}_1+\varepsilon)}{\varepsilon_1\varepsilon_2}.\end{aligned} \tag{5.3.26}$$

Then the differential equation satisfied by the modified partition function $\widetilde{\mathcal{Z}}^R_\beta$ is

$$\begin{aligned}0 = \Bigg[&\varepsilon_1^2\partial^2 - \varepsilon_1\varepsilon_2\frac{2z-1}{z(z-1)}\partial + \varepsilon_1\varepsilon_2\frac{\mathfrak{q}-1}{z(z-1)(z-\mathfrak{q})}\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} \\ &+\varepsilon_1\varepsilon_2\left(\frac{\Delta_0}{z^2}+\frac{\Delta'_1}{(z-1)^2}+\frac{\Delta'_{\mathfrak{q}}}{(z-\mathfrak{q})^2}-\frac{-\frac{2\varepsilon+\varepsilon_2}{4\varepsilon_1}+\Delta'_1+\Delta'_{\mathfrak{q}}+\Delta_0-\Delta_\infty}{z(z-1)}\right)\Bigg]\widetilde{\mathcal{Z}}^R_\beta.\end{aligned} \tag{5.3.27}$$

Note that this differential equation is precisely the equation (5.3.17) for $\hat{\hat{\mathfrak{D}}}_2$, except $\Delta_1 \to \Delta'_1, \Delta_{\mathfrak{q}} \to \Delta'_{\mathfrak{q}}$. To equate these quantities to get the same equation, we have to clarify how the Coulomb moduli of the two theories are associated. This will be the subject of section 5.4.2.2.

Remarks

- This time, $0 < |\mathfrak{q}_1|, |\mathfrak{q}_2| < 1$ implies the convergence domain $0 < |z| < |\mathfrak{q}| < 1$. Thus the domains for the solutions $\widetilde{\mathcal{Z}}^L_\beta$ and $\widetilde{\mathcal{Z}}^R_\beta$ are disjoint.

5.3.1.2 $N = 3$

Since $\mathcal{Y}_0(x)$ is now a polynomial of degree 3, the x^{-1}-term of (5.3.6) can be written as

$$0 = \left\langle \mathcal{G}^{(-4)}(t) \right\rangle - \sum_{\alpha=1}^{3} a_{0,\alpha} \left\langle \mathcal{G}^{(-3)}(t) \right\rangle + \sum_{1\leq\alpha<\beta\leq 3} a_{0,\alpha} a_{0,\beta} \left\langle \mathcal{G}^{(-2)}(t) \right\rangle - \prod_{\alpha=1}^{3} a_{0,\alpha} \left\langle \mathcal{G}^{(-1)}(t) \right\rangle. \tag{5.3.28}$$

In addition to (5.3.8), we utilize the following relation from the single-columnedness of $\boldsymbol{\lambda}^{(1)}$

$$\begin{aligned} \left\langle \sum_{\square\in\boldsymbol{\lambda}^{(1)}} c_\square^2 \right\rangle &= (a_{0,\beta}-\varepsilon)^2 \left\langle k_1 \right\rangle + \varepsilon_1 (a_{0,\beta}-\varepsilon) \left\langle k_1(k_1+1) \right\rangle + \frac{\varepsilon_1^2}{6} \left\langle k_1(k_1+1)(2k_1+1) \right\rangle \\ &= \frac{1}{\mathcal{Z}^L_\beta} \left[\frac{\varepsilon_1^2}{3} \left(\mathfrak{q}_1 \frac{\partial}{\partial \mathfrak{q}_1} \right)^3 + \left(\frac{\varepsilon_1^2}{2} + \varepsilon_1 (a_{0,\beta}-\varepsilon) \right) \left(\mathfrak{q}_1 \frac{\partial}{\partial \mathfrak{q}_1} \right)^2 + \left(\frac{\varepsilon_1^2}{6} + (a_{0,\beta}-\varepsilon)(a_{0,\beta}-\varepsilon_2) \right) \left(\mathfrak{q}_1 \frac{\partial}{\partial \mathfrak{q}_1} \right) \right] \mathcal{Z}^L_\beta. \end{aligned} \tag{5.3.29}$$

Using the relations (5.3.8) and (5.3.29), the residue of (5.3.28) at $t = -z_0^{-1}$ can be written as the following third order differential equation

$$
\begin{aligned}
0 = \Bigg[& -\varepsilon_1^3 \left(z_0 \frac{\partial}{\partial z_0} \right)^3 + \varepsilon_1^2 \left(3a_{0,\beta} - \sum_{\alpha=1}^{3} a_{0,\alpha} - \varepsilon_2 \frac{z_1}{z_1 - z_0} + \sum_{i=1}^{2} \frac{z_i}{z_i - z_0} \mathcal{A}_i^{(1)} \right) \left(z_0 \frac{\partial}{\partial z_0} \right)^2 \\
& - \varepsilon_1 \Bigg(\prod_{\alpha \neq \beta} (a_{0,\beta} - a_{0,\alpha}) + \frac{z_1 z_2}{(z_1 - z_0)(z_2 - z_0)} \mathcal{A}_2^{(1)} (\mathcal{A}_1^{(1)} - \varepsilon) - \varepsilon_2 \varepsilon \left(2a_{0,\beta} - \varepsilon - \varepsilon_1 \right) \frac{z_1}{z_1 - z_0} \\
& \qquad + \sum_{i=1}^{2} \frac{z_i}{z_i - z_0} \left(\left(2a_{0,\beta} - \sum_{\alpha=1}^{3} a_{0,\alpha} \right) \mathcal{A}_i^{(1)} - \varepsilon_1 \varepsilon_2 z_i \frac{\partial}{\partial z_i} + \frac{1}{2} \left(\mathcal{A}_i^{(2)} + \left(\mathcal{A}_i^{(1)} \right)^2 \right) \right) \Bigg) \left(z_0 \frac{\partial}{\partial z_0} \right) \\
& + \sum_{i=1}^{2} \frac{z_i}{z_i - z_0} \Bigg(\frac{1}{6} (\mathcal{A}_i^{(1)})^3 + \frac{1}{3} \mathcal{A}_i^{(3)} + \frac{1}{2} \mathcal{A}_i^{(1)} \mathcal{A}_i^{(2)} - \frac{1}{2} \left(\mathcal{A}_i^{(2)} + \left(\mathcal{A}_i^{(1)} \right)^2 \right) \left(\sum_{\alpha=1}^{3} a_{0,\alpha} - a_{0,\beta} \right) \\
& \qquad + \varepsilon_1 \varepsilon_2 \left(\sum_{\alpha=1}^{3} a_{0,\alpha} - a_{0,\beta} - \mathcal{A}_i^{(1)} \right) \left(z_i \frac{\partial}{\partial z_i} \right) + \prod_{\alpha \neq \beta} a_{0,\alpha} \, \mathcal{A}_i^{(1)} - \varepsilon_1 \varepsilon_2 \varepsilon \; z_2 \partial_2 \Bigg) \\
& - 2\varepsilon_1 \varepsilon_2 \frac{z_0 (z_1 - z_2)}{(z_0 - z_1)(z_0 - z_2)} \left\langle \sum_{\square \in \boldsymbol{\lambda}^{(2)}} c_\square \right\rangle_{A_2} \\
& + \frac{z_1 z_2}{(z_1 - z_0)(z_2 - z_0)} \Bigg(\frac{1}{2} \left(\mathcal{A}_1^{(1)} - 2\varepsilon \right) \left(\mathcal{A}_2^{(2)} + \left(\mathcal{A}_2^{(1)} \right)^2 \right) + \frac{1}{2} \mathcal{A}_2^{(1)} \left(\mathcal{A}_1^{(2)} + \left(\mathcal{A}_1^{(1)} \right)^2 \right) \\
& \qquad - \mathcal{A}_2^{(1)} \left(\mathcal{A}_1^{(1)} - \varepsilon \right) \left(\sum_{\alpha=1}^{3} a_{0,\alpha} - a_{0,\beta} + \varepsilon \right) - \varepsilon_1 \varepsilon_2 \left(\mathcal{A}_2^{(1)} z_1 \frac{\partial}{\partial z_1} + (\mathcal{A}_1^{(1)} - 2\varepsilon) z_2 \frac{\partial}{\partial z_2} \right) \Bigg) \Bigg] \mathcal{Z}_\beta^L,
\end{aligned}
\tag{5.3.30}
$$

where we have used (5.3.10). We modify the partition function by multiplying the prefactors,

$$
\widetilde{\mathcal{Z}}_3 \equiv \prod_{i=0}^{2} z_i^{L_i} \prod_{0 \le i < j \le 2} \left(1 - \frac{z_j}{z_i} \right)^{T_{ij}} \mathcal{Z}_3, \tag{5.3.31}
$$

where

$$
\begin{aligned}
L_i \equiv & \frac{(a_{i+1,1} - a_{i+1,2})^2 + (a_{i+1,1} - a_{i+1,3})^2 - (a_{i+1,1} - a_{i+1,2})(a_{i+1,1} - a_{i+1,3})}{3\varepsilon_1 \varepsilon_2} \\
& - \frac{(a_{i,1} - a_{i,2})^2 + (a_{i,1} - a_{i,3})^2 - (a_{i,1} - a_{i,2})(a_{i,1} - a_{i,3})}{3\varepsilon_1 \varepsilon_2} \\
& + \frac{3(\bar{a}_i - \bar{a}_{i+1} + \varepsilon)(\bar{a}_i - \bar{a}_{i+1})}{\varepsilon_1 \varepsilon_2}, \qquad\qquad i = 0, 1, 2, \\
T_{ij} \equiv & \frac{3(\bar{a}_j - \bar{a}_{j+1} + \varepsilon)(\bar{a}_i - \bar{a}_{i+1})}{\varepsilon_1 \varepsilon_2}, \quad i, j = 0, 1, 2.
\end{aligned}
\tag{5.3.32}
$$

With the constraints (5.2.4), the prefactors simplify. We also set $z_0 = z$, $z_1 = 1$, and $z_2 = \mathfrak{q}$. Then the prefactor for $\mathcal{Z}^L_\beta$ becomes

$$\begin{gathered}\left(-\frac{1}{z}\right)^{-r_{L,\beta}} \mathfrak{q}^{-\Delta_{\mathfrak{q}}-\Delta_0+\frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2-\frac{(a_{2,1}-a_{2,2})^2+(a_{2,1}-a_{2,3})^2-(a_{2,1}-a_{2,2})(a_{2,1}-a_{2,3})}{3}\right)} \\ \left(1-\frac{1}{z}\right)^{\frac{3\bar{a}_0-3\bar{a}_2+3\varepsilon-\varepsilon_2}{3\varepsilon_1}} \left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{\bar{a}_2-\bar{a}_3+\varepsilon}{\varepsilon_1}} (1-\mathfrak{q})^{\frac{(\bar{a}_2-\bar{a}_3+\varepsilon)(3\bar{a}\_3\bar{a}_2-\varepsilon_2)}{\varepsilon_1\varepsilon_2}} ,\end{gathered} \tag{5.3.33}$$

where the exponents are

$$(r_{L,\beta})^3_{\beta=1} \equiv \left(\frac{-3a_{0,\beta}+\sum_{\gamma=1}^3 a_{0,\gamma}+3\varepsilon_1+5\varepsilon_2}{3\varepsilon_1}\right)^3_{\beta=1} , \tag{5.3.34}$$

and

$$\begin{aligned}\Delta_0 &\equiv \frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2-\frac{(a_{3,1}-a_{3,2})^2+(a_{3,1}-a_{3,3})^2-(a_{3,1}-a_{3,2})(a_{3,1}-a_{3,3})}{3}\right) \\ \Delta_{\mathfrak{q}} &\equiv -\frac{3(\bar{a}_2-\bar{a}_3)(\bar{a}_2-\bar{a}_3+\varepsilon)}{\varepsilon_1\varepsilon_2} \\ \Delta_1 &\equiv -\frac{(3\bar{a}_0-3\bar{a}_2-\varepsilon_2)(3\bar{a}_0-3\bar{a}_2+3\varepsilon-\varepsilon_2)}{3\varepsilon_1\varepsilon_2} \\ \Delta_\infty &\equiv \frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2-\frac{(a_{0,1}-a_{0,2})^2+(a_{0,1}-a_{0,3})^2-(a_{0,1}-a_{0,2})(a_{0,1}-a_{0,3})}{3}\right).\end{aligned} \tag{5.3.35}$$

It is also convenient to define the quantities

$$\begin{aligned}\Lambda_0 &\equiv \frac{(2a_{3,1}-a_{3,2}-a_{3,3})(-a_{3,1}+2a_{3,2}-a_{3,3})(-a_{3,1}-a_{3,2}+2a_{3,3})}{27\varepsilon_1^3} \\ \Lambda_{\mathfrak{q}} &\equiv \frac{(\bar{a}_2-\bar{a}_3)(\bar{a}_2-\bar{a}_3+\varepsilon)(2\bar{a}_2-2\bar{a}_3+\varepsilon)}{\varepsilon_1^3} \\ \Lambda_1 &\equiv \frac{1}{\varepsilon_1^3}\left(\bar{a}_0-\bar{a}_2-\frac{\varepsilon_2}{3}\right)\left(\bar{a}_0-\bar{a}_2+\varepsilon-\frac{\varepsilon_2}{3}\right)\left(2\bar{a}_0-2\bar{a}_2+\varepsilon-\frac{2\varepsilon_2}{3}\right) \\ \Lambda_\infty &\equiv \frac{(2a_{0,1}-a_{0,2}-a_{0,3})(-a_{0,1}+2a_{0,2}-a_{0,3})(-a_{0,1}-a_{0,2}+2a_{0,3})}{27\varepsilon_1^3}.\end{aligned} \tag{5.3.36}$$

Then under the modification, the differential equation (5.3.30) defines an operator $\widehat{\widehat{\mathfrak{D}}}_3$ annihilating the partition function $\widetilde{\mathcal{Z}}_\beta^L$,

$$0 = \left[\varepsilon_1^3 \partial^3 - \varepsilon_1^2 \varepsilon_2 \frac{5z-3}{z(z-1)} \partial^2 + \varepsilon_1 \widehat{t}_2(z, \mathfrak{q}) \partial + \widehat{t}_3(z, \mathfrak{q})\right] \widetilde{\mathcal{Z}}_\beta^L, \tag{5.3.37}$$

where we have defined the meromorphic operators,

$$\begin{aligned} \widehat{t}_2(z, \mathfrak{q}) \equiv \varepsilon_1 \varepsilon_2 &\left(\frac{\Delta_0}{z^2} + \frac{\Delta_{\mathfrak{q}}}{(z-\mathfrak{q})^2} + \frac{\Delta_1}{(z-1)^2} + \frac{\Delta_\infty - \Delta_1 - \Delta_{\mathfrak{q}} - \Delta_0 + \frac{3\varepsilon+\varepsilon_2}{3\varepsilon_1}}{z(z-1)} + \frac{\widehat{H}_1}{z(z-\mathfrak{q})(z-1)} \right) \\ &+ \varepsilon_2 \left(\frac{3\varepsilon_1 + 2\varepsilon_2}{z^2} - \frac{2(\bar{a}_2 - \bar{a}_3 + \varepsilon)}{(z-\mathfrak{q})^2} + \frac{\bar{a}_0 - \bar{a}_2 + \varepsilon - \frac{\varepsilon_2}{3}}{(z-1)^2} \right. \\ &\qquad \left. - \frac{3\bar{a}_0 - 9\bar{a}_2 + 6\bar{a}_3 - 16\varepsilon_2}{3z(z-1)} - \frac{2(1-\mathfrak{q})(\bar{a}_2 - \bar{a}_3 + \varepsilon)}{z(z-\mathfrak{q})(z-1)} \right) \end{aligned} \tag{5.3.38a}$$

$$\begin{aligned} \widehat{t}_3(z, \mathfrak{q}) \equiv &\frac{\varepsilon_1^3 \Lambda_0}{z^3} + \frac{\varepsilon_1^3 \Lambda_{\mathfrak{q}}}{(z-\mathfrak{q})^3} + \frac{\varepsilon_1^3 \Lambda_1}{(z-1)^3} + \frac{\varepsilon_1^3 (\Lambda_\infty - \Lambda_0 - \Lambda_{\mathfrak{q}} - \Lambda_1 - \frac{\varepsilon_2 (3\varepsilon+\varepsilon_2)(3\varepsilon+2\varepsilon_2)}{27\varepsilon_1^3})}{z(z-\mathfrak{q})(z-1)} \\ &+ \frac{(1-\mathfrak{q})(6\bar{a}_0 - 6\bar{a}_2 + 3\varepsilon_1 + \varepsilon_2)}{6z(z-\mathfrak{q})(z-1)^2} \varepsilon_1 \varepsilon_2 \left(-\Delta_\infty + \Delta_0 + \Delta_{\mathfrak{q}} + \Delta_1 - \frac{3\varepsilon + \varepsilon_2}{3\varepsilon_1} \right) \\ &- \frac{1}{2z(z-\mathfrak{q})(z-1)} \varepsilon_1 \varepsilon_2 \left(\frac{2\bar{a}_2 - 2\bar{a}_3 + \varepsilon}{z - \mathfrak{q}} + \frac{6\bar{a}_0 - 6\bar{a}_2 + 3\varepsilon_1 + \varepsilon_2}{3(z-1)} \right) \widehat{H}_1 \\ &+ \frac{\widehat{H}_2}{z^2 (z-\mathfrak{q})(z-1)} + \frac{\varepsilon_1}{2} \partial \left(\widehat{t}_2(z, \mathfrak{q}) \right) + \varepsilon_2 (\cdots) . \end{aligned} \tag{5.3.38b}$$

We have omitted the last term in $\widehat{t}_3(z, \mathfrak{q})$ which is rather lengthy but is constant and subleading in ε_2. This term decouples in the limit $\varepsilon_2 \to 0$. Also, we have defined

$$\widehat{H}_1 \equiv -\mathfrak{q}(1-\mathfrak{q}) \frac{\partial}{\partial \mathfrak{q}} \tag{5.3.39a}$$

$$\begin{aligned} \widehat{H}_2 \equiv &-(1-\mathfrak{q}) \left(\frac{1}{3} \left\langle \mathcal{O}_3 \right\rangle_{A_2} + \varepsilon_1 \varepsilon_2 \left(3\bar{a}_2 - \bar{a}_3 + \frac{3\varepsilon_1 + 2\varepsilon_2}{2} \right) \mathfrak{q} \frac{\partial}{\partial \mathfrak{q}} + \cdots \right) \\ &+ \mathfrak{q} \left(\varepsilon_1 \varepsilon_2 (3\bar{a}_0 - 6\bar{a}_2 + 3\bar{a}_3 - 2\varepsilon - \varepsilon_2) \mathfrak{q} \frac{\partial}{\partial \mathfrak{q}} + \cdots \right) \\ &+ \frac{\mathfrak{q}^2}{1-\mathfrak{q}} (3\bar{a}_0 - 6\bar{a}_2 + 3\bar{a}_3 - 2\varepsilon - \varepsilon_2)(3\bar{a}_0 - 3\bar{a}_2 - \varepsilon_2)(\bar{a}_2 - \bar{a}_3 + \varepsilon). \end{aligned} \tag{5.3.39b}$$

It is not very instructive to write down the full lengthy expression of $\widehat{H}_2$ here, but we emphasize that it is fully expressed in gauge theoretical terms. In particular, it includes the expectation value of the chiral observable

$$\mathcal{O}_3 = \mathrm{Tr}\phi_2^3 = \sum_{\alpha=1}^{3} a_{2,\alpha}^3 - 3\varepsilon_1\varepsilon_2\varepsilon k_2 - 6\varepsilon_1\varepsilon_2 \sum_{\square\in\boldsymbol{\lambda}^{(2)}} c_\square \tag{5.3.40}$$

of the A_2-theory. We present the full expression for $\widehat{H}_2$ in the appendix D.

In the Nekrasov-Shatashvili limit, the partiton function exhibits the asymptotics:

$$\widetilde{\mathcal{Z}}_\beta^L(\mathbf{a}_2 \equiv \mathbf{a}, z, \mathfrak{q}) = e^{\frac{\widetilde{\mathcal{W}}(\mathbf{a},\mathfrak{q})}{\varepsilon_2}} \left(\chi_\beta(\mathbf{a}, z, \mathfrak{q}) + \mathcal{O}(\varepsilon_2)\right), \tag{5.3.41}$$

where $\widetilde{\mathcal{W}}$ is a part of the effective twisted superpotential of the underlying A_1-gauge theory,

$$\begin{aligned}\widetilde{\mathcal{W}} &\equiv \lim_{\varepsilon_2\to 0} \varepsilon_2 \log \widetilde{\mathcal{Z}}_\beta^L \\ &= \widetilde{\mathcal{W}}^{\text{classical}} + \widetilde{\mathcal{W}}^{\text{inst}} + \widetilde{\mathcal{W}}^{\text{extra}}.\end{aligned} \tag{5.3.42}$$

Each piece is given as

$$\widetilde{\mathcal{W}}^{\text{classical}} \equiv -\frac{(a_1-a_2)^2 + (a_1-a_3)^2 - (a_1-a_2)(a_1-a_3)}{3\varepsilon_1} \log \mathfrak{q} \tag{5.3.43a}$$

$$\widetilde{\mathcal{W}}^{\text{inst}} \equiv \lim_{\varepsilon_2\to 0} \varepsilon_2 \log \mathcal{Z}_{A_1}^{\text{inst}} \tag{5.3.43b}$$

$$\widetilde{\mathcal{W}}^{\text{extra}} \equiv \varepsilon_1 \left(1 - \delta_{\mathfrak{q}} - \delta_0\right) \log \mathfrak{q} + \frac{3(\bar{a} - \bar{a}_3 + \varepsilon)(\bar{a}_0 - \bar{a})}{\varepsilon_1} \log(1-\mathfrak{q}), \tag{5.3.43c}$$

where $\widetilde{\mathcal{W}}^{\text{inst}}$ is the is fully determined by the Young diagram expansions. Also we have defined the limit,

$$\varepsilon_2 \Delta_i \xrightarrow{\varepsilon_2\to 0} \varepsilon_1 \delta_i, \quad i = 0, \mathfrak{q}, 1, \infty. \tag{5.3.44}$$

It is convenient to define also

$$\Lambda_i \xrightarrow{\varepsilon_2 \to 0} \lambda_i, \quad i = 0, \mathfrak{q}, 1, \infty. \tag{5.3.45}$$

It is clear that δ_i and λ_i are written in gauge theoretical terms by their definitions. Now, the equation (5.3.37) for the operator $\widehat{\widehat{\mathfrak{D}}}_3$ becomes

$$0 = \left[\partial^3 + t_2(z)\partial + t_3(z)\right] \chi_\beta, \tag{5.3.46}$$

where the meromorphic functions $t_i(z)$ are obtained by taking the limit to the meromorphic operators $\widehat{t_i}(z, \mathfrak{q})$,

$$t_2(z) \equiv \frac{\delta_0}{z^2} + \frac{\delta_\mathfrak{q}}{(z-\mathfrak{q})^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_1 - \delta_\mathfrak{q} - \delta_0}{z(z-1)} + \frac{H_1}{z(z-\mathfrak{q})(z-1)}, \tag{5.3.47a}$$

$$\begin{aligned} t_3(z) \equiv\ & \frac{\lambda_0}{z^3} + \frac{\lambda_\mathfrak{q}}{(z-\mathfrak{q})^3} + \frac{\lambda_1}{(z-1)^3} + \frac{\lambda_\infty - \lambda_0 - \lambda_\mathfrak{q} - \lambda_1}{z(z-\mathfrak{q})(z-1)} \\ & - \frac{H_1}{2z(z-\mathfrak{q})(z-1)} \frac{1}{\varepsilon_1} \left(\frac{2\bar{a} - 2\bar{a}_3 + \varepsilon_1}{z-\mathfrak{q}} + \frac{2\bar{a}_0 - 2\bar{a} + \varepsilon_1}{z-1} \right) + \frac{H_2}{z^2(z-\mathfrak{q})(z-1)} \\ & + \frac{(1-\mathfrak{q})(2\bar{a}_0 - 2\bar{a}_2 + \varepsilon_1)}{2z(z-\mathfrak{q})(z-1)} \frac{1}{\varepsilon_1} (-\delta_\infty + \delta_0 + \delta_\mathfrak{q} + \delta_1) + \frac{1}{2} t_2'(z). \end{aligned} \tag{5.3.47b}$$

This is exactly the equation for the $SL(3)$-oper $\widehat{\mathfrak{D}}_3$ on the four-punctured sphere $\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$.[7] In particular, the monodromies of $\widehat{\mathfrak{D}}_3$ around the punctures exhibit the desired semi-simplicity and degeneracy of the eigenvalues, as verified by the analytic properties of the solutions χ obtained from the surface defect partition functions (see section 5.6). The variety $\mathcal{O}_3[\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}]$

[7] The equation (5.3.46) matches exactly the one for the generalized Heun oper in [98], where it is derived from the constraints for the minimal punctures.

of such opers is parametrized by the accessory parameters,

$$\begin{aligned} H_1 &\equiv -\mathfrak{q}(1-\mathfrak{q})\frac{1}{\varepsilon_1}\frac{\partial\widetilde{\mathcal{W}}}{\partial\mathfrak{q}}, \qquad (5.3.48a)\\ &= (1-\mathfrak{q})\left(\frac{1}{2\varepsilon_1^2}\lim_{\varepsilon_2\to0}\left\langle \mathcal{O}_2\right\rangle_{A_1} - 1 + \delta_{\mathfrak{q}} + \delta_0\right) + \frac{3(\bar{a}-\bar{a}_3+\varepsilon_1)(\bar{a}_0-\bar{a})}{\varepsilon_1^2}\mathfrak{q} \\ H_2 &\equiv -(1-\mathfrak{q})\left(\frac{1}{3\varepsilon_1^3}\lim_{\varepsilon_2\to0}\left\langle \mathcal{O}_3\right\rangle_{A_1} + \frac{1}{\varepsilon_1^2}\left(3\bar{a}-\bar{a}_3+\frac{3\varepsilon_1}{2}\right)\mathfrak{q}\frac{\partial\widetilde{\mathcal{W}}}{\partial\mathfrak{q}} + \cdots\right) \qquad (5.3.48b)\\ &\quad + \mathfrak{q}\left(\frac{1}{\varepsilon_1^2}(3\bar{a}_0-6\bar{a}+3\bar{a}_3-2\varepsilon_1)\mathfrak{q}\frac{\partial\widetilde{\mathcal{W}}}{\partial\mathfrak{q}} + \cdots\right) \\ &\quad + \frac{\mathfrak{q}^2}{1-\mathfrak{q}}\frac{3(\bar{a}_0-\bar{a})(\bar{a}-\bar{a}_3+\varepsilon_1)(3\bar{a}_0-6\bar{a}+3\bar{a}_3-2\varepsilon_1)}{\varepsilon_1^3}. \end{aligned}$$

We present the full expression for H_2 in appendix D. Notice that the accessory parameters are expanded as series in $\mathfrak{q}$ whose coefficients are completely determined in gauge theoretical terms. In particular, the series begin with

$$\begin{aligned} H_1 &= \frac{(a_1-a_2)^2+(a_1-a_3)^2-(a_1-a_2)(a_1-a_3)}{3\varepsilon_1^2} - 1 + \delta_{\mathfrak{q}} + \delta_0 + \mathcal{O}(\mathfrak{q}) \\ H_2 &= \lambda_0 - \frac{\lambda_{\mathfrak{q}}}{2} - \frac{(2a_1-a_2-a_3)(-a_1+2a_2-a_3)(-a_1-a_2+2a_3)}{27\varepsilon_1^3} \\ &\quad + \frac{2\bar{a}-2\bar{a}_3+\varepsilon_1}{2\varepsilon_1}\left(\delta_0 - 1 + \frac{(a_1-a_2)^2+(a_1-a_3)^2-(a_1-a_2)(a_1-a_3)}{3\varepsilon_1^2}\right) + \mathcal{O}(\mathfrak{q}). \end{aligned} \qquad (5.3.49)$$

Thus holomorphic coordinates on the variety $\mathcal{O}_3[\mathbb{P}^1\backslash\{0,\underline{\mathfrak{q}},\underline{1},\infty\}]$ of opers are given by the expectation values of the chiral observables in the A_1-theory, $\mathcal{O}_2$ and $\mathcal{O}_3$ [8], in the limit $\varepsilon_2 \to 0$. Hence we observe that the variety $\mathcal{O}_3[\mathbb{P}^1\backslash\{0,\underline{\mathfrak{q}},\underline{1},\infty\}]$ of opers gives a quantization of the Coulomb moduli space of $\mathcal{T}[A_2,\mathbb{P}^1\backslash\{0,\underline{\mathfrak{q}},\underline{1},\infty\}]$. The Bethe/gauge correspondence identifies

[8]Here, we are using the fact that

$$\lim_{\varepsilon_2\to0}\left\langle \mathcal{O}_3\right\rangle_{A_2} = \lim_{\varepsilon_2\to0}\frac{\left\langle \mathcal{I}_\beta^L\,\mathcal{O}_3\right\rangle_{A_1}}{\left\langle \mathcal{I}_\beta^L\right\rangle_{A_1}} = \lim_{\varepsilon_2\to0}\left\langle \mathcal{O}_3\right\rangle_{A_1},$$

since the expectation value is dominated by the limit shape when $\varepsilon_2 \to 0$.

the Nekrasov-Shatashvili limits of the expectation values of $\mathcal{O}_2$ and $\mathcal{O}_3$ with the off-shell spectra of the Hamiltonians of the quantum Hitchin system on $\mathbb{P}^1\backslash\{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$. Thus, the relations (5.3.48) establish the connection between the holomorphic functions on the variety $\mathcal{O}_3[\mathbb{P}^1\backslash\{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}]$ of opers and the off-shell spectra of the quantum Hitchin Hamiltonians.

Remarks

- The gauge theoretical derivation of the series expansions (5.3.48) for the accessory parameters guarantee their validity to all orders in the gauge coupling $\mathfrak{q}$. It would be nice to mimick the procedure in [110, 111] and check the series expansions by directly computing the monodromy of the oper $\widehat{\mathfrak{D}}_3$ (5.3.46) along the A-cycle on $\mathbb{P}^1\backslash\{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$ (see Figure 5.5).

- From the point of view of the AGT correspondence [63], the expectation value of the higher chiral observable $\mathcal{O}_3$ corresponds to the conformal block with a $\mathcal{W}$-descendant (we briefly mention this issue in section 5.7). It is not very obvious how we should relate the semi-classical conformal block with a $\mathcal{W}$-descendant to the off-shell spectrum of the higher quantum Hitchin Hamiltonian. In the gauge theoretical perspective, the Bethe/gauge correspondence immediately establishes the relation between the expectation value of $\mathcal{O}_3$ and the off-shell spectrum of the higher quantum Hitchin Hamiltonian. Thus, the relation between the accessory parameter H_2 and the off-shell spectrum of the higher quantum Hitchin Hamiltonian is also revealed through (5.3.48b).

Similarly, we can start by imposing other constraints, e.g. (5.2.34) on the A_2-theory. Hence we consider the partition function $\mathcal{Z}_\beta^R$ (5.2.37). Again, we modify the partition function as (5.3.31) with the prefactors (5.3.32), yet this time under the constraint (5.2.34) and the re-definition (5.2.36). The final form of the prefactor is

$$\begin{aligned}
&\left(-\frac{\mathfrak{q}}{z}\right)^{-r_{R,\beta}} \mathfrak{q}^{\frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2-\frac{(a_{1,1}-a_{1,2})^2+(a_{1,1}-a_{1,3})^2-(a_{1,1}-a_{1,2})(a_{1,1}-a_{1,3})}{3}\right)-\Delta'_{\mathfrak{q}}-\Delta_0+\frac{3\varepsilon+\varepsilon_2}{3\varepsilon_1}} \\
&(1-\mathfrak{q})^{\frac{(\bar{a}_0-\bar{a}_1+\varepsilon)(3\bar{a}_1-3\bar{a}_3-\varepsilon_2)}{\varepsilon_1\varepsilon_2}}(1-z)^{\frac{\bar{a}_0-\bar{a}_1+\varepsilon}{\varepsilon_1}}\left(1-\frac{z}{\mathfrak{q}}\right)^{\frac{3\bar{a}_1-3\bar{a}_3+3\varepsilon-\varepsilon_2}{3\varepsilon_1}},
\end{aligned} \tag{5.3.50}$$

where we have defined

$$(r_{R,\beta})_{\beta=1}^3 \equiv \left(\frac{-3a_{3,\beta}+\sum_{\gamma=1}^3 a_{3,\gamma}+3\varepsilon}{3\varepsilon_1}\right)_{\beta=1}^3 \tag{5.3.51}$$

and

$$\begin{aligned}
\Delta'_{\mathfrak{q}} &\equiv -\frac{(3\bar{a}_1-3\bar{a}_3-\varepsilon_2)(3\bar{a}_1-3\bar{a}_3+3\varepsilon-\varepsilon_2)}{3\varepsilon_1\varepsilon_2} \\
\Delta'_1 &\equiv -\frac{(\bar{a}_0-\bar{a}_1)(\bar{a}_0-\bar{a}_1+\varepsilon)}{\varepsilon_1\varepsilon_2}.
\end{aligned} \tag{5.3.52}$$

Let us also define

$$\begin{aligned}
\Lambda'_{\mathfrak{q}} &\equiv \frac{(3\bar{a}_1-3\bar{a}_3+3\varepsilon-\varepsilon_2)(3\bar{a}_1-3\bar{a}_3-\varepsilon_2)(6\bar{a}_1-6\bar{a}_3-3\varepsilon+2\varepsilon_2)}{27\varepsilon_1^3} \\
\Lambda'_1 &\equiv \frac{(\bar{a}_0-\bar{a}_1)(\bar{a}_0-\bar{a}_1+\varepsilon)(2\bar{a}_0-2\bar{a}_1+\varepsilon)}{\varepsilon_1^3}.
\end{aligned} \tag{5.3.53}$$

Then the modified partition function $\widetilde{\mathcal{Z}}^R_\beta$ satisfies the equation of the form (5.3.37), after substituting $\Delta_{\mathfrak{q},1} \to \Delta'_{\mathfrak{q},1}$ and $\Lambda_{\mathfrak{q},1} \to \Lambda'_{\mathfrak{q},1}$.

5.3.2 The $(N-1,1)$-type $\mathbb{Z}_2$-orbifold

We construct the surface defect on the A_1-theory by placing it on $\mathbb{Z}_p$-orbifold. Due to the orbifolding, the bulk $\mathcal{Y}$-observable fractionalizes into p observables,

$$\mathcal{Y}_\omega(x)[\boldsymbol{\lambda}] = \prod_{\alpha\in c^{-1}(\omega)}(x-a_\alpha)\prod_{\square\in K_\omega}\frac{x-c_\square-\varepsilon_1}{x-c_\square}\prod_{\square\in K_{\omega-1}}\frac{x-c_\square-\varepsilon_2}{x-c_\square-\varepsilon}. \tag{5.3.54}$$

The fundamental refined qq-characters are given by [15]

$$\mathcal{X}_\omega(x) = \mathcal{Y}_{\omega+1}(x+\varepsilon) + \mathfrak{q}_\omega \frac{P_\omega(x)}{\mathcal{Y}_\omega(x)}. \tag{5.3.55}$$

It is often possible to derive a useful equation for the partition function for specific p and the coloring function c from the non-perturbative Dyson-Schwinger equations of (5.3.55). We now describe how this is be done for the $(N-1,1)$-type $\mathbb{Z}_2$-orbifold. The details of the computation for the non-regular parts of $\mathcal{X}_\omega$ is given in the appendix E. Below we focus on the results.

5.3.2.1 $N = 2$

For $N = 2$, we consider $(1,1)$-type $\mathbb{Z}_2$-orbifold. This case is special since the coloring function is one-to-one. Let us define

$$c^{-1}(1) = \beta, \quad c^{-1}(0) = \bar{\beta}, \tag{5.3.56}$$

without any loss of generality. Each of the non-perturbative Dyson-Schwinger equations

$$[x^{-1}]\Big\langle \mathcal{X}_0(x) \Big\rangle = [x^{-1}]\Big\langle \mathcal{X}_1(x) \Big\rangle = 0 \tag{5.3.57}$$

involoves the unwanted term

$$\Bigg\langle \sum_{\square \in K_0} c_\square - \sum_{\square \in K_1} c_\square \Bigg\rangle, \tag{5.3.58}$$

but they can be combined to cancel this term and to yield the following closed equation

$$
\begin{aligned}
0 = \Bigg[&\varepsilon_1^2 (z\partial)^2 - \varepsilon_1 \left\{ -2\widetilde{a}_{\bar{\beta}} + \sum_{\alpha=1,2} \widetilde{m}_{+,\alpha} - \frac{\mathfrak{q}}{z-\mathfrak{q}} \sum_{\alpha=1,2} (\widetilde{a}_\alpha - \widetilde{m}_{-,\alpha}) + \frac{1}{1-z} \sum_{\alpha=1,2} (\widetilde{a}_\alpha - \widetilde{m}_{+,\alpha}) \right\} (z\partial) \\
&+ \varepsilon_1 \widetilde{\varepsilon}_2 \frac{z(1-\mathfrak{q})}{(1-z)(z-\mathfrak{q})} \mathfrak{q} \frac{\partial}{\partial \mathfrak{q}} + \frac{1}{2} \left(\widetilde{a}_{\bar{\beta}} - \sum_{\alpha=1,2} \widetilde{m}_{+,\alpha} \right)^2 + \frac{1}{2} \widetilde{a}_{\bar{\beta}}^2 - \frac{1}{2} \sum_{\alpha=1,2} \widetilde{m}_{+,\alpha}^2 \\
&- \frac{1}{2(1-z)} \left[\left(\widetilde{a}_{\bar{\beta}} - \sum_{\alpha=1,2} \widetilde{m}_{+,\alpha} \right)^2 + \widetilde{a}_{\bar{\beta}}^2 - \sum_{\alpha=1,2} \widetilde{m}_{+,\alpha}^2 \right] \\
&- \frac{\mathfrak{q}}{2(z-\mathfrak{q})} \left[\left(\widetilde{a}_\beta - \sum_{\alpha=1,2} \widetilde{m}_{-,\alpha} - \frac{\widetilde{\varepsilon}_2}{2} \right)^2 + \left(\widetilde{a}_\beta + \frac{\widetilde{\varepsilon}_2}{2} \right)^2 - \sum_{\alpha=1,2} \left(\widetilde{m}_{-,\alpha} + \frac{\widetilde{\varepsilon}_2}{2} \right)^2 \right] \Bigg] \mathcal{Z}_\beta^{\mathbb{Z}_2},
\end{aligned}
\tag{5.3.59}
$$

where we have re-defined the couplings as in (5.2.64), $\mathfrak{q}_0 = -z$ and $\mathfrak{q}_1 = -\frac{\mathfrak{q}}{z}$ (up to the sign which is not very important). Now, let us also re-define the parameters as

$$
\widetilde{a}_\alpha = a_{2,\alpha}, \quad \widetilde{m}_{+,\alpha} = a_{0,\alpha}, \quad \widetilde{m}_{-,\alpha} = a_{3,\alpha} - \varepsilon_1 - \widetilde{\varepsilon}_2, \quad \alpha = 1, 2. \tag{5.3.60}
$$

Then we decouple multiplicative prefactors

$$
\begin{aligned}
\widetilde{\mathcal{Z}}_\beta^{\mathbb{Z}_2} \equiv &- \left(-\frac{1}{z} \right)^{-r_\beta^{\mathbb{Z}_2}} \mathfrak{q}^{\frac{\varepsilon^2 - (a_{2,1} - a_{2,2})^2}{4\varepsilon_1 \varepsilon_2} - \Delta_0 - \Delta_{\mathfrak{q}}} \\
&(1-z)^{\frac{\varepsilon(2\bar{a}_0 - 2\bar{a}_2 - \varepsilon_2)}{2\varepsilon_1 \varepsilon_2}} (1-\mathfrak{q})^{\frac{(2\bar{a}_0 - 2\bar{a}_2 - \varepsilon_2)(\bar{a}_2 - \bar{a}_3 + \varepsilon)}{\varepsilon_1 \varepsilon_2}} \left(1 - \frac{\mathfrak{q}}{z} \right)^{\frac{\bar{a}_2 - \bar{a}_3 + \varepsilon}{\varepsilon_1}} \mathcal{Z}_\beta^{\mathbb{Z}_2},
\end{aligned}
\tag{5.3.61}
$$

where

$$
\left(r_\beta^{\mathbb{Z}_2} \right)_{\beta=1,2} \equiv \left(\frac{-a_{2,1} + a_{2,2} + \varepsilon}{2\varepsilon_1}, \ \frac{a_{2,1} - a_{2,2} + \varepsilon}{2\varepsilon_1} \right), \tag{5.3.62}
$$

and the other exponents have been defined in the previous section. The differential equation (5.3.59) then becomes

$$
\begin{aligned}
0 = \Bigg[& \varepsilon_1^2 \partial^2 - \varepsilon_1\varepsilon_2 \frac{2z-1}{z(z-1)} \partial + \varepsilon_1\varepsilon_2 \frac{\mathfrak{q}-1}{z(z-1)(z-\mathfrak{q})} \mathfrak{q}\frac{\partial}{\partial \mathfrak{q}} \\
& + \varepsilon_1\varepsilon_2 \left(\frac{\Delta_0}{z^2} + \frac{\Delta_1}{(z-1)^2} + \frac{\Delta_{\mathfrak{q}}}{(z-\mathfrak{q})^2} - \frac{-\frac{2\varepsilon+\varepsilon_2}{4\varepsilon_1} + \Delta_1 + \Delta_{\mathfrak{q}} + \Delta_0 - \Delta_\infty}{z(z-1)} \right) \Bigg] \widetilde{\mathcal{Z}}_\beta^{\mathbb{Z}_2},
\end{aligned}
\tag{5.3.63}
$$

which is precisely the differential equation (5.3.17) for $\hat{\hat{\mathfrak{D}}}_2$.

Remarks

- The convergence domain for the partition function is $0 < |\mathfrak{q}_0|, |\mathfrak{q}_1| < 1$. This implies the solutions $\widetilde{\mathcal{Z}}_\beta^{\mathbb{Z}_2}$ are in yet another *intermediate* domain $0 < |\mathfrak{q}| < |z| < 1$.

5.3.2.2 $N = 3$

For $N = 3$, the computation is more involved. First, recall that the $(2,1)$-type $\mathbb{Z}_2$-orbifold surface defect partition function (5.2.74) is split into the underlying A_1-theory part and the surface defect part. The fixed points of the instanton moduli space of the underlying A_1-theory are enumerated by the Young diagrams $\boldsymbol{\Lambda}$ (5.2.55), whose weights are encoded in the space $\widetilde{K} = \widetilde{K}_1$ (5.2.54). Thus the observables in the underlying A_1-theory descends from the observables in the space K_1 of the original theory on the $\mathbb{Z}_2$-orbifold . In particular, we have

$$
\sum_{\square \in K_1} c_\square = \sum_{\square \in \boldsymbol{\Lambda}} \tilde{c}_\square + \frac{1}{2} \tilde{\varepsilon}_2 k_1,
\tag{5.3.64}
$$

where

$$
\tilde{c}_\square \equiv \tilde{a}_\alpha + (i-1)\varepsilon_1 + (j-1)\tilde{\varepsilon}_2, \quad \text{for} \quad \square_{(i,j)} \in \Lambda^{(\alpha)}.
\tag{5.3.65}
$$

We will reduce the non-perturbative Dyson-Schwinger equations so that the final equation only involves the expectation value of this observable, since it comprises the chiral observable

$$\mathcal{O}_3[\mathbf{\Lambda}] = \sum_{\alpha=1}^{3} \widetilde{a}_\alpha^3 - 3\varepsilon_1\widetilde{\varepsilon}_2(\varepsilon_1 + \widetilde{\varepsilon}_2)k_1 - 6\varepsilon_1\widetilde{\varepsilon}_2 \sum_{\square\in\mathbf{\Lambda}} \widetilde{c}_\square, \tag{5.3.66}$$

of the underlying A_1-theory. The non-perturbative Dyson-Schwinger equations that we utilize are

$$[x^{-1}]\Big\langle \mathcal{X}_1(x) \Big\rangle = [x^{-1}]\Big\langle \mathcal{X}_0(x) \Big\rangle = [x^{-2}]\Big\langle \mathcal{X}_0(x) \Big\rangle = 0. \tag{5.3.67}$$

The second equation can be used to cancel the unwanted terms

$$\Big\langle \sum_{\square\in K_0} c_\square \Big\rangle, \quad \Big\langle (k_0 - k_1) \Big(\sum_{\square\in K_0} c_\square - \sum_{\square\in K_1} c_\square \Big) \Big\rangle, \tag{5.3.68}$$

while the first and the third equations can be combined to cancel the unwanted term

$$\Big\langle \sum_{\square\in K_0} c_\square^2 - \sum_{\square\in K_1} c_\square^2 \Big\rangle, \tag{5.3.69}$$

The final equation only involves the partition function itself and the expectation value $\left\langle \sum_{\square\in\boldsymbol{\Lambda}} \widetilde{c}_\square \right\rangle$:

$$
\begin{aligned}
0 = \Bigg[& -\varepsilon_1^3 (z\partial)^3 + \varepsilon_1^2 \left(3\widetilde{a}_\beta - 3\bar{\widetilde{a}} + \frac{3\mathfrak{q}}{z-\mathfrak{q}}\left(-\bar{\widetilde{a}} + \bar{\widetilde{m}}_-\right) - \frac{6z}{z-1}\left(\bar{\widetilde{a}} - \bar{\widetilde{m}}_+\right)\right)(z\partial)^2 \\
& - \varepsilon_1 \Bigg\{ -\frac{z}{1-z}\left(\left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - \sum_{\alpha=1}^{3}\widetilde{m}_{+,\alpha}\right)^2 + \sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta}^2 - \sum_{\alpha=1}^{3}\widetilde{m}_{+,\alpha}^2\right) - \frac{6\varepsilon_1\left(\bar{a} - \bar{\widetilde{m}}_+\right)z}{(1-z)^2} + \frac{\widetilde{\varepsilon}_2}{2}\left(\varepsilon_1 + \frac{\widetilde{\varepsilon}_2}{2}\right) \\
& + \frac{\prod_{\bar\beta\neq\beta}\left(\varepsilon_1 + \frac{\widetilde{\varepsilon}_2}{2} - \widetilde{a}_{\bar\beta}\right)z}{z-\mathfrak{q}} + \frac{\varepsilon_1(\varepsilon_1 - \widetilde{a}_\beta)}{2(1-z)} - \frac{z(\varepsilon_1 + \widetilde{\varepsilon}_2)\left(2\varepsilon_1 + \widetilde{\varepsilon}_2 - \sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta}\right)}{2(z-\mathfrak{q})} \\
& - \frac{\mathfrak{q}}{2(z-\mathfrak{q})}\left(\left(\widetilde{a}_\beta - 3\bar{\widetilde{m}}_- - \widetilde{\varepsilon}_2\right)^2 + (\varepsilon_1 + \widetilde{\varepsilon}_2)(\widetilde{a}_\beta - 3\bar{\widetilde{m}}_- - \widetilde{\varepsilon}_2) + \left(\widetilde{a}_\beta + \frac{\widetilde{\varepsilon}_2}{2}\right)^2 - \sum_{\alpha=1}^{3}\left(\widetilde{m}_{-,\alpha} + \frac{\widetilde{\varepsilon}_2}{2}\right)^2\right) \\
& + \frac{z}{2(1-z)}\left(\left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - 3\bar{\widetilde{m}}_+\right)^2 + \sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta}^2 - \sum_{\alpha=1}^{3}\widetilde{m}_{+,\alpha}^2 + \varepsilon_1\left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - \sum_{\alpha=1}^{3}\widetilde{m}_{+,\alpha}\right)\right) \\
& - \varepsilon_1\widetilde{\varepsilon}_2\frac{z(1-\mathfrak{q})}{(z-1)(z-\mathfrak{q})}\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} + \left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - 3\bar{\widetilde{m}}_+ + \frac{\varepsilon_1}{2} + \frac{3(\bar{\widetilde{a}} - \bar{\widetilde{m}}_+)}{z-1}\right)\left(2\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - 3\bar{\widetilde{m}}_+ + \frac{3(\bar{\widetilde{a}} - \bar{\widetilde{m}}_+) + \varepsilon_1}{z-1} + \frac{3\mathfrak{q}(\bar{\widetilde{a}} - \bar{\widetilde{m}}_-)}{z-\mathfrak{q}}\right)\Bigg\}(z\partial \\
& + \Bigg\{ -\frac{2\varepsilon_1 z}{(1-z)^2} + \frac{1+z}{2(1-z)}\left(-2\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} + 3\bar{\widetilde{m}}_+ - \frac{3(\bar{\widetilde{a}} - \bar{\widetilde{m}}_+) + \varepsilon_1}{z-1} - \frac{3\mathfrak{q}(\bar{\widetilde{a}} - \bar{\widetilde{m}}_-)}{z-\mathfrak{q}}\right) - \varepsilon_1\frac{z(1-\mathfrak{q})}{(1-z)(z-\mathfrak{q})} \\
& - \varepsilon_1 - \frac{\widetilde{\varepsilon}_2}{2} + \frac{\varepsilon_1 - \widetilde{a}_\beta}{2(1-z)} + \frac{z(2\varepsilon_1 + \widetilde{\varepsilon}_2 - \sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta})}{2(z-\mathfrak{q})} + \frac{z\left(-\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} + 3\bar{\widetilde{m}}_+\right)}{2(z-1)} + \frac{(\widetilde{a}_\beta - \widetilde{\varepsilon}_2 - 3\bar{\widetilde{m}}_-)\mathfrak{q}}{2(z-\mathfrak{q})}\Bigg\}\varepsilon_1\widetilde{\varepsilon}_2\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} \\
& + \frac{z(1-\mathfrak{q})}{(1-z)(z-\mathfrak{q})}\left(2\varepsilon_1\widetilde{\varepsilon}_2\left\langle\sum_{\square\in\boldsymbol{\Lambda}}\widetilde{c}_\square\right\rangle + \varepsilon_1\widetilde{\varepsilon}_2(\varepsilon_1 + \widetilde{\varepsilon}_2)\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}}\right) + \frac{z\varepsilon_1}{(1-z)^2}\left(\left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - 3\bar{\widetilde{m}}_+\right)^2 + \sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta}^2 - \sum_{\alpha=1}^{3}\widetilde{m}_{+,\alpha}^2\right) \\
& + \frac{z}{2(1-z)}\left(\left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - 3\bar{\widetilde{m}}_+\right)^2 + \sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta}^2 - \sum_{\alpha=1}^{3}\widetilde{m}_{+,\alpha}^2\right)\left(2\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - 3\bar{\widetilde{m}}_+ + \frac{3(\bar{\widetilde{a}} - \bar{\widetilde{m}}_+) + \varepsilon_1}{z-1} + \frac{3\mathfrak{q}(\bar{\widetilde{a}} - \bar{\widetilde{m}}_-)}{z-\mathfrak{q}}\right) \\
& - \frac{\mathfrak{q}}{z-\mathfrak{q}}\Bigg(\frac{\left(\widetilde{a}_\beta - 3\bar{\widetilde{m}}_- - \widetilde{\varepsilon}_2\right)^3}{6} + \frac{\left(\widetilde{a}_\beta + \frac{\widetilde{\varepsilon}_2}{2}\right)^3 - \sum_{\alpha=1}^{3}\left(\widetilde{m}_{-,\alpha} + \frac{\widetilde{\varepsilon}_2}{2}\right)^3}{3} \\
& \qquad + \frac{\left(\widetilde{a}_\beta - 3\bar{\widetilde{m}}_- - \widetilde{\varepsilon}_2\right)\left(\left(\widetilde{a}_\beta + \frac{\widetilde{\varepsilon}_2}{2}\right)^2 - \sum_{\alpha=1}^{3}\left(\widetilde{m}_{-,\alpha} + \frac{\widetilde{\varepsilon}_2}{2}\right)^2\right)}{2}\Bigg) \\
& - \frac{z}{1-z}\left(\frac{\left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - 3\bar{\widetilde{m}}_+\right)^3}{6} + \frac{\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta}^3 - \sum_{\alpha=1}^{3}\widetilde{m}_{+,\alpha}^3}{3} + \frac{\left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta} - 3\bar{\widetilde{m}}_+\right)\left(\sum_{\bar\beta\neq\beta}\widetilde{a}_{\bar\beta}^2 - \sum_{\alpha=1}^{3}\widetilde{m}_{+,\alpha}^2\right)}{2}\right)\Bigg]\mathcal{Z}_\beta^{\mathbb{Z}_2},
\end{aligned}
\tag{5.3.70}
$$

where we have re-defined the couplings as $\mathfrak{q}_0 = -z$ and $\mathfrak{q}_1 = -\frac{\mathfrak{q}}{z}$. Let us also re-define the other parameters as

$$\widetilde{a}_\alpha = a_{2,\alpha}, \quad \widetilde{m}_{+,\alpha} = a_{0,\alpha}, \quad \widetilde{m}_{-,\alpha} = a_{3,\alpha} - \varepsilon_1 - \widetilde{\varepsilon}_2, \quad \alpha = 1,2,3. \tag{5.3.71}$$

and modify the partition function by the prefactors,

$$\begin{aligned} \widetilde{\mathcal{Z}}_\beta^{\mathbb{Z}_2} \equiv - \left(-\frac{1}{z}\right)^{-r_\beta^{\mathbb{Z}_2}} \mathfrak{q}^{\frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2 - \frac{(a_{2,1}-a_{2,2})^2+(a_{2,1}-a_{2,3})^2-(a_{2,1}-a_{2,2})(a_{2,1}-a_{2,3})}{3}\right)-\Delta_{\mathfrak{q}}-\Delta_0} \\ (1-z)^{-\frac{2(3\bar{a}_0-3\bar{a}_2-\varepsilon_2)}{3\varepsilon_1}} (1-\mathfrak{q})^{\frac{(3\bar{a}_0-3\bar{a}_2-\varepsilon_2)(\bar{a}_2-\bar{a}_3+\varepsilon)}{\varepsilon_1\varepsilon_2}} \left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{\bar{a}_2-\bar{a}_3+\varepsilon}{\varepsilon_1}} \mathcal{Z}_\beta^{\mathbb{Z}_2}. \end{aligned} \tag{5.3.72}$$

Here, we have defined the critical exponent for z as

$$\left(r_\beta^{\mathbb{Z}_2}\right)_{\beta=1}^3 = \left(\frac{-3a_{2,\beta} + \sum_{\gamma=1}^3 a_{2,\gamma} + 3\varepsilon}{3\varepsilon_1}\right)_{\beta=1}^3 . \tag{5.3.73}$$

Then the equation satisfied by the modified partition function $\widetilde{\mathcal{Z}}_\beta^{\mathbb{Z}_2}$ becomes of the form (5.3.37).

5.4 Analytic continuation and gluing

To compute the monodromies of the solutions to the quantized opers, it is necessary to know how to connect the solutions in different convergence domains. We accomplish this by analytically continuing the surface defect partition functions to different convergence domains, and gluing those continuations in the intermediate regime.

5.4.1 Analytic continuation

We use the duality transformation similar to the one described on p.13 of [115]. There, one traded the sum over the fluxes of the two dimensional abelian gauge field (magnetic fluxes) for the sum over a dual integral variable (electric flux), which could be viewed as the label enumerating the sheets of the (possibly disconnected) effective target space.

5.4.1.1 Gauged linear sigma model

Let us begin with the two-dimensional gauged linear sigma model (GLSM), which would generate the surface defect when coupled to the four-dimensional A_1-theory. In section 5.2, we have shown that the 2d GLSM responsible for the quiver surface defect and the $(N-1,1)$-type $\mathbb{Z}_2$-orbifold surface defect is the one which flows to the non-linear sigma model on the $\text{Hom}(\mathcal{O}(-1),\mathbb{C}^N)$-bundle over $\mathbb{P}^{N-1}$. This theory is the $\mathcal{N}=(2,2)$ supersymmetric $U(1)$ gauge theory with the field contents

$$\begin{aligned} \text{Twisted chiral}:&\quad \Sigma=(\sigma,A) \\ \text{Fundamental chiral}:&\quad Q_\alpha \quad \alpha=1,\cdots,N, \\ \text{Anti-fundamental chiral}:&\quad \widetilde{Q}_\alpha \quad \alpha=1,\cdots,N, \end{aligned} \tag{5.4.1}$$

where we have only denoted the bosonic component fields. By weakly gauging the $\left(U(N)\times U(N)\right)/U(1)$ flavor symmetry, the fundamental and the anti-fundamental acquire the twisted masses which we denote as $(a_{0,\alpha})_{\alpha=1}^N$ and $(a_{2,\alpha})_{\alpha=1}^N$ respectively, for the reason to be clarified soon. Note that we may re-define σ by a constant amount so that the twisted masses appear as if weakly gauging the full $U(N)\times U(N)$ symmetry. Due to the twisted masses all the chiral multiplets can be integrated out. The resulting effective theory is the $\mathcal{N}=(2,2)$ $U(1)$ gauge theory with the effective twisted superpotential

$$\widetilde{\mathcal{W}}(\sigma)=-t\sigma-\sum_{\alpha=1}^{N}(\sigma-a_{0,\alpha})\left(\log(\sigma-a_{0,\alpha})-1\right)-\sum_{\alpha=1}^{N}(-\sigma+a_{2,\alpha})\left(\log(-\sigma+a_{2,\alpha})-1\right), \tag{5.4.2}$$

where we have introduced the complex coupling $t = r - i\theta$ from the Fayet-Illiopoulos parameter r and the two-dimensional θ-angle. Hence the vacuum equation reads

$$\prod_{\alpha=1}^{N} \frac{-\sigma + a_{2,\alpha}}{\sigma - a_{0,\alpha}} = e^t = z, \tag{5.4.3}$$

with the Kähler modulus defined by $z \equiv e^t$. Note that the Fayet-Illiopoulos parameter r is not renormalized since the total charge of the chiral multiplets is zero, and we can imagine flowing from the region $r \gg 0$ to the region $r \ll 0$. The GLSM in both regions gives rise to the non-linear sigma model on the $\mathrm{Hom}(\mathcal{O}(-1), \mathbb{C}^N)$-bundle over $\mathbb{P}^{N-1}$, yet with the base and the fiber exchanged with each other as we cross $r = 0$. The classical singularity at $r = 0$ is actually shifted by the quantum effect, leaving only a single point $\theta = N\pi$ (mod 2π) singular. Hence the flow can be smoothly continued to the other region, connecting the two sigma models. The vacuum equation (5.4.3) implies that the N-vacua continuously flow from $\sigma \sim a_{0,\alpha}$ at $r \gg 0$ to $\sigma \sim a_{2,\alpha}$ at $r \ll 0$.

Upon the Ω-deformation on the two-dimensional plane, the partition function of the GLSM can be exactly computed by the equivariant localization. The effective twisted superpotential only exhibits the leading singular term in the partition function, so we investigate how the flow of z appears at the level of the partition function. The partition function localizes on the generalized vortex configurations,

$$\begin{aligned} &D_{\bar{z}} Q \equiv \partial_{\bar{z}} Q + A_{\bar{z}} Q = 0 \\ &D_{\bar{z}} \widetilde{Q} = 0 \\ &F_{z\bar{z}} + |Q|^2 - |\widetilde{Q}|^2 = r. \end{aligned} \tag{5.4.4}$$

Depending on the sign of r, we are forced to localize on either vortices or anti-vortices. Let us assume $r > 0$ for now. The asymptotics of the D-term equation forbids the anti-fundamental $\widetilde{Q}$ to generate any bosonic moduli, and only allows its fermionic zero-modes [116]. The final form of the partition function is precisely the expression (5.2.29) without the coupling to the four-dimension,

$$
\begin{aligned}
\mathcal{Z}_\beta^{\mathrm{GLSM}} &= \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} \frac{\prod_{\alpha=1}^N \left(1+\frac{a_{0,\beta}-a_{2,\alpha}}{\varepsilon_1}\right)_k}{\prod_{\alpha\neq\beta}\left(1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)_k} \\
&= {}_NF_{N-1}\left(\left(1+\frac{a_{0,\beta}-a_{2,\alpha}}{\varepsilon_1}\right)_{\alpha=1,\cdots,N};\ \left(1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)_{\alpha\neq\beta};\ z^{-1}\right),
\end{aligned}
\tag{5.4.5}
$$

where we have chosen the vacuum at the infinity as $\sigma = a_{0,\beta}$. The effective twisted superpotential evaluated at this vacuum can be obtained by taking the asymptotics of the partition function,

$$
\mathcal{Z}_\beta^{\mathrm{GLSM}} = e^{\frac{\widetilde{\mathcal{W}}_\beta}{\varepsilon_1}} \left(1+\mathcal{O}(\varepsilon_1)\right). \tag{5.4.6}
$$

Once we flow to the region $r<0$, the above series expansion is no longer valid. However, we can still study the asymptotics of the partition function, i.e., the effective twisted superpotential, in this region by applying the Picard-Lefschetz theory to the integral representation of the partition function [117]. To illustrate the idea, let us consider the case $N=2$. Also let us assume $\mathrm{Re}\left(1+\frac{a_{0,1}-a_{0,2}}{\varepsilon_1}\right) > \mathrm{Re}\left(1+\frac{a_{0,1}-a_{2,2}}{\varepsilon_1}\right) > 0$ for simplicity. Then the Euler integral representation for the hypergeometric function gives

$$
\mathcal{Z}_1^{\mathrm{GLSM}} = \frac{\Gamma\left(1+\frac{a_{0,1}-a_{0,2}}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,1}-a_{2,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{2,2}-a_{0,2}}{\varepsilon_1}\right)} \int_0^1 dt\, t^{\frac{a_{0,1}-a_{0,2}}{\varepsilon_1}} (1-t)^{-1+\frac{a_{2,2}-a_{0,2}}{\varepsilon_1}} (1-z^{-1}t)^{-1-\frac{a_{0,1}-a_{2,1}}{\varepsilon_1}}. \tag{5.4.7}
$$

We now promote the real integral to an integral on the complex t-plane. We can represent the integral as

$$\int_{C=[0,1]} dt\; g(t) e^{\frac{S(t)}{\varepsilon_1}},$$

$$g(t) = (1-t)^{-1}(1-z^{-1}t)^{-1}$$
$$S(t) = (a_{0,1} - a_{0,2}) \log t + (a_{2,2} - a_{0,2}) \log(1-t) - (a_{0,1} - a_{2,1}) \log(1 - z^{-1}t). \tag{5.4.8}$$

The critical points of $S(t)$ are at

$$S'(t) = \frac{a_{0,1} - a_{2,2}}{t} - \frac{a_{2,2} - a_{0,2}}{1-t} + \frac{(a_{0,1} - a_{2,1})z^{-1}}{1 - z^{-1}t} = 0. \tag{5.4.9}$$

Let us denote the critical points as $t_\pm$, namely, $S'(t_\pm) = 0$. Let us assume that the masses are generic enough so that the critical points $t_\pm$ are distinct. We would like to deform the integration contour C into a union of paths, in which each path passes through one of the critical points and the imaginary part $\mathrm{Im}S(t)$ is constant along the path. Such paths are called *the Lefschetz thimbles*, and can be obtained by treating the imaginary part of $S(t)$ as a Hamiltonian

$$H(t) \equiv \mathrm{Im}S(t) = \frac{1}{2i}(S(t) - \bar{S}(t)), \tag{5.4.10}$$

which defines the gradient flow by the equation

$$\dot{t} = \{H, \bar{t}\} = \omega^{ab}\partial_a H \partial_b \bar{t} = -\frac{\partial S(t)}{\partial t}, \tag{5.4.11}$$

where the symplectic form on the t-plane is given by $\omega = \frac{1}{2i} dt \wedge d\bar{t}$. The Lefschetz thimble $\mathcal{J}_\pm$ is defined as the union of these paths emanating from the critical points $t_\pm$. Note that $\mathrm{Re}S(t)$ monotonically decreases along the flow (5.4.11), so that the integral along $\mathcal{J}_\pm$ would show good convergence. Now the problem is decomposing the contour C into a union of those

Lefschetz thimbles, and this procedure can be done as follows. Note that the integration contour C defines an element of the relative homology $H_1(\mathbb{C}, \mathbb{C}_{-T}; \mathbb{Z})$, where

$$\mathbb{C}_{-T} \equiv \{t \in \mathbb{C} \mid \mathrm{Re}S(t) \leq -T\}, \tag{5.4.12}$$

for $T \gg 1$. The Lefschetz thimbles are defined as the paths emanating from the critical points, in which $\mathrm{Re}S(t)$ decreases along the flow. Hence the Lefschetz thimbles also define elements of the relative homology, $\mathcal{J}_\pm \in H_1(\mathbb{C}, \mathbb{C}_{-T}; \mathbb{Z})$, and moreover they actually form a basis of this relative homology. Thus we can express C as a linear combination of the basis elements $\mathcal{J}_\pm$, say, $C = \sum_\pm n_\pm \mathcal{J}_\pm$. Then the integral in the partition function can be expressed as

$$\sum_\pm n_\pm \int_{\mathcal{J}_\pm} dt\, g(t) e^{\frac{S(t)}{\varepsilon_1}}. \tag{5.4.13}$$

The remaining problem is to find the number $n_\pm$. For this, let us consider the relative homology $H_1(\mathbb{C}, \mathbb{C}^T; \mathbb{Z})$, where

$$\mathbb{C}^T \equiv \{t \in \mathbb{C} \mid \mathrm{Re}S(t) \geq T\}, \tag{5.4.14}$$

for $T \gg 1$. This relative homology is generated by *the dual Lefschetz thimbles*, $\mathcal{K}_\pm$, which are defined as the union of the paths (5.4.11) converging to the critical point $t_\pm$. Note that we have the intersection pairing

$$\langle \mathcal{J}_\tau, \mathcal{K}_{\tau'} \rangle = \delta_{\tau,\tau'}, \quad \tau, \tau' = \pm, \tag{5.4.15}$$

under an appropriate orientation on these thimbles, since $\mathcal{J}_\pm$ and $\mathcal{K}_\pm$ intersect at $t_\pm$ and

Re$S(t)$ only decreases or increases along these thimbles. Therefore, we derive

$$n_\pm = \langle C, \mathcal{K}_\pm \rangle, \tag{5.4.16}$$

and the final form of the integral is

$$\sum_\pm \langle C, \mathcal{K}_\pm \rangle \int_{\mathcal{J}_\pm} dt\, g(t) e^{\frac{S(t)}{\varepsilon_1}}. \tag{5.4.17}$$

When $r > 0$ ($|z| > 1$), it can be checked that only one dual thimble, say, $\mathcal{K}_+$, intersects with the original contour $C = [0,1]$. Hence the integral can be performed in the WKB sense as

$$\sqrt{-\frac{\pi \varepsilon_1}{S''(t_+)}}\, g(t_+)\, e^{\frac{S(t_+)}{\varepsilon_1}} \left(1 + \sum_{k=1}^{\infty} c_{+,k}\, \varepsilon_1^k\right) \tag{5.4.18}$$

In particular, the effective twisted superpotential is essentially $S(t_+)$. This confirms that we have a contribution from the single vacuum $\sigma = a_{0,\beta}$. However, when $r < 0$ ($|z| < 1$) the topology of thimbles change so that both dual thimbles $\mathcal{K}_\pm$ intersect with the contour $C = [0,1]$. Hence the integral is rather performed as

$$\sqrt{-\frac{\pi \varepsilon_1}{S''(t_+)}}\, g(t_+)\, e^{\frac{S(t_+)}{\varepsilon_1}} \left(1 + \sum_{k=1}^{\infty} c_{+,k}\, \varepsilon_1^k\right) + \sqrt{-\frac{\pi \varepsilon_1}{S''(t_-)}}\, g(t_-)\, e^{\frac{S(t_-)}{\varepsilon_1}} \left(1 + \sum_{k=1}^{\infty} c_{-,k}\, \varepsilon_1^k\right), \tag{5.4.19}$$

In other words, we start to get a contribution from the other vacuum, represented by the thimble $\mathcal{J}_-$. The continuous flow the the vacua (5.4.3) only exhibits the leading contribution from $\mathcal{J}_+$, but the Picard-Lefschetz analysis shows that the contribution from the other vacuum also emerges as we flow to the region $r < 0$.

For higher ranks $N \geq 3$, we have to deal with the Euler integral representation for the generalized hypergeometric function ${}_N F_{N-1}$ which is $N-1$-complex dimensional. It is more difficult to visualize, but the basic idea is the same. When we fix a vacuum in the region

$r > 0$ and flow to the region $r < 0$, the exponentially suppressed contributions from the other $N-1$-vacua start to emerge. It can be also understood as the manifestation of the analytic continuation of the generalized hypergeometric function. In the domain $|z| < 1$, the generalized hypergeometric function (5.4.5) is still well-defined by the analytic continuation, and the proper series expansion for this analytic continuation is simply obtained by the connection formula,

$$
\begin{aligned}
&{}_N F_{N-1}\left(\left(1+\frac{a_{0,\beta}-a_{2,\gamma}}{\varepsilon_1}\right)_{\gamma=1,\cdots,N};\ \left(1+\frac{a_{0,\beta}-a_{0,\beta'}}{\varepsilon_1}\right)_{\beta'\neq\beta};\ z^{-1}\right) \\
&= -\sum_{\alpha=1}^{N}\prod_{\beta'\neq\beta}\frac{\Gamma\left(1+\frac{a_{0,\beta}-a_{0,\beta'}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{2,\alpha}-a_{0,\beta'}}{\varepsilon_1}\right)}\prod_{\alpha'\neq\alpha}\frac{\Gamma\left(\frac{a_{2,\alpha}-a_{2,\alpha'}}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,\beta}-a_{2,\alpha'}}{\varepsilon_1}\right)}(-z)^{1+\frac{a_{0,\beta}-a_{2,\alpha}}{\varepsilon_1}} \\
&\quad {}_N F_{N-1}\left(\left(1+\frac{a_{0,\gamma}-a_{2,\alpha}}{\varepsilon_1}\right)_{\gamma=1,\cdots,N};\left(1+\frac{a_{2,\alpha'}-a_{2,\alpha}}{\varepsilon_1}\right)_{\alpha'\neq\alpha};\ z\right).
\end{aligned}
\tag{5.4.20}
$$

The Picard-Lefschetz analysis provides a physical interpretation of this formula, i.e., the emergence of other $N-1$-vacua as a consequence of the flow from $r > 0$ to $r < 0$.

5.4.1.2 Four-dimensional theory with surface defect

The analytic continuation along the flow of the Kähler modulus can be conducted in a more general setting: the two-dimensional gauged linear sigma model coupled to the four-dimensional gauge theory. Let us start with the quiver surface defect partition function (5.2.5) with the constraints (5.2.4), namely,

$$
\mathcal{Z}^L_\beta = \mathcal{Z}_{A_2}\left(\mathbf{a}_0;\, a_{1,\alpha} = a_{0,\alpha} - \varepsilon_2\delta_{\alpha,\beta};\, \mathbf{a}_2;\, \mathbf{a}_3 \mid \varepsilon_1,\varepsilon_2 \mid \mathfrak{q}_1 = z^{-1}, \mathfrak{q}_2 = \mathfrak{q}\right). \tag{5.4.21}
$$

We recall that this can be expressed in terms of the $\mathcal{Q}$-observables (5.2.32). Thus (5.2.5) can be written as

$$\begin{aligned}\mathcal{Z}^L_\beta &= \sum_{\boldsymbol{\lambda}^{(2)}} \mathfrak{q}_2^{|\boldsymbol{\lambda}^{(2)}|} \boldsymbol{\mu}_{\boldsymbol{\lambda}^{(2)}} \sum_{k=0}^{\infty} \mathfrak{q}_1^k \prod_{\alpha=1}^{N} \frac{(-1)^k\, \Gamma\left(1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)\Gamma\left(-\frac{a_{0,\beta}-a_{2,\alpha}}{\varepsilon_1}\right)}{\Gamma\left(k+1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)\Gamma\left(-k-\frac{a_{0,\beta}-a_{2,\alpha}}{\varepsilon_1}\right)} \\ &\qquad \prod_{\square\in K_2} \frac{a_{0,\beta}+k\varepsilon_1-c_\square-\varepsilon_2}{a_{0,\beta}+k\varepsilon_1-c_\square} \frac{a_{0,\beta}-c_\square}{a_{0,\beta}-c_\square-\varepsilon_2} \\ &= \sum_{\boldsymbol{\lambda}^{(2)}} \mathfrak{q}_2^{|\boldsymbol{\lambda}^{(2)}|} \boldsymbol{\mu}_{\boldsymbol{\lambda}^{(2)}} \sum_{k=0}^{\infty} \mathfrak{q}_1^k \prod_{\alpha=1}^{N} \frac{\Gamma\left(1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)\Gamma\left(k+1+\frac{a_{0,\beta}-a_{2,\alpha}}{\varepsilon_1}\right)}{\Gamma\left(k+1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)\Gamma\left(1+\frac{a_{0,\beta}-a_{2,\alpha}}{\varepsilon_1}\right)} \\ &\qquad \prod_{\square\in K_2} \frac{a_{0,\beta}+k\varepsilon_1-c_\square-\varepsilon_2}{a_{0,\beta}+k\varepsilon_1-c_\square} \frac{a_{0,\beta}-c_\square}{a_{0,\beta}-c_\square-\varepsilon_2},\end{aligned} \tag{5.4.22}$$

where we have used the reflection formula $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$ in the second equality. It is crucial to notice that the partition function now can be represented as a contour integral

$$\begin{aligned}\mathcal{Z}^L_\beta &= -\prod_{\alpha=1}^{N} \frac{\Gamma\left(1+\frac{a_{0,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,\beta}-a_{2,\alpha}}{\varepsilon_1}\right)} (-\mathfrak{q}_1)^{-\frac{a_{0,\beta}}{\varepsilon_1}} \sum_{\boldsymbol{\lambda}^{(2)}} \mathfrak{q}_2^{|\boldsymbol{\lambda}^{(2)}|} \widetilde{\boldsymbol{\mu}}_{\boldsymbol{\lambda}^{(2)}} \\ &\oint_{\mathcal{C}} dx\, (-\mathfrak{q}_1)^{\frac{x}{\varepsilon_1}} \frac{\Gamma\left(-\frac{x-a_{0,\beta}}{\varepsilon_1}\right)\prod_{\alpha=1}^{N}\Gamma\left(1+\frac{x-a_{2,\alpha}}{\varepsilon_1}\right)}{\prod_{\alpha\neq\beta}\Gamma\left(1+\frac{x-a_{0,\alpha}}{\varepsilon_1}\right)} \prod_{\alpha=1}^{N} \prod_{i=1}^{l\left(\lambda^{(2,\alpha)}\right)} \frac{x-a_{2,\alpha}-(i-1)\varepsilon_1-\lambda_i^{(2,\alpha)}\varepsilon_2}{x-a_{2,\alpha}-(i-1)\varepsilon_1},\end{aligned} \tag{5.4.23}$$

where we have defined

$$\begin{aligned}\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\lambda}^{(2)}} &\equiv \boldsymbol{\mu}_{\boldsymbol{\lambda}^{(2)}} \prod_{\square\in\boldsymbol{\lambda}^{(2)}} \frac{a_{0,\beta}-c_\square}{a_{0,\beta}-c_\square-\varepsilon_2} \\ &= \epsilon\left[N_2K_2^* + q_{12}N_2^*K_2 - P_{12}K_2K_2^* - M_0K_2^* - q_{12}M_3^*K_2\right].\end{aligned} \tag{5.4.24}$$

The contour $\mathcal{C}$ is described in Figure 5.2. Here, we are assuming the Coulomb moduli $\mathbf{a}_2 = (a_{2,\alpha})_{\alpha=1}^N$ and the masses of hypermultiplets $\mathbf{a}_0 = (a_{0,\alpha})_{\alpha=1}^N$ (and $\mathbf{a}_3 = (a_{3,\alpha})_{\alpha=1}^N$ for $\mathcal{Z}^R_\beta$) are generic, so that the simple poles do not overlap with each other. Note that this contour integral is analogous to the famous *Barnes integral*. It is straightforward to prove that the integral (5.4.23) uniformly converges as long as $\mathrm{Arg}(-\mathfrak{q}_1) < \pi$, i.e., $\mathfrak{q}_1 \notin \mathbb{R}^+$, using

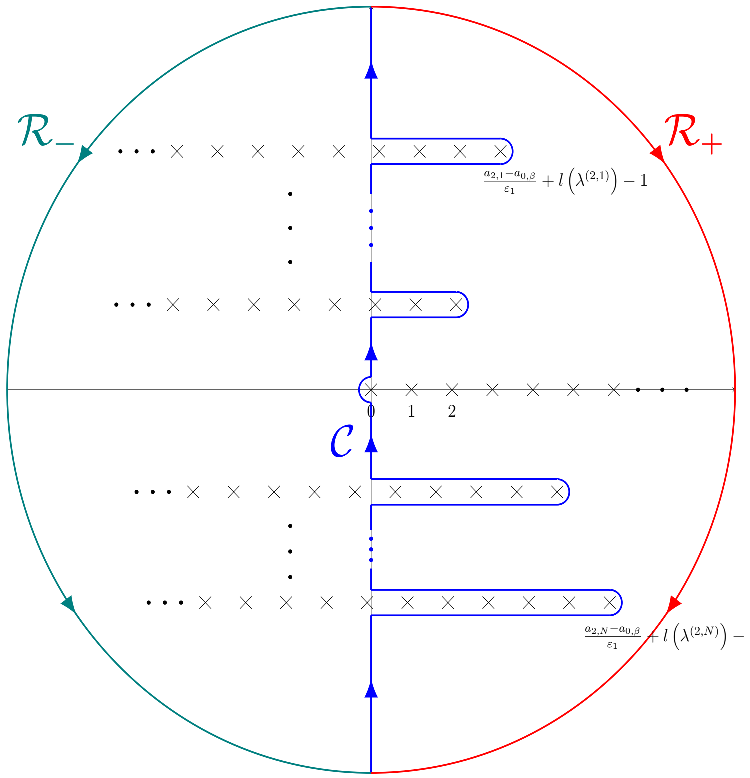

Figure 5.2: The contour $\mathcal{C}$ on the $\frac{x-a_{0,\beta}}{\varepsilon_1}$-plane.

the asymptotics of the Γ-functions. The equality in (5.4.23) is obtained as we close the contour by adding the semi-circle $\mathcal{R}_+$ at the infinity, picking only the poles at $x = a_{0,\beta} + k\varepsilon_1$, $k \in \mathbb{Z}^{\geq 0}$. It can be shown that the integral along $\mathcal{R}_+$ uniformly converges to zero in the regime $|\mathfrak{q}_1| < 1$, and therefore it is safe to add $\mathcal{R}_+$ to the contour $\mathcal{C}$.

Now, we take the contour integral representation (5.4.23) as the analytic continuation of the partition function $\mathcal{Z}^L_\beta$. In particular, the partition function assumes a different series expansion in the regime $|\mathfrak{q}_1| > 1$, and it can be computed as we close the contour by adding a semi-circle $\mathcal{R}_-$ on the opposite side. It is possible to show that the integral along $\mathcal{R}_-$ uniformly converges to zero in the regime $|\mathfrak{q}_1| > 1$, and hence it is safe to add $\mathcal{R}_-$ to the contour $\mathcal{C}$. The resulting contour encloses the rest of the poles, i.e., $x = a_{2,\alpha} + \left(l\left(\lambda^{(2,\alpha)}\right) - k - 1\right)\varepsilon_1$ where $\alpha = 1, \cdots, N$ and $k \in \mathbb{Z}^{\geq 0}$. First note that the denominator in the contour integral can be absorbed into the Γ-functions, yielding

$$\oint_{\mathcal{C}} dx\,(-\mathfrak{q}_1)^{\frac{x}{\varepsilon_1}} \frac{\Gamma\left(-\frac{x-a_{0,\beta}}{\varepsilon_1}\right)\prod_{\alpha=1}^{N}\Gamma\left(-l\left(\lambda^{(2,\alpha)}\right)+1+\frac{x-a_{2,\alpha}}{\varepsilon_1}\right)}{\prod_{\alpha\neq\beta}\Gamma\left(1+\frac{x-a_{0,\alpha}}{\varepsilon_1}\right)} \prod_{\alpha=1}^{N}\prod_{i=1}^{l\left(\lambda^{(2,\alpha)}\right)} \frac{x-a_{2,\alpha}-(i-1)\varepsilon_1-\lambda_i^{(2,\alpha)}\varepsilon_2}{\varepsilon_1}. \tag{5.4.25}$$

Then we can pick up the residues of the N-rays of poles at $x = a_{2,\alpha} + \left(l\left(\lambda^{(2,\alpha)}\right)-k-1\right)\varepsilon_1$, $\alpha = 1, \cdots, N$ and $k \in \mathbb{Z}^{\geq 0}$. We can write the resulting series expansion for the analytically continued partition function as a sum over these N-rays,

$$\mathcal{Z}_\beta^L = \sum_{\alpha=1}^{N}\prod_{\beta'\neq\beta}\frac{\Gamma\left(1+\frac{a_{0,\beta}-a_{0,\beta'}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{2,\alpha}-a_{0,\beta'}}{\varepsilon_1}\right)}\prod_{\alpha'\neq\alpha}\frac{\Gamma\left(\frac{a_{2,\alpha}-a_{2,\alpha'}}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,\beta}-a_{2,\alpha'}}{\varepsilon_1}\right)}\mathfrak{q}_1^{-1}(-\mathfrak{q}_1)^{\frac{a_{2,\alpha}-a_{0,\beta}}{\varepsilon_1}}\mathcal{Z}_\alpha^{L\to M}, \tag{5.4.26}$$

where we have defined the basis function in the regime $|\mathfrak{q}_1| > 1$, which is independent of the choice of β in the constraints (5.2.4), by

$$\begin{aligned}\mathcal{Z}_\alpha^{L\to M}(\mathbf{a}_2) \equiv &\sum_{\boldsymbol{\lambda}^{(2)}}\mathfrak{q}_2^{|\boldsymbol{\lambda}^{(2)}|}\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\lambda}^{(2)}}\sum_{k=0}^{\infty}\mathfrak{q}_1^{-k+l\left(\lambda^{(2,\alpha)}\right)}\frac{(-1)^k}{k!}\\ &\prod_{\alpha'\neq\alpha}\frac{\Gamma\left(-k+l\left(\lambda^{(2,\alpha)}\right)-l\left(\lambda^{(2,\alpha')}\right)+\frac{a_{2,\alpha}-a_{2,\alpha'}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{2,\alpha}-a_{2,\alpha'}}{\varepsilon_1}\right)}\prod_{\gamma=1}^{N}\frac{\Gamma\left(\frac{a_{2,\alpha}-a_{0,\gamma}}{\varepsilon_1}\right)}{\Gamma\left(-k+l\left(\lambda^{(2,\alpha)}\right)+\frac{a_{2,\alpha}-a_{0,\gamma}}{\varepsilon_1}\right)}\\ &\prod_{\gamma=1}^{N}\prod_{i=1}^{l\left(\lambda^{(2,\gamma)}\right)}\frac{a_{2,\alpha}-a_{2,\gamma}+\left(l\left(\lambda^{(2,\alpha)}\right)-k-i\right)\varepsilon_1-\lambda_i^{(2,\gamma)}\varepsilon_2}{\varepsilon_1},\end{aligned} \tag{5.4.27}$$

so that the choice of β only affects the coefficients of the continuation formula (5.4.26). We will explicitly write the argument of $\mathcal{Z}_\alpha^{L\to M}$ only when we emphasize its Coulomb moduli, but otherwise we omit it. Note that the basis function can be expressed as the expectation value of an infinite sum of $\mathcal{Q}$-observables (5.2.32),

$$\begin{aligned}\mathcal{Z}_\alpha^{L\to M} = \sum_{\boldsymbol{\lambda}^{(2)}} \mathfrak{q}_2^{|\boldsymbol{\lambda}^{(2)}|} \widetilde{\boldsymbol{\mu}}_{\boldsymbol{\lambda}^{(2)}} \prod_{\square\in\boldsymbol{\lambda}^{(2)}} \frac{a_{2,\alpha}-\varepsilon-c_\square}{a_{2,\alpha}-\varepsilon_1-c_\square} \sum_{k=0}^{\infty} \mathfrak{q}_1^{-k+l\left(\lambda^{(2,\alpha)}\right)} \varepsilon_1^{N\left(k-l\left(\lambda^{(2,\alpha)}\right)\right)} \\ \prod_{\gamma=1}^{N} \frac{\Gamma\left(\frac{a_{2,\alpha}-a_{0,\gamma}}{\varepsilon_1}\right)}{\Gamma\left(-k+l\left(\lambda^{(2,\alpha)}\right)+\frac{a_{2,\alpha}-a_{0,\gamma}}{\varepsilon_1}\right)} \frac{\mathcal{Q}_2\left(a_{2,\alpha}+\left(l\left(\lambda^{(2,\alpha)}\right)-k-1\right)\varepsilon_1\right)}{\mathcal{Q}_2\left(a_{2,\alpha}-\varepsilon_1\right)}.\end{aligned} \tag{5.4.28}$$

Remarks

- The ratios of the Γ-functions in (5.4.27) and (5.4.28) can be expressed as Pochhammer symbols, but they may appear either in the numerator or in the denominator depending on k and $l\left(\lambda^{(2,\alpha)}\right)$'s.

- While the exponent of $\mathfrak{q}_2$ is always positive, the exponent of $\mathfrak{q}_1$ can either be positive or negative depending on k and $l\left(\lambda^{(2,\alpha)}\right)$. The convergence regime is $0<|\mathfrak{q}_2|<|\mathfrak{q}_1^{-1}|<1$. We may introduce new coupling constants

$$\mathfrak{q}_1 \equiv \mathfrak{q}_1'^{\,-1}, \quad \mathfrak{q}_2 \equiv \mathfrak{q}_1'\mathfrak{q}_2', \tag{5.4.29}$$

  so that the the convergence regime becomes $0<|\mathfrak{q}_1'|,|\mathfrak{q}_2'|<1$. Indeed, the exponent of the new coupling constant $\mathfrak{q}_1'$ is $k+|\boldsymbol{\lambda}^{(2)}|-l\left(\lambda^{(2,\alpha)}\right)\geq k\geq 0$, i.e., bounded below. The first inequality is saturated if and only if $\boldsymbol{\lambda}^{(2)}$ is single-columned, namely, $\lambda^{(2,\alpha')}=\varnothing$ for all $\alpha'\neq\alpha$ and $\lambda^{(2,\alpha)}$ is single-columned. This suggests the basis function $\mathcal{Z}_\alpha$ is related to the A_2-theory in which the Coulomb moduli of the two gauge nodes are subject to certain constraints. We come back to this question in section 5.4.2.1.

- The reparametrization of the couplings $\mathfrak{q}_1=z^{-1}$ and $\mathfrak{q}_2=\mathfrak{q}$ of (5.2.5) were introduced to be consistent with the convention in (5.3.17). Note that $\mathfrak{q}_1'=z$ and $\mathfrak{q}_2'=\frac{\mathfrak{q}}{z}$ under the reparametrization.

- Let $\mathcal{O}$ be an observable lying only on the second gauge node, i.e., $\mathcal{O}[\boldsymbol{\lambda}]=\mathcal{O}[\boldsymbol{\lambda}^{(2)}]$. The expectation value of such observables can similarly be analytically continued. We simply need to insert the observable inside (5.4.27), along with the measure $\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\lambda}^{(2)}}$.

Similarly, we can analytically continue the quiver surface defect partition function (5.2.38). After imposing the constraints (5.2.34) and the re-definition of parameters (5.2.36) we consider

$$\mathcal{Z}_{\beta}^{R} = \mathcal{Z}_{A_2}\left(-a_{0,\alpha}-\varepsilon;\ -a_{1,\alpha};\ -a_{3,\alpha}+\varepsilon-\varepsilon_2\delta_{\alpha,\beta};\ -a_{3,\alpha}+2\varepsilon \mid \varepsilon_1,\varepsilon_2 \mid \mathfrak{q}_1=\mathfrak{q},\mathfrak{q}_2=\mathfrak{q}^{-1}z\right), \tag{5.4.30}$$

The partition function can be analytically continued in the same way,

$$\mathcal{Z}_{\beta}^{R} = \sum_{\alpha=1}^{N}\prod_{\beta'\neq\beta}\frac{\Gamma\left(1+\frac{a_{3,\beta'}-a_{3,\beta}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{3,\beta'}-a_{1,\alpha}}{\varepsilon_1}\right)}\prod_{\alpha'\neq\alpha}\frac{\Gamma\left(\frac{a_{1,\alpha'}-a_{1,\alpha}}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{1,\alpha'}-a_{3,\beta}}{\varepsilon_1}\right)}\mathfrak{q}_2^{-1}(-\mathfrak{q}_2)^{\frac{a_{3,\beta}-a_{1,\alpha}}{\varepsilon_1}}\mathcal{Z}_{\alpha}^{R\to M}, \tag{5.4.31}$$

where

$$\begin{aligned}
\mathcal{Z}_{\alpha}^{R\to M}(\mathbf{a}_1) = &\sum_{\boldsymbol{\lambda}^{(1)}}\mathfrak{q}_1^{|\boldsymbol{\lambda}^{(1)}|}\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\lambda}^{(1)}}\sum_{k=0}^{\infty}\mathfrak{q}_2^{-k+l\left(\lambda^{(1,\alpha)}\right)}\frac{(-1)^k}{k!}\\
&\prod_{\alpha'\neq\alpha}\frac{\Gamma\left(-k+l\left(\lambda^{(1,\alpha)}\right)-l\left(\lambda^{(1,\alpha')}\right)+\frac{a_{1,\alpha'}-a_{1,\alpha}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{1,\alpha'}-a_{1,\alpha}}{\varepsilon_1}\right)}\prod_{\gamma=1}^{N}\frac{\Gamma\left(\frac{a_{3,\gamma}-a_{1,\alpha}}{\varepsilon_1}\right)}{\Gamma\left(-k+l\left(\lambda^{(1,\alpha)}\right)+\frac{a_{3,\gamma}-a_{1,\alpha}}{\varepsilon_1}\right)}\\
&\prod_{\gamma=1}^{N}\prod_{i=1}^{l\left(\lambda^{(1,\gamma)}\right)}\frac{-a_{1,\alpha}+a_{1,\gamma}+\left(l\left(\lambda^{(1,\alpha)}\right)-k-i\right)\varepsilon_1-\lambda_i^{(1,\gamma)}\varepsilon_2}{\varepsilon_1}.\\
= &\sum_{\boldsymbol{\lambda}^{(1)}}\mathfrak{q}_1{}^{|\boldsymbol{\lambda}^{(1)}|}\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\lambda}^{(1)}}\prod_{\boldsymbol{\lambda}^{(1)}}\frac{-a_{1,\alpha}-c_{\square}-\varepsilon}{-a_{1,\alpha}-c_{\square}-\varepsilon_1}\sum_{k=0}^{\infty}\mathfrak{q}_2^{-k+l\left(\lambda^{(1,\alpha)}\right)}\varepsilon_1^{N\left(k-l\left(\lambda^{(1,\alpha)}\right)\right)}\\
&\prod_{\gamma=1}^{N}\frac{\Gamma\left(\frac{a_{3,\gamma}-a_{1,\alpha}}{\varepsilon_1}\right)}{\Gamma\left(-k+l\left(\lambda^{(1,\alpha)}\right)+\frac{a_{3,\gamma}-a_{1,\alpha}}{\varepsilon_1}\right)}\frac{\mathcal{Q}_1\left(-a_{1,\alpha}+\left(l\left(\lambda^{(1,\alpha)}\right)-k-1\right)\varepsilon_1\right)}{\mathcal{Q}_1\left(-a_{1,\alpha}-\varepsilon_1\right)},
\end{aligned} \tag{5.4.32}$$

We have defined the modified measure

$$\begin{aligned}
\widetilde{\boldsymbol{\mu}}_{\boldsymbol{\lambda}^{(1)}} &\equiv \boldsymbol{\mu}_{\boldsymbol{\lambda}^{(1)}}\prod_{\square\in K_1}\frac{-a_{3,\beta}-c_{\square}}{-a_{3,\beta}-c_{\square}-\varepsilon_2}\\
&= \epsilon\left[N_1K_1^*+q_{12}N_1^*K_1-P_{12}K_1K_1^*-M_0K_1^*-q_{12}^2M_3^*K_1\right].
\end{aligned} \tag{5.4.33}$$

Remarks

- The convergence regime of (5.4.32) is $0 < |\mathfrak{q}_1| < |\mathfrak{q}_2^{-1}| < 1$. We may define new coupling constants

$$\mathfrak{q}_1 \equiv \mathfrak{q}_1' \mathfrak{q}_2', \quad \mathfrak{q}_2 \equiv {\mathfrak{q}_2'}^{-1}, \tag{5.4.34}$$

  so that the convergence regime becomes $0 < |\mathfrak{q}_1'|, |\mathfrak{q}_2'| < 1$.

- The reparametrizations of coupling constants (5.2.37) were introduced to be consistent with the convention in (5.3.27). Note that the new coupling constants $\mathfrak{q}_1' = z$ and $\mathfrak{q}_2' = \frac{\mathfrak{q}}{z}$ match with the previous ones. Thus, both analytically continued partition functions lie in the intermediate domain, $0 < |\mathfrak{q}| < |z| < 1$.

5.4.2 Gluing the partition functions

5.4.2.1 The connection matrix

Recall that the surface defect partition functions are annihilated by the operators $\hat{\hat{\mathfrak{D}}}$ obtained in section 5.3. The uniqueness of the analytic continuation guarantees that the continued functions satisfy the same differential equations. Therefore we may regard the analytically continued partition functions as the extentions of the solutions to other convergence domains. Motivated by the analytic continuation formulas (5.4.26) and (5.4.31), let us define *the connection matrices*

$$(\mathbf{C}_\infty)_{\alpha\beta} \equiv \prod_{\alpha' \neq \alpha} \frac{\Gamma\left(1 + \frac{a_{0,\alpha} - a_{0,\alpha'}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{2,\beta} - a_{0,\alpha'}}{\varepsilon_1}\right)} \prod_{\beta' \neq \beta} \frac{\Gamma\left(\frac{a_{2,\beta} - a_{2,\beta'}}{\varepsilon_1}\right)}{\Gamma\left(1 + \frac{a_{0,\alpha} - a_{2,\beta'}}{\varepsilon_1}\right)}, \tag{5.4.35a}$$

$$(\mathbf{C}_0)_{\alpha\beta} \equiv \prod_{\alpha' \neq \alpha} \frac{\Gamma\left(1 + \frac{a_{3,\alpha'} - a_{3,\alpha}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{3,\alpha'} - a_{1,\beta}}{\varepsilon_1}\right)} \prod_{\beta' \neq \beta} \frac{\Gamma\left(\frac{a_{1,\beta'} - a_{1,\beta}}{\varepsilon_1}\right)}{\Gamma\left(1 + \frac{a_{1,\beta'} - a_{3,\alpha}}{\varepsilon_1}\right)}. \tag{5.4.35b}$$

We will scrutinize below how the connection matrices associate the solutions to $\hat{\hat{\mathfrak{D}}}$ in different convergence domians, for each $N \geq 2$.

$\boldsymbol{N = 2}$ We have shown in section 5.3.1.1 that the modified surface defect partition functions,

$$\widetilde{\mathfrak{Z}}^{L} \equiv \left(\widetilde{\mathcal{Z}}^{L}_{\alpha} \right)_{\alpha=1,2}, \tag{5.4.36}$$

solve the differential equation (5.3.17) given by $\hat{\hat{\mathfrak{D}}}_2$, with the prefactors in (5.3.13). These functions provided the solutions of the form

$$\sum_{k_1,k_2=0}^{\infty} c_{k_1,k_2}\, z^{r_L - k_1}\, \mathfrak{q}^{L_2+k_2}, \tag{5.4.37}$$

in the domain $0 < |\mathfrak{q}| < 1 < |z|$. It is not so difficult to show that they are the only solutions once the critical exponent L_2 is given. Indeed, by directly acting $\hat{\hat{\mathfrak{D}}}_2$ to the ansatz and expanding in z^{-1} and $\mathfrak{q}$, we get a recursive relations for the coefficients c_{k_1,k_2}. In particular, the zeroth order equation is

$$0 = \left(\varepsilon_1^2 r_L^2 - \varepsilon_1(\varepsilon_1 + 2\varepsilon_2) r_L + \varepsilon_1\varepsilon_2 \left(\frac{2\varepsilon + \varepsilon_2}{4\varepsilon_1} + \Delta_1 \right) \right) c_{0,0}. \tag{5.4.38}$$

The existence of the solution ($c_{0,0} \neq 0$) implies that we are restricted to only two choices for the critical exponent r_L,

$$\left(r_{L,\alpha}\right)_{\alpha=1,2} = \left(\frac{-a_{0,1} + a_{0,2} + \varepsilon + \varepsilon_2}{2\varepsilon_1},\ \frac{a_{0,1} - a_{0,2} + \varepsilon + \varepsilon_2}{2\varepsilon_1} \right), \tag{5.4.39}$$

which are precisely (5.3.15). Once r_L is chosen, the recursive relations fully determine all the coefficients c_{k_1,k_2}. Since the partition functions $\widetilde{\mathfrak{Z}}^{L}$ already provide two solutions, we conclude that the surface defect partition functions $\widetilde{\mathfrak{Z}}^{L}$ provide all solutions to $\hat{\hat{\mathfrak{D}}}_2$ in the domain $0 < |\mathfrak{q}| < 1 < |z|$, for each fixed L_2.

With the modification of the partition function by the multiplication of the prefactors

(5.3.13), the analytic continuation formula (5.4.26) becomes

$$\begin{aligned}\widetilde{\mathfrak{Z}}^L_\alpha = - \sum_{\beta=1,2} (\mathbf{C}_\infty)_{\alpha\beta} \left(-\frac{1}{z}\right)^{-r_{L,\alpha}+\frac{2\bar{a}_0-2\bar{a}_2+\varepsilon_2}{2\varepsilon_1}+\frac{a_{2,\beta}-a_{0,\alpha}}{\varepsilon_1}} \; \mathfrak{q}^{-\Delta_\mathfrak{q}-\Delta_0+\frac{\varepsilon^2-(a_{2,1}-a_{2,2})^2}{4\varepsilon_1\varepsilon_2}} \\ (1-z)^{\frac{2\bar{a}_0-2\bar{a}_2+2\varepsilon_1+\varepsilon_2}{2\varepsilon_1}} \left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{\bar{a}_2-\bar{a}_3+\varepsilon}{\varepsilon_1}} (1-\mathfrak{q})^{\frac{2(\bar{a}_2-\bar{a}_3+\varepsilon)(2\bar{a}_0-2\bar{a}_2-\varepsilon_2)}{\varepsilon_1\varepsilon_2}} \, \mathfrak{Z}^{L\to M}_\beta .\end{aligned} \tag{5.4.40}$$

Note that the critical exponent of z is independent of α, namely,

$$\begin{aligned}(r_{L\to M,\beta})_{\beta=1,2} &\equiv \left(r_{L,\alpha} - \frac{2\bar{a}_0-2\bar{a}_2+\varepsilon_2}{2\varepsilon_1} - \frac{a_{2,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)_{\beta=1,2} \\ &= \left(\frac{-a_{2,1}+a_{2,2}+\varepsilon}{2\varepsilon_1}, \; \frac{a_{2,1}-a_{2,2}+\varepsilon}{2\varepsilon_1}\right).\end{aligned} \tag{5.4.41}$$

Finally, we define the modified basis functions as

$$\begin{aligned}\widetilde{\mathfrak{Z}}^{L\to M}_\beta &\equiv -\left(-\frac{1}{z}\right)^{-r_{L\to M,\beta}} \; \mathfrak{q}^{-\Delta_\mathfrak{q}-\Delta_0+\frac{\varepsilon^2-(a_{2,1}-a_{2,2})^2}{4\varepsilon_1\varepsilon_2}} \\ &\qquad (1-z)^{\frac{2\bar{a}_0-2\bar{a}_2+2\varepsilon_1+\varepsilon_2}{2\varepsilon_1}} \left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{\bar{a}_2-\bar{a}_3+\varepsilon}{\varepsilon_1}} (1-\mathfrak{q})^{\frac{2(\bar{a}_2-\bar{a}_3+\varepsilon)(2\bar{a}_0-2\bar{a}_2-\varepsilon_2)}{\varepsilon_1\varepsilon_2}} \, \mathfrak{Z}^{L\to M}_\beta , \\ \widetilde{\boldsymbol{\mathfrak{Z}}}^{L\to M} &\equiv \left(\widetilde{\mathfrak{Z}}^{L\to M}_\beta\right)_{\beta=1,2} .\end{aligned} \tag{5.4.42}$$

The uniqueness of analytic continuation guarantees that $\widetilde{\boldsymbol{\mathfrak{Z}}}^{L\to M}$ also provides solutions to $\hat{\hat{\mathfrak{D}}}_2$. Therefore, the analytic continuation formula,

$$\widetilde{\boldsymbol{\mathfrak{Z}}}^L = \mathbf{C}_\infty \widetilde{\boldsymbol{\mathfrak{Z}}}^{L\to M}, \tag{5.4.43}$$

connects the solutions to the differential operators $\hat{\hat{\mathfrak{D}}}_2$ in different convergence domains, through the connection matrix defined in (5.4.35a).

Remarks

- In the limit $\varepsilon_2 \to 0$, the modified functions $\boldsymbol{\mathfrak{Z}}^{L\to M}$ produce solutions to the oper $\widehat{\mathfrak{D}}$. It is evident from the expression (5.4.28) that the solutions are again expressed as sums of the Baxter $\mathfrak{Q}$-functions.

- We observe that the critical exponent $r_{L\to M,\beta}$ of z for $\widetilde{\mathcal{Z}}_{\beta}^{L\to M}$ is precisely the L_1 in (5.3.13) subject to the constraint

$$\begin{cases} a_{1,\beta} = a_{2,\beta} + \varepsilon_2 \\ a_{1,\alpha} = a_{2,\alpha} \ \ (\alpha \neq \beta) \end{cases} . \tag{5.4.44}$$

This strongly indicates the identity,

$$\mathcal{Z}_{\beta}^{L\to M} = (1-z)^{-\frac{2\varepsilon}{\varepsilon_1\varepsilon_2}(\bar{a}_0-\bar{a}_2-\varepsilon_2)}\, \mathcal{Z}_{A_2}\left(a_{1,\alpha} = a_{2,\alpha} + \varepsilon_2 \delta_{\alpha,\beta}\right), \tag{5.4.45}$$

between the two seemingly distinct partition functions. Even though this identity is rather obvious in the point of view of AGT [63], its rigorous proof in the gauge theory side is not. As is clear from the definition of each side, this identity implies a lot of non-trivial combinatoric identities. It would be nice to directly prove the identity, perhaps by using the non-perturbative Dyson-Schwinger equations, but it is not necessary for our study so we do not attempt it here.

Similarly, we have shown that the modified surface defect partition functions,

$$\widetilde{\boldsymbol{\mathcal{Z}}}^{R} \equiv \left(\widetilde{\mathcal{Z}}_{\alpha}^{R}\right)_{\alpha=1,2}, \tag{5.4.46}$$

give solutions to $\hat{\hat{\hat{\mathfrak{D}}}}_2$ in the domain $0<|z|<|\mathfrak{q}|<1$, which are now of the form

$$\sum_{k_1,k_2=0}^{\infty} c_{k_1,k_2}\ z^{r_R+k_2}\ \mathfrak{q}^{L_1+k_1-k_2} = \sum_{k_1,k_2=0}^{\infty} c_{k_1,k_2}\ \mathfrak{q}^{L_1+r_R+k_1}\left(\frac{z}{\mathfrak{q}}\right)^{r_R+k_2} . \tag{5.4.47}$$

We can act with $\hat{\hat{\hat{\mathfrak{D}}}}_2$ on this series and expand in $\mathfrak{q}$ and $\frac{z}{\mathfrak{q}}$, to find the indicial equation,

$$0 = \varepsilon_1^2 r_R^2 - \varepsilon_1\varepsilon\, r_R + \varepsilon_1\varepsilon_2\Delta_0, \tag{5.4.48}$$

whose solutions are precisely (5.3.25), namely,

$$(r_{R,\alpha})_{\alpha=1,2} \equiv \left(\frac{-a_{3,1} + a_{3,2} + \varepsilon}{2\varepsilon_1}, \ \frac{a_{3,1} - a_{3,2} + \varepsilon}{2\varepsilon_1} \right). \tag{5.4.49}$$

Once r_R is chosen, all the coefficients c_{k_1,k_2} are determined recursively. Thus we conclude that $\widetilde{\boldsymbol{\mathfrak{Z}}}^R$ provide the only two solutions to $\hat{\hat{\mathfrak{D}}}_2$ in the domain $0 < |z| < |\mathfrak{q}| < 1$, for each fixed $L_1 + r_R$. With the prefactors (5.3.24), the analytic continuation formula (5.4.31) becomes

$$\begin{aligned} \widetilde{\mathfrak{Z}}^R_\alpha = - \sum_{\beta=1,2} (\mathbf{C}_0)_{\alpha\beta} & \left(-\frac{\mathfrak{q}}{z}\right)^{-r_{R,\alpha} - \frac{2\bar{a}_1 - 2\bar{a}_3 + \varepsilon_2}{2\varepsilon_1} - \frac{a_{3,\alpha} - a_{1,\beta}}{\varepsilon_1}} \mathfrak{q}^{\frac{\varepsilon^2 - (a_{1,1} - a_{1,2})^2}{4\varepsilon_1\varepsilon_2} - \Delta_0 - \Delta'_{\mathfrak{q}} + \frac{2\varepsilon + \varepsilon_2}{4\varepsilon_1}} \\ & (1-\mathfrak{q})^{\frac{(\bar{a}_0 - \bar{a}_1 + \varepsilon)(2\bar{a}_1 - 2\bar{a}_3 - \varepsilon_2)}{\varepsilon_1\varepsilon_2}} (1-z)^{\frac{\bar{a}_0 - \bar{a}_1 + \varepsilon}{\varepsilon_1}} \left(1 - \frac{\mathfrak{q}}{z}\right)^{\frac{2\bar{a}_1 - 2\bar{a}_3 + 2\varepsilon_1 + \varepsilon_2}{2\varepsilon_1}} \mathfrak{Z}^{R\to M}_\beta . \end{aligned} \tag{5.4.50}$$

Note that the critical exponent for z becomes, again, independent of α,

$$\begin{aligned} (r_{R\to M,\beta})_{\beta=1,2} &\equiv \left(r_{R,\alpha} + \frac{2\bar{a}_1 - 2\bar{a}_3 + \varepsilon_2}{2\varepsilon_1} + \frac{a_{3,\alpha} - a_{1,\beta}}{\varepsilon_1} \right)_{\beta=1,2} \\ &= \left(\frac{-a_{1,1} + a_{1,2} + \varepsilon + \varepsilon_2}{2\varepsilon_1}, \ \frac{a_{1,1} - a_{1,2} + \varepsilon + \varepsilon_2}{2\varepsilon_1} \right). \end{aligned} \tag{5.4.51}$$

Hence we define the modified basis function by

$$\begin{aligned} \widetilde{\mathfrak{Z}}^{R\to M}_\beta \equiv & - \left(-\frac{\mathfrak{q}}{z}\right)^{-r_{R\to M,\beta}} \mathfrak{q}^{\frac{\varepsilon^2 - (a_{1,1} - a_{1,2})^2}{4\varepsilon_1\varepsilon_2} - \Delta_0 - \Delta'_{\mathfrak{q}} + \frac{2\varepsilon + \varepsilon_2}{4\varepsilon_1}} \\ & (1-\mathfrak{q})^{\frac{(\bar{a}_0 - \bar{a}_1 + \varepsilon)(2\bar{a}_1 - 2\bar{a}_3 - \varepsilon_2)}{\varepsilon_1\varepsilon_2}} (1-z)^{\frac{\bar{a}_0 - \bar{a}_1 + \varepsilon}{\varepsilon_1}} \left(1 - \frac{\mathfrak{q}}{z}\right)^{\frac{2\bar{a}_1 - 2\bar{a}_3 + 2\varepsilon_1 + \varepsilon_2}{2\varepsilon_1}} \mathfrak{Z}^{R\to M}_\beta , \\ \widetilde{\boldsymbol{\mathfrak{Z}}}^{R\to M} \equiv & \left(\widetilde{\mathfrak{Z}}^{R\to M}_\beta \right)_{\beta=1,2} . \end{aligned} \tag{5.4.52}$$

By the uniqueness of the analytic continuation, we conclude that $\widetilde{\boldsymbol{\mathfrak{Z}}}^{R\to M}$ gives the solutions to $\hat{\hat{\mathfrak{D}}}_2$ in the domain $0 < |\mathfrak{q}| < |z| < 1$. The analytic continuation formula,

$$\widetilde{\boldsymbol{\mathfrak{Z}}}^R = \mathbf{C}_0 \, \widetilde{\boldsymbol{\mathfrak{Z}}}^{R\to M}, \tag{5.4.53}$$

connects the solutions in different convergence domains, through the connection matrix defined in (5.4.35b).

***N* = 3** Under the modification (5.3.31) with the prefactors (5.3.32), the analytic continuation formula (5.4.26) becomes

$$\widetilde{\mathcal{Z}}_{\alpha}^{L} = -\sum_{\beta=1}^{3} (\mathbf{C}_{\infty})_{\alpha\beta} \left(-\frac{1}{z}\right)^{-r_{L,\alpha}+\frac{3\bar{a}_0-3\bar{a}_2+2\varepsilon_2}{3\varepsilon_1}+\frac{a_{2,\beta}-a_{0,\alpha}}{\varepsilon_1}} \tag{5.4.54}$$

$$\mathfrak{q}^{-\Delta_{\mathfrak{q}}-\Delta_0+\frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2-\frac{(a_{2,1}-a_{2,2})^2+(a_{2,1}-a_{2,3})^2-(a_{2,1}-a_{2,2})(a_{2,1}-a_{2,3})}{3}\right)} \tag{5.4.55}$$

$$(1-z)^{\frac{3\bar{a}_0-3\bar{a}_2+3\varepsilon-\varepsilon_2}{3\varepsilon_1}} \left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{\bar{a}_2-\bar{a}_3+\varepsilon}{\varepsilon_1}} (1-\mathfrak{q})^{\frac{(\bar{a}_2-\bar{a}_3+\varepsilon)(3\bar{a}_-3\bar{a}_2-\varepsilon_2)}{\varepsilon_1\varepsilon_2}} \mathcal{Z}_{\beta}^{L\to M}. \tag{5.4.56}$$

Again, the critical exponent of z is independent of α,

$$\begin{aligned}(r_{L\to M,\beta})_{\beta=1}^{3} &\equiv \left(r_{L,\alpha} - \frac{3\bar{a}_0-3\bar{a}_2+2\varepsilon_2}{3\varepsilon_1} - \frac{a_{2,\beta}-a_{0,\alpha}}{\varepsilon_1}\right)_{\beta=1}^{3} \\ &= \left(\frac{-3a_{2,\beta}+\sum_{\gamma=1}^{3} a_{2,\gamma}+3\varepsilon}{3\varepsilon_1}\right)_{\beta=1}^{3}.\end{aligned} \tag{5.4.57}$$

Hence, we define the modified basis functions as

$$\begin{aligned}\widetilde{\mathcal{Z}}_{\beta}^{L\to M} &\equiv -\left(-\frac{1}{z}\right)^{-r_{L\to M,\beta}} \mathfrak{q}^{-\Delta_{\mathfrak{q}}-\Delta_0+\frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2-\frac{(a_{2,1}-a_{2,2})^2+(a_{2,1}-a_{2,3})^2-(a_{2,1}-a_{2,2})(a_{2,1}-a_{2,3})}{3}\right)} \\ &\quad (1-z)^{\frac{3\bar{a}_0-3\bar{a}_2+3\varepsilon-\varepsilon_2}{3\varepsilon_1}} \left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{\bar{a}_2-\bar{a}_3+\varepsilon}{\varepsilon_1}} (1-\mathfrak{q})^{\frac{(\bar{a}_2-\bar{a}_3+\varepsilon)(3\bar{a}_-3\bar{a}_2-\varepsilon_2)}{\varepsilon_1\varepsilon_2}} \mathcal{Z}_{\beta}^{L\to M}, \\ \widetilde{\boldsymbol{\mathfrak{Z}}}^{L\to M} &\equiv \left(\widetilde{\mathcal{Z}}_{\beta}^{L\to M}\right)_{\beta=1}^{3}.\end{aligned} \tag{5.4.58}$$

Then the analytic continuation formula,

$$\widetilde{\boldsymbol{\mathfrak{Z}}}^{L} = \mathbf{C}_{\infty} \widetilde{\boldsymbol{\mathfrak{Z}}}^{L\to M}, \tag{5.4.59}$$

connects the solutions to $\hat{\hat{\hat{\mathfrak{D}}}}_3$ in different converence domains.

Likewise, under the multiplication of the prefactors (5.3.50), the analytic continuation formula (5.4.31) becomes

$$\widetilde{\mathcal{Z}}_{\alpha}^{R}=-\sum_{\beta=1}^{3}(\mathbf{C}_0)_{\alpha\beta}\left(-\frac{\mathfrak{q}}{z}\right)^{-r_{R,\alpha}-\frac{3\bar{a}_1-3\bar{a}_3+2\varepsilon_2}{3\varepsilon_1}-\frac{a_{3,\alpha}-a_{1,\beta}}{\varepsilon_1}}(1-\mathfrak{q})^{\frac{(\bar{a}_0-\bar{a}_1+\varepsilon)(3\bar{a}_1-3\bar{a}_3-\varepsilon_2)}{\varepsilon_1\varepsilon_2}}(1-z)^{\frac{(\bar{a}_0-\bar{a}_1+\varepsilon)}{\varepsilon_1}}$$
$$\mathfrak{q}^{\frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2-\frac{(a_{1,1}-a_{1,2})^2+(a_{1,1}-a_{1,3})^2-(a_{1,1}-a_{1,2})(a_{1,1}-a_{1,3})}{3}\right)-\Delta_{\mathfrak{q}}'-\Delta_0+\frac{3\varepsilon+\varepsilon_2}{3\varepsilon_1}}\left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{3\bar{a}_1-3\bar{a}_3+3\varepsilon-\varepsilon_2}{3\varepsilon_1}}$$
$$\mathcal{Z}_{\beta}^{R\to M}. \tag{5.4.60}$$

Note that the critical exponent of z becomes independent of α, namely,

$$\begin{aligned}(r_{R\to M,\beta})_{\beta=1}^{3}&\equiv\left(r_{R,\alpha}+\frac{3\bar{a}_1-3\bar{a}_3+2\varepsilon_2}{3\varepsilon_1}+\frac{a_{3,\alpha}-a_{1,\beta}}{\varepsilon_1}\right)_{\beta=1}^{3}\\&=\left(\frac{-3a_{1,\beta}+\sum_{\gamma=1}^{3}a_{1,\gamma}+3\varepsilon+2\varepsilon_2}{3\varepsilon_1}\right)_{\beta=1}^{3}.\end{aligned} \tag{5.4.61}$$

Therefore we modify the basis function by

$$\widetilde{\mathcal{Z}}_{\beta}^{R\to M}\equiv-\left(-\frac{\mathfrak{q}}{z}\right)^{-r_{R\to M,\beta}}(1-\mathfrak{q})^{\frac{(\bar{a}_0-\bar{a}_1+\varepsilon)(3\bar{a}_1-3\bar{a}_3-\varepsilon_2)}{\varepsilon_1\varepsilon_2}}(1-z)^{\frac{\bar{a}_0-\bar{a}_1+\varepsilon}{\varepsilon_1}}\left(1-\frac{\mathfrak{q}}{z}\right)^{\frac{3\bar{a}_1-3\bar{a}_3+3\varepsilon-\varepsilon_2}{3\varepsilon_1}}$$
$$\mathfrak{q}^{\frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2-\frac{(a_{1,1}-a_{1,2})^2+(a_{1,1}-a_{1,3})^2-(a_{1,1}-a_{1,2})(a_{1,1}-a_{1,3})}{3}\right)-\Delta_{\mathfrak{q}}'-\Delta_0+\frac{3\varepsilon+\varepsilon_2}{3\varepsilon_1}}\mathcal{Z}_{\beta}^{R\to M}$$
$$\widetilde{\boldsymbol{\mathcal{Z}}}^{R\to M}\equiv\left(\widetilde{\mathcal{Z}}_{\beta}^{R\to M}\right)_{\beta=1}^{3}. \tag{5.4.62}$$

We conclude that the connection formula,

$$\widetilde{\boldsymbol{\mathcal{Z}}}^{R}=\mathbf{C}_0\,\widetilde{\boldsymbol{\mathcal{Z}}}^{R\to M}, \tag{5.4.63}$$

associate the solutions in different domains, through the connection matrix (5.4.35b).

5.4.2.2 The shift matrix

We have verified in section 5.4.2.1 that the analytically continued partition functions $\widetilde{\mathfrak{Z}}^{L\to M}$ and $\widetilde{\mathfrak{Z}}^{R\to M}$ provide the solutions to the operator $\hat{\hat{\mathfrak{D}}}$ in the intermediate domain, $0 < |\mathfrak{q}| < |z| < 1$. Moreover, we have found in section 5.3.2 that the $(N-1,1)$-type $\mathbb{Z}_2$-orbifold surface defect partition functions $\mathfrak{Z}^{\mathbb{Z}_2}$ also provide the solutions to $\hat{\hat{\mathfrak{D}}}$ in the same domain. The question arises on how these solutions are associated to each other. Exact identities between these partition functions are established with the help of *the shift matrix*

$$\mathbf{S}_{\alpha\beta} \equiv e^{\varepsilon_2 \frac{\partial}{\partial a_\alpha}}\ \delta_{\alpha\beta}, \tag{5.4.64}$$

which is introduced to facilitate shifting the Coulomb moduli of the underlying A_1-theory. We proceed below with the derivation of the identities, for each $N \geq 2$.

***N* = 2** Let us consider the generic ansatz for $\hat{\hat{\mathfrak{D}}}_2$ in the intermediate domain $0 < |\mathfrak{q}| < |z| < 1$,

$$\sum_{k_1,k_2=0}^{\infty} c_{k_1,k_2}\ z^{r_M+k_1-k_2}\mathfrak{q}^{L_2+k_2} = \sum_{k_1,k_2=0}^{\infty} c_{k_1,k_2}\ z^{r_M+L_2+k_1}\left(\frac{\mathfrak{q}}{z}\right)^{L_2+k_2}. \tag{5.4.65}$$

By acting $\hat{\hat{\mathfrak{D}}}_2$ to the ansatz and expanding in z and $\frac{\mathfrak{q}}{z}$, we find the indicial equation

$$0 = \varepsilon_1^2 r_M^2 - \varepsilon_1\varepsilon\, r_M + \varepsilon_1\varepsilon_2(\Delta_{\mathfrak{q}} + \Delta_0) + \varepsilon_1\varepsilon_2 L_2. \tag{5.4.66}$$

Once the critical exponents r_1 and L_2 are chosen to satisfy the indicial equation, all the coefficients c_{k_1,k_2} are determined recursively. The solution is unique in this sense.

We have seen that $\widetilde{\mathfrak{Z}}^{L\to M}$, $\widetilde{\mathfrak{Z}}^{R\to M}$, and $\widetilde{\mathfrak{Z}}^{\mathbb{Z}_2}$ are annihilated by $\hat{\hat{\mathfrak{D}}}_2$, and therefore their critical exponents evidently satisfy the indicial equation (5.4.66). Moreover, we observe from

(5.4.41), (5.4.51), (5.3.62), (5.3.16), and (5.3.26) that

$$(r_{L\to M,\alpha})_{\alpha=1,2} = \left(r_{R\to M,\alpha} \Big|_{a_{1,\alpha}\to a_{2,\alpha}+\varepsilon_2} \right)_{\alpha=1,2} = \left(r_\alpha^{\mathbb{Z}_2} \right)_{\alpha=1,2}, \tag{5.4.67}$$

$$\Delta_{\mathtt{q}} = \Delta'_{\mathtt{q}} \Big|_{a_{1,\alpha}\to a_{2,\alpha}+\varepsilon_2}, \tag{5.4.68}$$

so that the indicial equation guarantees that those solutions are identical under the shift of the Coulomb moduli, namely,

$$\widetilde{\mathfrak{Z}}^{L\to M}(\mathbf{a}) = \mathbf{S}\, \widetilde{\mathfrak{Z}}^{R\to M}(\mathbf{a}) = \widetilde{\mathfrak{Z}}^{\mathbb{Z}_2}(\mathbf{a}). \tag{5.4.69}$$

Note that the re-definitions (5.2.36) of the Coulomb moduli and the masses of the hypermultiplets for $\widetilde{\mathfrak{Z}}^R$ were carefully designed to yield this equality. Consequently, we conclude that the analytically continued partition functions agree in the intermediate domain, and this is also identical to the orbifold surface defect partition function.

$\boldsymbol{N=3}$ From (5.3.35), (5.3.36), (5.3.52), and (5.3.53), we observe that

$$\begin{aligned} \Delta_{\mathtt{q},1} &= \Delta'_{\mathtt{q},1} \Big|_{a_{1,\alpha}\to a_{2,\alpha}+\varepsilon_2} \\ \Lambda_{\mathtt{q},1} &= \Lambda'_{\mathtt{q},1} \Big|_{a_{1,\alpha}\to a_{2,\alpha}+\varepsilon_2} \end{aligned} \tag{5.4.70}$$

Also, from (5.4.57), (5.4.61), and (5.3.73), we have

$$(r_{L\to M,\alpha})_{\alpha=1}^3 = \left(r_{R\to M,\alpha} \Big|_{a_{1,\alpha}\to a_{2,\alpha}+\varepsilon_2} \right)_{\alpha=1}^3 = \left(r_\alpha^{\mathbb{Z}_2} \right)_{\alpha=1}^3. \tag{5.4.71}$$

Although these relations look promising, they do not guarantee the equality of the partition functions this time. The problem is that the equation for $\hat{\hat{\hat{\mathfrak{D}}}}_3$ involves the expectation value $\left\langle \mathcal{O}_3 \right\rangle$, which is an object independent of the partition function itself. Without an additional information on equating the expectation values analytically continued from different domains, the single equation of $\hat{\hat{\mathfrak{D}}}_3$ is not enough to fully determine the partition function. Nevertheless, in the limit $\varepsilon_2 \to 0$ the equation is reduced to the oper $\widehat{\mathfrak{D}}_3$ on $\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$, and the relations (5.4.70) and (5.4.71) are indeed enough to guarantee that the solutions agree wih each other. This is because, as we have seen earlier, the expectation value $\left\langle \mathcal{O}_3 \right\rangle$ is dominated by the limit shape and becomes a series only in $\mathfrak{q}$, comprising an accessory parameter for the oper $\widehat{\mathfrak{D}}_3$ which is unambiguously determined once the monodromy along the A-cycle is fixed.

We furthermore suspect that even for generic values of ε_2, there is a proper matching between the analytically continued expectation values in the intermediate domain, so that the identities,

$$\widetilde{\mathfrak{Z}}^{L\to M}(\mathbf{a}) = \mathbf{S}\, \widetilde{\mathfrak{Z}}^{R\to M}(\mathbf{a}) = \widetilde{\mathfrak{Z}}^{\mathbb{Z}_2}(\mathbf{a}), \tag{5.4.72}$$

persist to be true. We have checked the identities at low orders in the gauge couplings z and $\mathfrak{q}$. We discuss more on this issue in section 5.7.

Remarks

- The duality between the quiver-type and the orbifold-type surface defects was realized in [48] as the M-theory brane transition, for the A_1-theories. It would be interesting to study the relation between the higher rank generalization of the duality in [48] and the exact identification of the partition functions (5.4.72).

5.5 Darboux coordinates

Recall that the main assertion of [91] is that the generating function for the variety of opers with respect to the NRS coordinate system is identical to the effective twisted superpotential of a class $\mathcal{S}$ theory:

$$\mathcal{S}\left[\mathcal{O}_N[\underline{\mathcal{C}}]\right] = \frac{1}{\varepsilon_1}\left(\widetilde{\mathcal{W}}\left[\mathcal{T}[A_{N-1}, \underline{\mathcal{C}}]\right] - \widetilde{\mathcal{W}}_\infty\right). \tag{5.5.1}$$

We need a generalization of the NRS coordinates for $N > 2$ to give any meaning to the left hand side of the correspondence.

Here, we propose a Darboux coordinate system on the moduli space of flat $SL(N)$-connections on the $r+3$-punctured sphere $\mathbb{P}^1_{2,\underline{r+1}}$ with two maximal and $r+1$ minimal punctures, for the arbitrary higher rank $N-1$. The proposed coordinates reduce to the usual NRS coordinate system in $N = 2$ on a specific patch of the moduli space of flat connections.

In this section $\underline{\mathcal{C}}_r$ denotes $\mathbb{P}^1_{2,\underline{r+1}} = \mathbb{P}^1\backslash\{z_{-1}, \underline{z_0, \ldots, z_r}, z_{r+1}\}$. We often set $z_{-1} = \infty$, $z_{r+1} = 0$, and $z_0 = 1$.

5.5.1 Construction of Darboux coordinates

5.5.1.1 Definition

We construct Darboux coordinates on a patch of the moduli space of flat $SL(N)$-connections on the $r+3$-punctured sphere $\underline{\mathcal{C}}_r$, which reduces to the NRS coordinates in the $N = 2$ case. Our main example of the four-punctured sphere is the case $r = 1$. As in (5.1.15), the moduli space $\mathcal{M}_{\text{flat}}(SL(N), \underline{\mathcal{C}}_r)$ is the space of (stable) equivalence classes of the homomorphisms of the fundamental group of the punctured Riemann sphere to $SL(N)$, in which the loops encircling each puncture are mapped to the prescribed conjugacy classes in $SL(N)$. In particular, the two maximal punctures correspond to generic semisimple conjugacy classes in $SL(N)$, while the $r+1$ minimal punctures correspond to semisimple conjugacy classes in $SL(N)$ with maximally degenerate eigenvalues. We fix the conjugacy classes by specifying the eigenvalues of the holonomy matrices g_i, $i = -1, 0, 1, \ldots, r+1$. The moduli space is given by:

$$
\begin{aligned}
&\mathcal{M}_{\text{flat}}(SL(N), \underline{\mathcal{C}}_r) \\
&= \left\{ (g_i)_{i=-1}^{r+1} \;\middle|\; \begin{array}{l} g_i \in SL(N)\,, \\ \mathrm{Det}(g_i - x) = (\mathfrak{m}_i - x)^{N-1}(\mathfrak{m}_i^{1-N} - x)\,,\ i = 0, \ldots, r \\ \\ \mathrm{Det}(g_i - x) = \prod_{\alpha=1}^{N} \left(\left(\mathfrak{m}_i^{(\alpha)}\right)^{-\mathrm{sgn}(i)} - x \right)\,,\ i = -1, r+1 \\ g_{-1} g_0 \cdots g_{r+1} = \mathbb{1}_N \end{array} \right\}^{\text{stable}} \Bigg/ SL(N).
\end{aligned}
\tag{5.5.2}
$$

The stability condition chooses an open subset in the set of matrices g_i obeying all of the conditions above. We shall not need to specify the stability condition since we are going to work on an open patch of the moduli space which belongs to the stable subset.

The holonomies $g_i \sim \mathrm{diag}(\mathfrak{m}_i, \cdots, \mathfrak{m}_i, \mathfrak{m}_i^{-N+1})$ around the minimal punctures require more detailed notation. We can form such an element of $SL(N)$ by setting

$$
g_i = \mathfrak{m}_i \left(\mathbb{1}_N + \left(\mathfrak{m}_i^{-N} - 1 \right) E_i \otimes \tilde{E}_i \right), \tag{5.5.3}
$$

where

$$
\begin{aligned}
& E_i \in \mathbb{C}^N,\ \tilde{E}_i \in \left(\mathbb{C}^N\right)^* \ \text{(dual space)} \\
& \tilde{E}_i(E_i) = 1,
\end{aligned}
\tag{5.5.4}
$$

which are defined up to rescaling $\left(E_i, \tilde{E}_i\right) \mapsto \left(t_i E_i, t_i^{-1} \tilde{E}_i\right)$, $t_i \in \mathbb{C}^\times$. For fixed $\tilde{E}_i$, its null subspace in $\mathbb{C}^N$ is $N-1$-dimensional. Hence we have $N-1$-dimensional eigenspace of g_i with the eigenvalue $\mathfrak{m}_i$. The one last eigenvector is given by E_i, with the eigenvalue $\mathfrak{m}_i^{-N+1}$ fixed by the normalization condition. The number of degrees of freedom in such a g_i is equal to

$$2N \text{ (from } E \text{ and } \tilde{E}) - 1 \text{ (normalization)} - 1 \text{ (rescaling)} = 2(N-1). \tag{5.5.5}$$

Therefore, a simple dimension count gives

$$\begin{aligned}\dim \mathcal{M}_{\text{flat}}(SL(N), \mathbb{P}^1_{2,\underline{r+1}}) &= 2\left((N^2-1)-(N-1)\right) + (r+1)\left(2(N-1)\right) - 2(N^2-1) \\ &= 2r(N-1).\end{aligned} \tag{5.5.6}$$

We need to define $r(N-1)$-pairs of coordinates which are canonical under the Poisson bracket. For this, it is convenient to parametrize the moduli space as follows. Let us define the projection operators

$$\begin{aligned}&\Pi_i = E_i \otimes \tilde{E}_i, \quad i = 0, 1, \cdots, r \\ &\Pi_i^2 = \Pi_i,\end{aligned} \tag{5.5.7}$$

formed by the eigenvector E_i (5.5.4) of g_i and its dual-vector. Then g_i is expressed as

$$g_i = \mathfrak{m}_i \left(\mathbb{1}_N + \left(\mathfrak{m}_i^{-N} - 1\right) \Pi_i\right), \qquad i = 0, 1, \ldots, r. \tag{5.5.8}$$

For later use, we also give the expression for its inverse:

$$g_i^{-1} = \mathfrak{m}_i^{-1} \left(\mathbb{1}_N + (\mathfrak{m}_i^N - 1)\Pi_i\right), \qquad i = 0, 1, \ldots, r. \tag{5.5.9}$$

Let us also define

$$M_i \equiv g_{-1} g_0 \ldots g_i \in SL(N), \quad i = -1, 0, 1, \cdots, r+1. \tag{5.5.10}$$

These matrices represent the holonomies along the curves on the $r+3$-punctured sphere

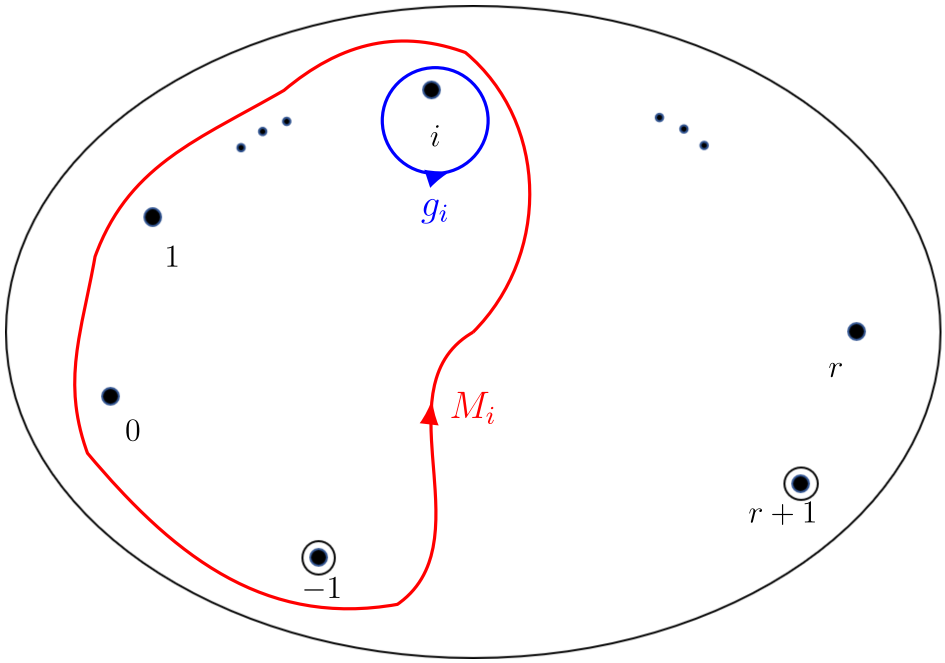

Figure 5.3: The $r+3$-punctured sphere $\mathbb{P}^1_{2,\underline{r+1}}$. The (-1)-th puncture (located at $z=\infty$) and the $(r+1)$-th puncture (located at $z=0$) are maximal, denoted by double circles, while all the other punctures are minimal, denoted by simple dots. The holonomy along the loop encircling each puncture is represented by g_i (blue line), while the holonomy along the loop enclosing $i+2$ punctures is represented by M_i (red line).

enclosing $i+2$ punctures (see Figure 5.3). In particular, it is immediate that we have $M_{-1}=g_{-1}$, $M_r=g_{r+1}^{-1}$, and $M_{r+1}=\mathbb{1}_N$. We can express these matrices as

$$M_i=\sum_{\alpha=1}^{N}\mathfrak{m}_i^{(\alpha)}\,\Pi_i^{(\alpha)}\,, \tag{5.5.11}$$

with the projection operators $\Pi_i^{(\alpha)}$ obeying

$$\Pi_i^{(\alpha)}\Pi_i^{(\beta)}=\delta_{\alpha,\beta}\,\Pi_i^{(\alpha)}, \tag{5.5.12}$$

each having rank one. Using the eigenbasis $E_i^{(\alpha)}\in\mathbb{C}^N$, $i=0,1,\ldots,r-1$ of M_i, and its dual-basis $\tilde{E}_i^{(\beta)}\in\left(\mathbb{C}^N\right)^*$, $\tilde{E}_i^{(\alpha)}(E_i^{(\beta)})=\delta_{\alpha,\beta}$, we can write

$$\Pi_i^{(\alpha)}=E_i^{(\alpha)}\otimes\tilde{E}_i^{(\alpha)}. \tag{5.5.13}$$

The basis vectors are defined up to rescalings $\left(E_i^{(\alpha)}, \tilde{E}_i^{(\alpha)}\right) \mapsto \left(t_i^{(\alpha)} E_i^{(\alpha)}, \left(t_i^{(\alpha)}\right)^{-1} \tilde{E}_i^{(\alpha)}\right)$, $t_i^{(\alpha)} \in \mathbb{C}^\times$, and reorderings $E_i^{(\alpha)} \mapsto E_i^{(\sigma_i(\alpha))}$, $\sigma_i \in \mathcal{S}(N)$.

Now we are ready to propose a Darboux coordinate system. We define the coordinates $\boldsymbol{\alpha}_i^{(\alpha)}$, $\tilde{\boldsymbol{\beta}}_i^{(\alpha)}$, $i = 0, 1, \ldots, r-1$, $\alpha = 1, \ldots, N$, subject to the constraints

$$\sum_{\alpha=1}^{N} \boldsymbol{\alpha}_i^{(\alpha)} = 0 \tag{5.5.14}$$

and defined up to the shifts

$$\tilde{\boldsymbol{\beta}}_i^{(\alpha)} \mapsto \tilde{\boldsymbol{\beta}}_i^{(\alpha)} + b_i\,, \qquad b_i \in \mathbb{C} \tag{5.5.15}$$

via

$$M_i\ E_i^{(\alpha)} = e^{2\pi \mathrm{i} \boldsymbol{\alpha}_i^{(\alpha)}}\ E_i^{(\alpha)} \tag{5.5.16}$$

and

$$e^{-\tilde{\boldsymbol{\beta}}_i^{(\alpha)} + \tilde{\boldsymbol{\beta}}_i} = \frac{\tilde{E}_i^{(\alpha)}(E_{i+1})}{\tilde{E}_i^{(\alpha)}(E_i)} \tilde{E}_{i+1}(E_i) = \frac{\mathrm{Tr}_N \Pi_i \Pi_i^{(\alpha)} \Pi_{i+1}}{\mathrm{Tr}_N \Pi_i \Pi_i^{(\alpha)}}, \tag{5.5.17}$$

where $\tilde{\boldsymbol{\beta}}_i$ is defined by:

$$e^{\tilde{\boldsymbol{\beta}}_i} = \sum_{\alpha=1}^{N} e^{\tilde{\boldsymbol{\beta}}_i^{(\alpha)}} \mathrm{Tr}_N \left(\Pi_{i+1} \Pi_i^{(\alpha)}\right). \tag{5.5.18}$$

Due to the constraint (5.5.14) and the ambiguity (5.5.15), the coordinates $\boldsymbol{\alpha}_i^{(\alpha)}$, $\tilde{\boldsymbol{\beta}}_i^{(\alpha)}$ are redundant. Thus we refine the coordinates by choosing mutually independent $r(N-1)$-pairs

$$\left(\boldsymbol{\alpha}_i^{(\alpha)},\ \boldsymbol{\beta}_i^{(\alpha)} \equiv \tilde{\boldsymbol{\beta}}_i^{(\alpha)} - \tilde{\boldsymbol{\beta}}_i^{(N)}\right), \quad i = 0, 1, \cdots, r-1,\ \alpha = 1, \cdots, N-1. \tag{5.5.19}$$

to form a proper coordinate system on $\mathcal{M}_{\text{flat}}(SL(N), \mathbb{P}^1_{2,\underline{r+1}})$.

5.5.1.2 Canonical Poisson relations

To show that $\left\{\boldsymbol{\alpha}_i^{(\alpha)}, \boldsymbol{\beta}_i^{(\alpha)} \mid i=0,1,\cdots,r-1,\ \alpha=1,\cdots,N-1\right\}$ forms a Darboux coordinate system on $\mathcal{M}_{\text{flat}}(SL(N),\mathbb{P}^1_{2,\underline{r+1}})$, we have to verify that the Poisson brackets [97] are canonical[9]

$$\begin{aligned} &\left\{\tilde{\boldsymbol{\beta}}_i^{(\alpha)}\,,\,\boldsymbol{\alpha}_j^{(\beta)}\right\}=\delta_{i,j}\delta_{\alpha,\beta} && i,j=0,1,\cdots,r-1,\\ &\left\{\boldsymbol{\alpha}_i^{(\alpha)},\boldsymbol{\alpha}_j^{(\beta)}\right\}=\left\{\tilde{\boldsymbol{\beta}}_i^{(\alpha)},\tilde{\boldsymbol{\beta}}_j^{(\beta)}\right\}=0, && \alpha,\beta=1,\cdots,N. \end{aligned} \tag{5.5.20}$$

The Poisson bracket on the space of all gauge fields

$$\left\{\mathcal{A}^a(x),\mathcal{A}^b(y)\right\}=\delta^{ab}\delta^{(2)}(x,y) \tag{5.5.21}$$

(the $\delta^{(2)}$ is a two-form on $\mathbb{P}^1_{2,\underline{r+1}}$) has a simple geometric description when represented on the holonomies. To illustrate, consider two distinct elements of the fundamental group $[\gamma_{1,2}]\in\pi_1\left(\mathbb{P}^1_{2,\underline{r+1}}\right)$. We can choose their representatives $\gamma_{1,2}$ to intersect transversally. We assign to each intersection point $x\in\gamma_1\cap\gamma_2$ a sign

$$s:\gamma_1\cap\gamma_2\longrightarrow\{\pm\} \tag{5.5.22}$$

according to the orientation of the curves $\gamma_{1,2}$ at x relative to the orientation of the sphere (see Figure 5.4). Then we define

$$(\gamma_1\cap\gamma_2)^{\pm}\equiv\{x\in\gamma_1\cap\gamma_2\mid s(x)=\pm\}\,. \tag{5.5.23}$$

At each intersection x, we compose a resolution $(\gamma_1\cup\gamma_2)_x$ of the union of the curves as described in Figure 5.4. Now the Poisson structure on the moduli space of flat connections

[9] It is clear that the Poisson brackets for the refined coordinates (5.5.19) are also canonical once (5.5.20) is proven.

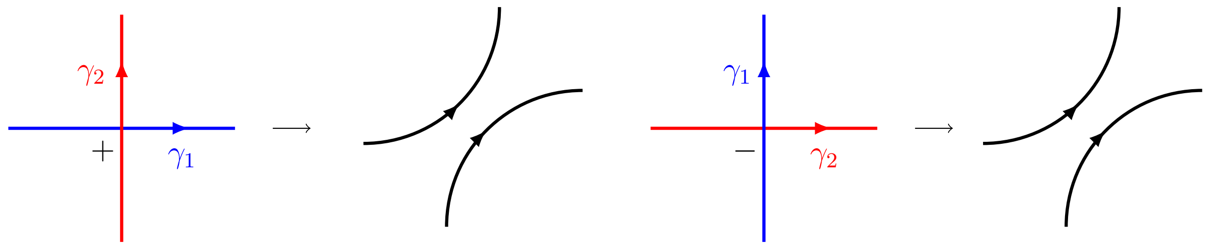

Figure 5.4: The sign assignment to intersection points and the resolution of the union of curves.

can be represented on the holonomies ρ along $\gamma_{1,2}$ by

$$\Big\{\rho([\gamma_1]), \rho([\gamma_2])\Big\} = \sum_{x\in(\gamma_1\cap\gamma_2)^+} \rho\left([(\gamma_1 \cup \gamma_2)_x]\right) - \sum_{x\in(\gamma_1\cap\gamma_2)^-} \rho\left([(\gamma_1 \cup \gamma_2)_x]\right). \tag{5.5.24}$$

Using the geometric description of the Poisson structure, we can show that the coordinates defined in (5.5.16) and (5.5.17) satisfy the canonical Poisson relations (5.5.20). Let us package (5.5.16) into the generating function:

$$\mathbb{A}_i(x) \equiv \mathrm{Tr}_N\ (x - M_i)^{-1} = \sum_{l=0}^{\infty} \frac{1}{x^{l+1}} \mathrm{Tr}_N\, M_i^l, \tag{5.5.25}$$

which has a simple geometric meaning as the generating function of the loops which wind along the same curve (whose holonomy is represented by M_i) multiple times. Since there is no intersection among these curves, it is clear that we have

$$\Big\{\mathbb{A}_i(x), \mathbb{A}_j(y)\Big\} = 0, \tag{5.5.26}$$

for any $i, j = 0, 1, \cdots, r-1$. Thus we derive

$$\Big\{\boldsymbol{\alpha}_i^{(\alpha)}, \boldsymbol{\alpha}_j^{(\beta)}\Big\} = 0, \tag{5.5.27}$$

for any $i, j = 0, 1, \cdots, r-1$, $\alpha, \beta = 1, \cdots, N$.

We can also package (5.5.17) into

$$\mathbb{B}_i(x) \equiv \mathrm{Tr}_N\, \Pi_i\, (x - M_i)^{-1}\, \Pi_{i+1} = e^{\tilde{\beta}_i} \sum_{\alpha=1}^{N} e^{-\tilde{\beta}_i^{(\alpha)}}\, \frac{\mathrm{Tr}_N\, \Pi_i \Pi_i^{(\alpha)}}{x - m_i^{(\alpha)}}. \tag{5.5.28}$$

We can re-express this via:

$$\begin{aligned} \mathbb{D}_i(x) &\equiv \mathrm{Tr}_N\, g_i\, (x - M_i)^{-1}\, g_{i+1} \\ &= \mathfrak{m}_i \mathfrak{m}_{i+1} (\mathfrak{m}_i^{-N} - 1)(\mathfrak{m}_{i+1}^{-N} - 1)\mathbb{B}_i(x) + \mathfrak{m}_i \mathfrak{m}_{i+1} x^{-1} \left(\frac{P_{i-1}(\mathfrak{m}_i^{-1} x)}{P_i(x)} - 1 \right) \\ &\quad - \mathfrak{m}_i \mathfrak{m}_{i+1}^{1-N} x^{-1} \left(\frac{P_{i+1}(\mathfrak{m}_{i+1} x)}{P_i(x)} - 1 \right) + \mathfrak{m}_i \mathfrak{m}_{i+1} \mathbb{A}_i(x), \end{aligned} \tag{5.5.29}$$

where $P_i(x)$ is the characteristic polynomial of M_i:

$$P_i(x) = \mathrm{Det}(x - M_i) = \prod_{\alpha=1}^{N} \left(x - \mathfrak{m}_i^{(\alpha)} \right). \tag{5.5.30}$$

In deriving the second equality of (5.5.29), we had simple manipulations on the determinants[10] and (5.5.11):

$$\frac{P_{i-1}(\mathfrak{m}_i^{-1} x)}{P_i(x)} - 1 = x \left(1 - \mathfrak{m}_i^{-N} \right) \mathrm{Tr}(M_i - x)^{-1} \Pi_i \tag{5.5.31}$$

$$\frac{P_i(\mathfrak{m}_i x)}{P_{i-1}(x)} - 1 = x \left(1 - \mathfrak{m}_i^{N} \right) \mathrm{Tr}(M_{i-1} - x)^{-1} \Pi_i. \tag{5.5.32}$$

The function $\mathbb{D}_i(x)$ has a simple geometric meaning:

$$\mathbb{D}_i(x) = \sum_{l=0}^{\infty} \frac{1}{x^{l+1}} \mathrm{Tr}_N\, g_i M_i^l g_{i+1} \tag{5.5.33}$$

from which it is obvious that $\{\mathbb{D}_i(x), \mathbb{A}_j(y)\} = 0$ for $i \neq j$ (the corresponding loops on the $r+3$-punctured sphere do not intersect), as well as that $\{\mathbb{D}_i(x), \mathbb{D}_j(y)\} = 0$ for $|i - j| > 1$.

[10] Use that for any rank one projector Π, and any operator A, $\mathrm{Det}(1 + A\Pi) = 1 + \mathrm{Tr}(A\Pi)$

From these, we derive

$$\begin{aligned} \left\{ \tilde{\boldsymbol{\beta}}_i^{(\alpha)}, \boldsymbol{\alpha}_j^{(\beta)} \right\} &= 0, \quad i \neq j, \ \alpha, \beta = 1, \cdots, N, \\ \left\{ \tilde{\boldsymbol{\beta}}_i^{(\alpha)}, \tilde{\boldsymbol{\beta}}_j^{(\beta)} \right\} &= 0, \quad |i-j| > 1, \ \alpha, \beta = 1, \cdots, N. \end{aligned} \tag{5.5.34}$$

It remains to compute:

$$\left\{ \mathbb{D}_i(x), \mathbb{A}_i(y) \right\}, \quad \left\{ \mathbb{D}_i(x), \mathbb{D}_{i+1}(y) \right\}, \quad \text{and} \quad \left\{ \mathbb{D}_i(x), \mathbb{D}_i(y) \right\}, \tag{5.5.35}$$

which are a bit more involoved. As we elaborate in appendix F in detail, the rest of the canonical Poisson relations (5.5.20) are obtained out of these brackets, confirming that the proposed coordinate system is indeed Darboux.

Remarks

- Other constructions generalizing the NRS-type coordinates were proposed in the $SL(2)$ case in [104], in the arbitrary group case in [103], and specifically in the $SL(3)$ case in [98]. In [104] and [98], the spectral coordinates are defined as the holonomies of a (twisted) flat $GL(1)$-connection on a line bundle over the N-fold branched covering Σ of the Riemann surface $\mathcal{C}$, which is a certain uplift (called *abelianization*) of the flat $SL(N)$-connection on $\mathcal{C}$ [101]. To give some credit to our construction, as described above, it produces the Darboux coordinates for arbitrary N in an elementary fashion, albeit only on a specific patch of the moduli space.

5.5.2 The four-punctured sphere

We consider our main example, the four-punctured sphere $\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$. The generalized NRS coordinate system on the moduli space $\mathcal{M}_{\text{flat}}(SL(N), \mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\})$ is just a special case $r = 1$ of the one defined in the previous section. In this special case, it is convenient to express the generalized NRS coordinates in terms of the trace invariants of the holonomies, and take these expressions as equivalent definitions for those coordinates. Since the dimension of the moduli space is $\dim \mathcal{M}_{\text{flat}}(SL(N), \mathbb{P}^1\backslash\{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}) = 2(N-1)$, it is enough to consider two independent cycles on $\mathbb{P}^1\backslash\{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$ which we choose to be the A-cycle and the B-cycle in Figure 5.5. We describe how the traces of the holonomies $M_{A,B}$ along these cycles are expressed in terms of the generalized NRS coordinates, for $N = 2$ and $N = 3$.

5.5.2.1 $SL(2)$

We start with the A-cycle. It is clear that we have

$$M_A = M_0^{-1} = \sum_{\alpha=1}^{2} \left(\mathfrak{m}_0^{(\alpha)}\right)^{-1} \Pi_0^{(\alpha)}. \tag{5.5.36}$$

Thus we find

$$\operatorname{Tr} M_A = \left(\mathfrak{m}_0^{(1)}\right)^{-1} + \left(\mathfrak{m}_0^{(2)}\right)^{-1} = e^{-2\pi i \boldsymbol{\alpha}_0^{(1)}} + e^{-2\pi i \boldsymbol{\alpha}_0^{(2)}}. \tag{5.5.37}$$

It is convenient to omit the superscript and write $\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}_0^{(1)} = -\boldsymbol{\alpha}_0^{(2)}$. Thus we have

$$\operatorname{Tr} M_A = 2\cos 2\pi\boldsymbol{\alpha}. \tag{5.5.38}$$

Next, we can express the holonomy along the B-cycle as

$$\begin{aligned} M_B &= g_2 g_{-1} = g_1^{-1} g_0^{-1} \\ &= \mathfrak{m}_0^{-1}\mathfrak{m}_1^{-1} \left(\mathbb{1}_2 + (\mathfrak{m}_1^2 - 1)\Pi_1\right)\left(\mathbb{1}_2 + (\mathfrak{m}_0^2 - 1)\Pi_0\right). \end{aligned} \tag{5.5.39}$$

Thus we find

$$\operatorname{Tr} M_B = \mathfrak{m}_0\mathfrak{m}_1^{-1} + \mathfrak{m}_0^{-1}\mathfrak{m}_1 + (\mathfrak{m}_0 - \mathfrak{m}_0^{-1})(\mathfrak{m}_1 - \mathfrak{m}_1^{-1}) \operatorname{Tr} \Pi_0\Pi_1 \tag{5.5.40}$$

Note that we can express the trace in the last term using the $\boldsymbol{\beta}$ coordinates,

$$\begin{aligned}
\operatorname{Tr}\Pi_0\Pi_1 &= \sum_{\alpha=1}^{2} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)}\Pi_1 \\
&= \sum_{\alpha=1}^{2} e^{-\tilde{\beta}_0^{(\alpha)}+\tilde{\beta}_0} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)} \\
&= \sum_{\alpha=1}^{2} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)} \operatorname{Tr}\Pi_1\Pi_0^{(\alpha)} \\
&\quad + e^{\tilde{\beta}_0^{(1)}-\tilde{\beta}_0^{(2)}} \operatorname{Tr}\Pi_0\Pi_0^{(2)} \operatorname{Tr}\Pi_1\Pi_0^{(1)} + e^{\tilde{\beta}_0^{(2)}-\tilde{\beta}_0^{(1)}} \operatorname{Tr}\Pi_0\Pi_0^{(1)} \operatorname{Tr}\Pi_1\Pi_0^{(2)},
\end{aligned} \tag{5.5.41}$$

where we have used (5.5.18) in the third equality. Using (5.5.31) and (5.5.32), we can express the rest of the traces in terms of the $\boldsymbol{\alpha}$ coordinates. For simplicity, let us define $\mathfrak{m}_{-1} \equiv \mathfrak{m}_{-1}^{(1)}$ and $\mathfrak{m}_2 \equiv \mathfrak{m}_1^{(2)}$.[11] Also we refine the $\boldsymbol{\beta}$ coordinate according to the definition (5.5.19): $\boldsymbol{\beta} \equiv \tilde{\boldsymbol{\beta}}_0^{(1)} - \tilde{\boldsymbol{\beta}}_0^{(2)}$. Then the final expression for the trace of the holonomy along the B-cycle is

$$\begin{aligned}
\operatorname{Tr} M_B &= \frac{\left(\mathfrak{m}_2+\mathfrak{m}_2^{-1}-\mathfrak{m}_1-\mathfrak{m}_1^{-1}\right)\left(\mathfrak{m}_{-1}+\mathfrak{m}_{-1}^{-1}-\mathfrak{m}_0-\mathfrak{m}_0^{-1}\right)}{8\sin^2\pi\boldsymbol{\alpha}} \\
&\quad + \frac{\left(\mathfrak{m}_2+\mathfrak{m}_2^{-1}+\mathfrak{m}_1+\mathfrak{m}_1^{-1}\right)\left(\mathfrak{m}_{-1}+\mathfrak{m}_{-1}^{-1}+\mathfrak{m}_0+\mathfrak{m}_0^{-1}\right)}{8\cos^2\pi\boldsymbol{\alpha}} \\
&\quad - \sum_{\pm} \frac{\left(e^{\mp 2\pi i\boldsymbol{\alpha}}-\mathfrak{m}_0\mathfrak{m}_{-1}\right)\left(\mathfrak{m}_0^{-1}e^{\mp 2\pi i\boldsymbol{\alpha}}-\mathfrak{m}_{-1}^{-1}\right)\left(e^{\pm 2\pi i\boldsymbol{\alpha}}-\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}\right)\left(\mathfrak{m}_1 e^{\pm 2\pi i\boldsymbol{\alpha}}-\mathfrak{m}_2\right)}{4\sin^2 2\pi\boldsymbol{\alpha}} e^{\pm\boldsymbol{\beta}}.
\end{aligned} \tag{5.5.42}$$

Thus we observe that the coordinates $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ determine the traces of the holonomies along the A-cycle and B-cycle on $\mathbb{P}^1\backslash\{0, \mathfrak{q}, 1, \infty\}$ by (5.5.38) and (5.5.42). Conversely, we may take these formulas as defining equations for the coordinates $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Remarks

[11] Not to be confused with the eigenvalue $\mathfrak{m}_2$ of g_2 which appears when $r > 1$. Here, we restrict ourselves only to the case $r = 1$ and there would be no confusion in notation.

- For a direct comparison with the coordinates defined in [91], let us define

$$x_1 = \mathfrak{m}_{-1} + \mathfrak{m}_{-1}^{-1}, \quad x_2 = \mathfrak{m}_0 + \mathfrak{m}_0^{-1}, \quad x_3 = \mathfrak{m}_2 + \mathfrak{m}_2^{-1}, \quad x_4 = \mathfrak{m}_1 + \mathfrak{m}_1^{-1}. \tag{5.5.43}$$

Also we use the abbreviation

$$A = \mathrm{Tr}\, M_A, \quad B = \mathrm{Tr}\, M_B. \tag{5.5.44}$$

Then we find that (5.5.42) becomes

$$\begin{aligned} B(A^2-4) = 2(x_1x_4 + x_2x_3) - A(x_1x_3 + x_2x_4) \\ + \left(e^{\boldsymbol{\beta}} + e^{-\boldsymbol{\beta}}\right)\sqrt{c_{12}(A)c_{34}(A)}, \end{aligned} \tag{5.5.45}$$

where

$$c_{ij}(A) \equiv A^2 - A\, x_i x_j + x_i^2 + x_j^2 - 4, \tag{5.5.46}$$

under the canonical transformation

$$\begin{aligned} \boldsymbol{\beta} \to \boldsymbol{\beta} + \frac{1}{2}\log\left(e^{-2\pi i\boldsymbol{\alpha}} - \mathfrak{m}_0\mathfrak{m}_{-1}\right)\left(\mathfrak{m}_0^{-1}e^{-2\pi i\boldsymbol{\alpha}} - \mathfrak{m}_{-1}^{-1}\right)\left(e^{2\pi i\boldsymbol{\alpha}} - \mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}\right)\left(\mathfrak{m}_1 e^{2\pi i\boldsymbol{\alpha}} - \mathfrak{m}_2\right) \\ - \frac{1}{2}\log\left(e^{2\pi i\boldsymbol{\alpha}} - \mathfrak{m}_0\mathfrak{m}_{-1}\right)\left(\mathfrak{m}_0^{-1}e^{2\pi i\boldsymbol{\alpha}} - \mathfrak{m}_{-1}^{-1}\right)\left(e^{-2\pi i\boldsymbol{\alpha}} - \mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}\right)\left(\mathfrak{m}_1 e^{-2\pi i\boldsymbol{\alpha}} - \mathfrak{m}_2\right). \end{aligned} \tag{5.5.47}$$

The relation (5.5.45) is precisely the defining equation for the NRS coordinate $\boldsymbol{\beta}$ for the four-punctured sphere. Thus we confirm that the Darboux coordinate system proposed in section 5.5.1 is a higher-rank generalization of the NRS coordinate system.

As we will see in section 5.6, the canonical transformation (5.5.47) amounts to change the boundary contribution to the effective twisted superpotential. Although the transformed coordinates may be natural in some context, we will find in section 5.6 that our original definition is more natural in the gauge theoretical context. Therefore we stick to our original definition of the generalized NRS coordinates in section 5.5.1 without making additional canonical transformation throughout the discussion.

5.5.2.2 $SL(3)$

We begin with the A-cycle holonomy which is clearly conjugate to

$$M_A = M_0^{-1} = \sum_{\alpha=1}^{3} \left(\mathfrak{m}_0^{(\alpha)}\right)^{-1} \Pi_0^{(\alpha)}. \tag{5.5.48}$$

Thus we find

$$\operatorname{Tr} M_A^{\pm 1} = \sum_{\alpha=1}^{3} \left(\mathfrak{m}_0^{(\alpha)}\right)^{\mp 1} = e^{\mp 2\pi i \boldsymbol{\alpha}_0^{(1)}} + e^{\mp 2\pi i \boldsymbol{\alpha}_0^{(2)}} + e^{\pm 2\pi i (\boldsymbol{\alpha}_0^{(1)} + \boldsymbol{\alpha}_0^{(2)})}. \tag{5.5.49}$$

For notational convenience, let us define the coordinates without superscripts,

$$\begin{aligned} &\boldsymbol{\alpha}_\alpha \equiv \boldsymbol{\alpha}_0^{(\alpha)}, \quad \alpha = 1,2,3. \\ &\boldsymbol{\alpha}_3 = -\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2, \end{aligned} \tag{5.5.50}$$

so that we have

$$\operatorname{Tr} M_A^{\pm 1} = e^{\mp 2\pi i \boldsymbol{\alpha}_1} + e^{\mp 2\pi i \boldsymbol{\alpha}_2} + e^{\pm 2\pi i (\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)}. \tag{5.5.51}$$

The expressions for the holonomy along the B-cycle and its inverse are

$$\begin{aligned} M_B^{\pm 1} &= \left(g_1^{-1} g_0^{-1}\right)^{\pm 1} \\ &= \mathfrak{m}_0^{\mp 1} \mathfrak{m}_1^{\mp 1} \left(\mathbb{1}_3 + (\mathfrak{m}_1^{\pm 3} - 1)\Pi_1\right) \left(\mathbb{1}_3 + (\mathfrak{m}_0^{\pm 3} - 1)\Pi_0\right). \end{aligned} \tag{5.5.52}$$

Thus we obtain

$$\operatorname{Tr} M_B^{\pm 1} = \mathfrak{m}_0^{\mp 1}\mathfrak{m}_1^{\mp 1}(1+\mathfrak{m}_0^{\pm 3}+\mathfrak{m}_1^{\pm 3}) + (\mathfrak{m}_0^{\pm 2}-\mathfrak{m}_0^{\mp 1})(\mathfrak{m}_1^{\pm 2}-\mathfrak{m}_1^{\mp 1}) \operatorname{Tr}\Pi_0\Pi_1. \tag{5.5.53}$$

Again, we can express the trace in the last term using the $\boldsymbol{\beta}$ coordinates,

$$\begin{aligned}\operatorname{Tr}\Pi_0\Pi_1 &= \sum_{\alpha=1}^{3} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)}\Pi_1 \\ &= \sum_{\alpha=1}^{3} e^{-\tilde{\boldsymbol{\beta}}_0^{(\alpha)}+\tilde{\boldsymbol{\beta}}_0} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)} \\ &= \sum_{\alpha=1}^{3} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)} \operatorname{Tr}\Pi_1\Pi_0^{(\alpha)} + \sum_{\alpha\neq\beta} e^{\tilde{\boldsymbol{\beta}}_0^{(\alpha)}-\tilde{\boldsymbol{\beta}}_0^{(\beta)}} \operatorname{Tr}\Pi_0\Pi_0^{(\beta)} \operatorname{Tr}\Pi_1\Pi_0^{(\alpha)}.\end{aligned} \tag{5.5.54}$$

The rest of the traces can be expressed in terms of the $\boldsymbol{\alpha}$ coordinates by using (5.5.31) and (5.5.32). We also write the refined $\boldsymbol{\beta}$ coordinates without superscripts,

$$\boldsymbol{\beta}_\alpha \equiv \tilde{\boldsymbol{\beta}}_0^{(\alpha)} - \tilde{\boldsymbol{\beta}}_0^{(3)}, \quad \alpha = 1, 2. \tag{5.5.55}$$

Then the final expressions for the traces of the holonomies along the B-cycle are

$$\begin{aligned}\operatorname{Tr} M_B^{\pm 1} = B_0^{\pm} &+ B_{12}^{\pm} e^{\boldsymbol{\beta}_1-\boldsymbol{\beta}_2} + B_{13}^{\pm} e^{\boldsymbol{\beta}_1} + B_{23}^{\pm} e^{\boldsymbol{\beta}_2} \\ &+ B_{21}^{\pm} e^{-\boldsymbol{\beta}_1+\boldsymbol{\beta}_2} + B_{31}^{\pm} e^{-\boldsymbol{\beta}_1} + B_{32}^{\pm} e^{-\boldsymbol{\beta}_2},\end{aligned} \tag{5.5.56}$$

where

$$\begin{aligned} B_0^{+} &= \mathfrak{m}_0^{-1}\mathfrak{m}_1^{-1} + \mathfrak{m}_0^{2}\mathfrak{m}_1^{-1} + \mathfrak{m}_0^{-1}\mathfrak{m}_1^{2} \\ &\quad - \frac{\mathfrak{m}_0^{2}\mathfrak{m}_1^{-1}}{16} \sum_{\alpha=1}^{3} \frac{\prod_{\gamma=1}^{3}\left(\mathfrak{m}_0^{-1}e^{\pi i\boldsymbol{\alpha}_\alpha} - \mathfrak{m}_{-1}^{(\gamma)}e^{-\pi i\boldsymbol{\alpha}_\alpha}\right)\left(\mathfrak{m}_1 e^{\pi i\boldsymbol{\alpha}_\alpha} - \mathfrak{m}_1^{(\gamma)}e^{-\pi i\boldsymbol{\alpha}_\alpha}\right)}{\prod_{\alpha'\neq\alpha}\sin^2\pi(\boldsymbol{\alpha}_\alpha - \boldsymbol{\alpha}_{\alpha'})} \\ B_0^{-} &= \mathfrak{m}_0\mathfrak{m}_1 + \mathfrak{m}_0^{-2}\mathfrak{m}_1 + \mathfrak{m}_0\mathfrak{m}_1^{-2} \\ &\quad - \frac{\mathfrak{m}_0\mathfrak{m}_1^{-2}}{16} \sum_{\alpha=1}^{3} \frac{\prod_{\gamma=1}^{3}\left(\mathfrak{m}_0^{-1}e^{\pi i\boldsymbol{\alpha}_\alpha} - \mathfrak{m}_{-1}^{(\gamma)}e^{-\pi i\boldsymbol{\alpha}_\alpha}\right)\left(\mathfrak{m}_1 e^{\pi i\boldsymbol{\alpha}_\alpha} - \mathfrak{m}_1^{(\gamma)}e^{-\pi i\boldsymbol{\alpha}_\alpha}\right)}{\prod_{\alpha'\neq\alpha}\sin^2\pi(\boldsymbol{\alpha}_\alpha - \boldsymbol{\alpha}_{\alpha'})} \end{aligned} \tag{5.5.57}$$

and

$$
\begin{aligned}
B^{+}_{\alpha\beta} &= -\frac{\mathfrak{m}_0^2\mathfrak{m}_1^{-1}}{16}\,\frac{\prod_{\gamma=1}^{3}\left(\mathfrak{m}_0^{-1}e^{\pi i\boldsymbol{\alpha}_\beta}-\mathfrak{m}_{-1}^{(\gamma)}e^{-\pi i\boldsymbol{\alpha}_\beta}\right)\left(\mathfrak{m}_1 e^{\pi i\boldsymbol{\alpha}_\alpha}-\mathfrak{m}_1^{(\gamma)}e^{-\pi i\boldsymbol{\alpha}_\alpha}\right)}{\prod_{\alpha'\neq\alpha}\sin\pi(\boldsymbol{\alpha}_{\alpha'}-\boldsymbol{\alpha}_\alpha)\prod_{\beta'\neq\beta}\sin\pi(\boldsymbol{\alpha}_\beta-\boldsymbol{\alpha}_{\beta'})}\\
B^{-}_{\alpha\beta} &= -\frac{\mathfrak{m}_0\mathfrak{m}_1^{-2}}{16}\,\frac{\prod_{\gamma=1}^{3}\left(\mathfrak{m}_0^{-1}e^{\pi i\boldsymbol{\alpha}_\beta}-\mathfrak{m}_{-1}^{(\gamma)}e^{-\pi i\boldsymbol{\alpha}_\beta}\right)\left(\mathfrak{m}_1 e^{\pi i\boldsymbol{\alpha}_\alpha}-\mathfrak{m}_1^{(\gamma)}e^{-\pi i\boldsymbol{\alpha}_\alpha}\right)}{\prod_{\alpha'\neq\alpha}\sin\pi(\boldsymbol{\alpha}_{\alpha'}-\boldsymbol{\alpha}_\alpha)\prod_{\beta'\neq\beta}\sin\pi(\boldsymbol{\alpha}_\beta-\boldsymbol{\alpha}_{\beta'})}.
\end{aligned}
\tag{5.5.58}
$$

Therefore, we obtain the traces of the holonomies along the A-cycle and B-cycle on $\mathbb{P}^1\backslash\{0,\underline{\mathfrak{q}},\underline{1},\infty\}$ expressed in terms of the generalized NRS coordinates in (5.5.51) and (5.5.56). We can conversely regard these formulas as the defining equations for the generalized NRS coordinates $\{\boldsymbol{\alpha}_\alpha,\boldsymbol{\beta}_\alpha\,|\,\alpha=1,2\}$.

Remarks

- After our work has been completed and submitted to the arXiv, we were informed that the generalized Fenchel-Nielsen spectral coordinates constructed in [98] are equivalent to the ones obtained here in (5.5.51) and (5.5.56), up to some simple shifts for the $\boldsymbol{\beta}$ coordinates. Since our construction is elementary and does not use the auxiliary constructs such as the Seiberg-Witten curve disguised in the form of the spectral network, we may hope that more general spectral network constructions of Darboux coordinates could be simplified as well. In this way we expect our coordinates to match with the (generalized) Fenchel-Nielsen spectral coordinates in [104, 98] and their higher-rank analogues [103], possibly up to some simple shifts.

5.6 Monodromies and generating functions of opers

Finally, we compute the monodromies of opers to find the expressions for the generalized NRS coordinates restricted to the variety of opers. Since the variety of opers is a Lagrangian submanifold in the moduli space of flat connections and the generalized NRS coordinates form a Darboux coordinate system, there exists generating function $\mathcal{S}\left[\mathcal{O}_N[\mathbb{P}^1_{2,\underline{r+1}}]\right]$ for the

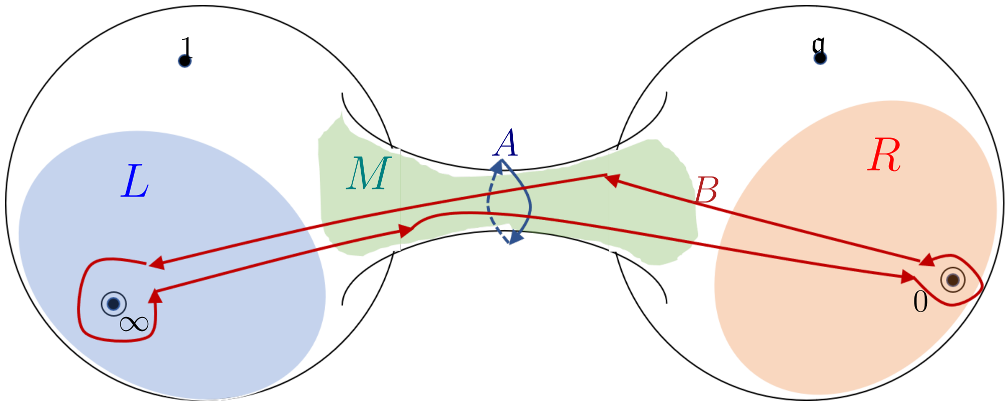

Figure 5.5: The A-cycle and the B-cycle on the four-punctured sphere $\mathbb{P}^1\backslash\{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$. The double circles represent the maximal punctures at 0 and ∞, while the simple dots represent the minimal punctures at $\mathfrak{q}$ and 1. The shaded regions represent the convergence domains L, M, and R, respectively. The A-cycle is represented by the dark blue line, while the B-cycle is represented by the dark red line.

variety $\mathcal{O}_N[\mathbb{P}^1_{2,\underline{r+1}}]$ of opers with respect to the generalized NRS coordinates:

$$\boldsymbol{\beta}_i^{(\alpha)} = \frac{\partial \mathcal{S}\left[\mathcal{O}_N[\mathbb{P}^1_{2,\underline{r+1}}]\right]}{\partial \boldsymbol{\alpha}_i^{(\alpha)}}, \quad i = 0, 1, \cdots r-1, \; \alpha = 1, \cdots, N-1. \tag{5.6.1}$$

We verify that the generating function for the variety of opers is identified with the effective twisted superpotential of the corresponding class $\mathcal{S}$ theory $\mathcal{T}\left[A_{N-1}, \mathbb{P}^1_{2,\underline{r+1}}\right]$, for the example of the four-punctured sphere $\mathbb{P}^1\backslash\{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$.

The strategy to compute the monodromies of the oper $\widehat{\mathfrak{D}}_N$ is to study the holonomy of the operator $\hat{\widehat{\mathfrak{D}}}_N$, and then take the limit $\varepsilon_2 \to 0$. The monodromy along the A-cycle is easy to compute: as noted in the section 5.4, the solution $\widetilde{\mathfrak{Z}}^{L\to M}$ is defined in the domain $0 < |\mathfrak{q}| < |z| < 1$ (it is easy to estimate the growth of coefficients of z-expansion to conclude it converges there). Thus we simply continue along the path

$$z \longrightarrow z\, e^{it} \quad \text{with} \quad 0 \leq t \leq 2\pi, \tag{5.6.2}$$

to enclose the punctures at 0 and $\mathfrak{q}$, thereby making the A-cycle. In this fashion we pick up the multiplicative factors from the non-integral part of the exponent of z in the perturbative prefactor, and thereby obtain the holonomy $\mathbf{M}_A(\widehat{\widehat{\mathfrak{D}}})$. The monodromy of the oper $\widehat{\mathfrak{D}}$ is then computed by taking the limit:

$$M_A(\widehat{\mathfrak{D}}) = \lim_{\varepsilon_2 \to 0} \mathbf{M}_A(\widehat{\widehat{\mathfrak{D}}}). \tag{5.6.3}$$

The monodromy along the B-cycle is more involved. First we need *the rotation matrices* $\mathbf{R}_0$ and $\mathbf{R}_\infty$ which are the monodromy matrices for the 2π-rotations around the punctures at 0 and ∞. As noted in section 5.3, the solutions $\widetilde{\mathfrak{Z}}^L$ and $\widetilde{\mathfrak{Z}}^R$ are defined by gauge theory as series expansions in the domains $0 < |\mathfrak{q}| < 1 < |z|$ and $0 < |z| < |\mathfrak{q}| < 1$, respectively. Thus we get $\mathbf{R}_0$ by following $\widetilde{\mathfrak{Z}}^R$ along the circle $z \mapsto z\, e^{2\pi i}$, and we get $\mathbf{R}_\infty$ by following $\widetilde{\mathfrak{Z}}^L$ along the circle $z \mapsto z\, e^{-2\pi i}$. For completeness, we give the expressions for the rotation matrices for the punctures at $\mathfrak{q}$ and 1 also:

$$\begin{aligned} \mathbf{R}_{\mathfrak{q}} &= \mathbf{M}_A\, \mathbf{C}_0^{-1}\, \mathbf{R}_0^{-1}\, \mathbf{C}_0 \\ \mathbf{R}_1 &= \mathbf{M}_A^{-1}\, \mathbf{C}_\infty^{-1}\, \mathbf{R}_\infty^{-1}\, \mathbf{C}_\infty. \end{aligned} \tag{5.6.4}$$

It is immediate to see, in the $N = 3$ case for example, that the eigenvalues of $\lim_{\varepsilon_2 \to 0} \mathbf{R}_{\mathfrak{q}}$ and $\lim_{\varepsilon_2 \to 0} \mathbf{R}_1$ are maximally degenerate, which means they correspond to minimal punctures. Now for the B-monodromy matrix, we start from the solution $\widetilde{\mathfrak{Z}}^L$. By concatenating the connection matrices, the shift matrices, and the rotation matrices, we construct the following sequence of continuations of the solutions

$$\widetilde{\mathfrak{Z}}^L \xrightarrow{\mathbf{C}_\infty} \widetilde{\mathfrak{Z}}^{L\to M} \xrightarrow{\mathbf{S}} \widetilde{\mathfrak{Z}}^{R\to M} \xrightarrow{\mathbf{C}_0^{-1}} \widetilde{\mathfrak{Z}}^R \xrightarrow{\mathbf{R}_0^{-1}} \widetilde{\mathfrak{Z}}^R \xrightarrow{\mathbf{C}_0} \widetilde{\mathfrak{Z}}^{R\to M} \xrightarrow{\mathbf{S}^{-1}} \widetilde{\mathfrak{Z}}^{L\to M} \xrightarrow{\mathbf{C}_\infty^{-1}} \widetilde{\mathfrak{Z}}^L \xrightarrow{\mathbf{R}_\infty^{-1}} \widetilde{\mathfrak{Z}}^L. \tag{5.6.5}$$

Hence the corresponding holonomy is

$$\mathbf{M}_B(\widehat{\widehat{\mathfrak{D}}}) = \mathbf{R}_\infty\, \mathbf{C}_\infty\, \mathbf{S}\, \mathbf{C}_0^{-1}\, \mathbf{R}_0\, \mathbf{C}_0\, \mathbf{S}^{-1}\, \mathbf{C}_\infty^{-1}. \tag{5.6.6}$$

We have seen in section 5.3.1 that under the Nekrasov-Shatashvili limit, the solutions $\widetilde{\mathfrak{Z}}^L$ for $\widehat{\widehat{\mathfrak{D}}}$ behaves as

$$\widetilde{\mathfrak{Z}}^L = e^{\frac{\widetilde{w}}{\varepsilon_2}}\ (\boldsymbol{\chi} + \mathcal{O}(\varepsilon_2))\,, \tag{5.6.7}$$

which leads to the equation for the oper, $\widehat{\mathfrak{D}}\boldsymbol{\chi} = 0$. Therefore, we compute the B-monodromy for the oper $\widehat{\mathfrak{D}}$ as

$$\begin{aligned} M_B(\widehat{\mathfrak{D}}) &= \lim_{\varepsilon_2\to 0} \mathbf{M}_B(\widehat{\widehat{\mathfrak{D}}})\ e^{\frac{\widetilde{w}}{\varepsilon_2}} \\ &= \lim_{\varepsilon_2\to 0} \mathbf{R}_\infty\ \mathbf{C}_\infty\ \mathbf{S}\ \mathbf{C}_0^{-1}\ \mathbf{R}_0\ \mathbf{C}_0\ \mathbf{S}^{-1}\ \mathbf{C}_\infty^{-1}\ e^{\frac{\widetilde{w}}{\varepsilon_2}}. \end{aligned} \tag{5.6.8}$$

Now we exhibit in detail how these computations can actually be done.

5.6.1 $SL(2)$-oper

We obtain the A-monodromy matrix for $\widehat{\widehat{\mathfrak{D}}}_2$ by letting z make the full circle $z \mapsto z\ e^{2\pi i}$ in the expression for $\widetilde{\mathfrak{Z}}^{L\to M}$. Since the critical exponent for z is given as (5.4.41), we find that

$$\mathbf{M}_A(\widehat{\widehat{\mathfrak{D}}}_2) = \mathrm{diag}\left(e^{2\pi i \frac{-a_1+a_2+\varepsilon}{2\varepsilon_1}},\ e^{2\pi i \frac{a_1-a_2+\varepsilon}{2\varepsilon_1}}\right). \tag{5.6.9}$$

Then by taking the Nekrasov-Shatashvili limit ($\varepsilon_1 \neq 0, \varepsilon_2 \to 0$), we obtain the A-monodromy matrix for the oper $\widehat{\mathfrak{D}}_2$,

$$\begin{aligned} M_A(\widehat{\mathfrak{D}}_2) &= \lim_{\varepsilon_2\to 0} \mathbf{M}_A(\widehat{\widehat{\mathfrak{D}}}_2) \\ &= \mathrm{diag}\left(e^{2\pi i \frac{-a_1+a_2+\varepsilon_1}{2\varepsilon_1}},\ e^{2\pi i \frac{a_1-a_2+\varepsilon_1}{2\varepsilon_1}}\right), \end{aligned} \tag{5.6.10}$$

so that

$$\mathrm{Tr}\ M_A(\widehat{\mathfrak{D}}_2) = 2\cos 2\pi \left(\frac{a_1 - a_2 + \varepsilon_1}{2\varepsilon_1}\right). \tag{5.6.11}$$

Comparing this with (5.5.38), we obtain

$$\boldsymbol{\alpha} = \frac{a_1 - a_2 + \varepsilon_1}{2\varepsilon_1}. \tag{5.6.12}$$

Next, we find the expression for the $\boldsymbol{\beta}$ coordinate by computing the B-monodromy matrix. It is necessary to compute the rotation matrices first, by shifting $z \mapsto z\, e^{-2\pi i}$ and $z \mapsto z\, e^{2\pi i}$ for $\widetilde{\mathfrak{Z}}^L$ and $\widetilde{\mathfrak{Z}}^R$, respectively. Since their critical exponents are given as (5.4.39) and (5.4.49), we immediately compute

$$\begin{aligned}
\mathbf{R}_\infty &= \mathrm{diag}\left(e^{\pi i \frac{a_{0,1}-a_{0,2}-\varepsilon_1-2\varepsilon_2}{\varepsilon_1}},\ e^{\pi i \frac{-a_{0,1}+a_{0,2}-\varepsilon_1-2\varepsilon_2}{\varepsilon_1}}\right),\\
\mathbf{R}_0 &= \mathrm{diag}\left(e^{\pi i \frac{-a_{3,1}+a_{3,2}+\varepsilon}{\varepsilon_1}},\ e^{\pi i \frac{a_{3,1}-a_{3,2}+\varepsilon}{\varepsilon_1}}\right).
\end{aligned} \tag{5.6.13}$$

The connection matrices and the shift matrices are given by (5.4.35) and (5.4.64). For $N = 2$, it is easier to write these matrices explicitly. In particular, the connection matrices are

$$\begin{aligned}
\mathbf{C}_\infty &= \begin{bmatrix} \frac{\Gamma\left(1+\frac{a_{0,1}-a_{0,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,1}-a_2}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_{0,2}}{\varepsilon_1}\right)} & \frac{\Gamma\left(1+\frac{a_{0,1}-a_{0,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,1}-a_1}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_{0,2}}{\varepsilon_1}\right)} \\ \frac{\Gamma\left(1+\frac{a_{0,2}-a_{0,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,2}-a_2}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_{0,1}}{\varepsilon_1}\right)} & \frac{\Gamma\left(1+\frac{a_{0,2}-a_{0,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,2}-a_1}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_{0,1}}{\varepsilon_1}\right)} \end{bmatrix},\\
\mathbf{C}_0 &= \begin{bmatrix} \frac{\Gamma\left(1+\frac{a_{3,2}-a_{3,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_2-a_{3,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,2}-a_1}{\varepsilon_1}\right)} & \frac{\Gamma\left(1+\frac{a_{3,2}-a_{3,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_1-a_{3,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,2}-a_2}{\varepsilon_1}\right)} \\ \frac{\Gamma\left(1+\frac{a_{3,1}-a_{3,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_2-a_{3,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,1}-a_1}{\varepsilon_1}\right)} & \frac{\Gamma\left(1+\frac{a_{3,1}-a_{3,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_1-a_{3,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,1}-a_2}{\varepsilon_1}\right)} \end{bmatrix}.
\end{aligned} \tag{5.6.14}$$

Their inverses can also be computed directly as

$$\begin{aligned}
\mathbf{C}_\infty^{-1} &= \frac{a_1 - a_2}{a_{0,1} - a_{0,2}} \begin{bmatrix} \frac{\Gamma\left(1+\frac{a_{0,2}-a_{0,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,2}-a_1}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_{0,1}}{\varepsilon_1}\right)} & -\frac{\Gamma\left(1+\frac{a_{0,1}-a_{0,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,1}-a_1}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_{0,2}}{\varepsilon_1}\right)} \\ -\frac{\Gamma\left(1+\frac{a_{0,2}-a_{0,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,2}-a_2}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_{0,1}}{\varepsilon_1}\right)} & \frac{\Gamma\left(1+\frac{a_{0,1}-a_{0,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_{0,1}-a_2}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_{0,2}}{\varepsilon_1}\right)} \end{bmatrix}, \\
\mathbf{C}_0^{-1} &= \frac{a_1 - a_2}{a_{3,1} - a_{3,2}} \begin{bmatrix} \frac{\Gamma\left(1+\frac{a_{3,1}-a_{3,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_1-a_{3,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,1}-a_2}{\varepsilon_1}\right)} & -\frac{\Gamma\left(1+\frac{a_{3,2}-a_{3,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_1-a_{3,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,2}-a_2}{\varepsilon_1}\right)} \\ -\frac{\Gamma\left(1+\frac{a_{3,1}-a_{3,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_2-a_{3,2}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,1}-a_1}{\varepsilon_1}\right)} & \frac{\Gamma\left(1+\frac{a_{3,2}-a_{3,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)}{\Gamma\left(1+\frac{a_2-a_{3,1}}{\varepsilon_1}\right)\Gamma\left(\frac{a_{3,2}-a_1}{\varepsilon_1}\right)} \end{bmatrix}.
\end{aligned} \tag{5.6.15}$$

Now it is straightforward to evaluate the B-monodromy matrix for $\widehat{\widehat{\mathfrak{D}}}_2$ by

$$\mathbf{M}_B(\widehat{\widehat{\mathfrak{D}}}_2) = \mathbf{R}_\infty\ \mathbf{C}_\infty\ \mathbf{S}\ \mathbf{C}_0^{-1}\ \mathbf{R}_0\ \mathbf{C}_0\ \mathbf{S}^{-1}\ \mathbf{C}_\infty^{-1}. \tag{5.6.16}$$

We need to evaluate the trace of the B-monodromy matrix for the $SL(2)$-oper $\widehat{\mathfrak{D}}_2$, which is obtained by taking the limit,

$$\mathrm{Tr}\ M_B(\widehat{\mathfrak{D}}_2) = \mathrm{Tr}\left(\lim_{\varepsilon_2\to 0} \mathbf{R}_\infty\ \mathbf{C}_\infty\ \mathbf{S}\ \mathbf{C}_0^{-1}\ \mathbf{R}_0\ \mathbf{C}_0\ \mathbf{S}^{-1}\ \mathbf{C}_\infty^{-1}\ e^{\frac{\widetilde{\mathcal{W}}}{\varepsilon_2}}\right). \tag{5.6.17}$$

A few pages of computation shows that

$$
\begin{aligned}
&\operatorname{Tr} M_B(\widehat{\mathfrak{D}}_2) \\
&= \frac{\left(\cos\pi\frac{a_{3,1}-a_{3,2}}{\varepsilon_1} - \cos\pi\frac{2(\bar{a}_3-\bar{a})}{\varepsilon_1}\right)\left(\cos\pi\frac{a_{0,1}-a_{0,2}}{\varepsilon_1} - \cos\pi\frac{2(\bar{a}_0-\bar{a})}{\varepsilon_1}\right)}{2\sin^2\pi\frac{a_1-a_2}{2\varepsilon_1}} \\
&+ \frac{\left(\cos\pi\frac{a_{3,1}-a_{3,2}}{\varepsilon_1} + \cos\pi\frac{2(\bar{a}_3-\bar{a})}{\varepsilon_1}\right)\left(\cos\pi\frac{a_{0,1}-a_{0,2}}{\varepsilon_1} + \cos\pi\frac{2(\bar{a}_0-\bar{a})}{\varepsilon_1}\right)}{2\cos^2\pi\frac{a_1-a_2}{2\varepsilon_1}} \\
&- 4\frac{\prod_{\gamma=1,2}\sin\pi\frac{a_2-a_{0,\gamma}}{\varepsilon_1}\ \sin\pi\frac{a_{3,\gamma}-a_1}{\varepsilon_1}}{\sin^2\pi\frac{a_1-a_2}{\varepsilon_1}} \frac{\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)^2}{\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)^2}\prod_{\gamma=1,2}\frac{\Gamma\left(\frac{a_{3,\gamma}-a_1}{\varepsilon_1}\right)\ \Gamma\left(\frac{a_2-a_{0,\gamma}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_1-a_{0,\gamma}}{\varepsilon_1}\right)\ \Gamma\left(\frac{a_{3,\gamma}-a_2}{\varepsilon_1}\right)} e^{\left(\frac{\partial}{\partial a_1}-\frac{\partial}{\partial a_2}\right)\widetilde{\mathcal{W}}} \\
&- 4\frac{\prod_{\gamma=1,2}\sin\pi\frac{a_1-a_{0,\gamma}}{\varepsilon_1}\ \sin\pi\frac{a_{3,\gamma}-a_2}{\varepsilon_1}}{\sin^2\pi\frac{a_1-a_2}{\varepsilon_1}} \left(\frac{\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)^2}{\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)^2}\prod_{\gamma=1,2}\frac{\Gamma\left(\frac{a_{3,\gamma}-a_1}{\varepsilon_1}\right)\ \Gamma\left(\frac{a_2-a_{0,\gamma}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_1-a_{0,\gamma}}{\varepsilon_1}\right)\ \Gamma\left(\frac{a_{3,\gamma}-a_2}{\varepsilon_1}\right)}\right)^{-1} e^{-\left(\frac{\partial}{\partial a_1}-\frac{\partial}{\partial a_2}\right)\widetilde{\mathcal{W}}}
\end{aligned}
\tag{5.6.18}
$$

It is crucial to note that the products of Γ-functions in the third and the fourth lines can be absorbed as the 1-loop part of the effective twisted superpotential of the A_1-theory computed under the ζ-function regularization (see (2.1.37) and its derivation above), namely,

$$
\left(\frac{\partial}{\partial a_1}-\frac{\partial}{\partial a_2}\right)\widetilde{\mathcal{W}}^{\text{1-loop}} = \log\ \frac{\Gamma\left(\frac{a_1-a_2}{\varepsilon_1}\right)^2}{\Gamma\left(\frac{a_2-a_1}{\varepsilon_1}\right)^2}\prod_{\gamma=1,2}\frac{\Gamma\left(\frac{a_{3,\gamma}-a_1}{\varepsilon_1}\right)\ \Gamma\left(\frac{a_2-a_{0,\gamma}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_1-a_{0,\gamma}}{\varepsilon_1}\right)\ \Gamma\left(\frac{a_{3,\gamma}-a_2}{\varepsilon_1}\right)}. \tag{5.6.19}
$$

Hence we define the full effective twisted superpotential by

$$
\widetilde{\mathcal{W}}^{\text{full}} \equiv \widetilde{\mathcal{W}}^{\text{classical}} + \widetilde{\mathcal{W}}^{\text{1-loop}} + \widetilde{\mathcal{W}}^{\text{inst}} + \widetilde{\mathcal{W}}^{\text{extra}}, \tag{5.6.20}
$$

where the 1-loop part is given in (2.1.33) and the other parts have been obtained in (5.3.20),

$$\widetilde{\mathcal{W}}^{\text{classical}} = -\frac{(a_1 - a_2)^2}{4\varepsilon_1} \log \mathfrak{q} \tag{5.6.21a}$$

$$\widetilde{\mathcal{W}}^{\text{1-loop}} = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \frac{\prod_{\alpha,\beta=1}^{2} \Gamma_2(a_\alpha - a_\beta; \varepsilon_1, \varepsilon_2)}{\prod_{\alpha,\beta=1}^{2} \Gamma_2(a_\alpha - a_{0,\beta}; \varepsilon_1, \varepsilon_2)\Gamma_2(a_{3,\alpha} - a_\beta; \varepsilon_1, \varepsilon_2)} \tag{5.6.21b}$$

$$\widetilde{\mathcal{W}}^{\text{inst}} = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \mathcal{Z}^{\text{inst}}_{A_1} \tag{5.6.21c}$$

$$\widetilde{\mathcal{W}}^{\text{extra}} = \varepsilon_1 \left(\frac{1}{4} - \delta_2 - \delta_3 \right) \log \mathfrak{q} + \frac{2(\bar{a}_0 - \bar{a})(\bar{a} - \bar{a}_3 + \varepsilon_1)}{\varepsilon_1} \log(1 - \mathfrak{q}). \tag{5.6.21d}$$

Thus the expression for the trace of the B-monodromy matrix is simplified with the full effective twisted superpotential $\widetilde{\mathcal{W}}^{\text{full}}$. Let us also make an overall shift for the Coulomb moduli and the masses of the hypermultiplets to recover the $SU(2)$ parameters (see section 5.2). The final form of the expression is

$$\begin{aligned}
\text{Tr}\, M_B(\widehat{\mathfrak{D}}_2) = & \frac{\left(\cos \pi \frac{a_{3,1}-a_{3,2}}{\varepsilon_1} - \cos \pi \frac{2\bar{a}_3}{\varepsilon_1}\right)\left(\cos \pi \frac{a_{0,1}-a_{0,2}}{\varepsilon_1} - \cos \pi \frac{2\bar{a}_0}{\varepsilon_1}\right)}{2 \cos^2 \pi \boldsymbol{\alpha}} \\
& + \frac{\left(\cos \pi \frac{a_{3,1}-a_{3,2}}{\varepsilon_1} + \cos \pi \frac{2\bar{a}_3}{\varepsilon_1}\right)\left(\cos \pi \frac{a_{0,1}-a_{0,2}}{\varepsilon_1} + \cos \pi \frac{2\bar{a}_0}{\varepsilon_1}\right)}{2 \sin^2 \pi \boldsymbol{\alpha}} \\
& - \sum_{\pm} 4\, \frac{\prod_{\gamma=1,2} \cos \pi \left(\mp \boldsymbol{\alpha} - \frac{a_{0,\gamma}}{\varepsilon_1} \right) \cos \pi \left(\frac{a_{3,\gamma}}{\varepsilon_1} \mp \boldsymbol{\alpha} \right)}{\sin^2 2\pi \boldsymbol{\alpha}} e^{\pm \frac{1}{\varepsilon_1} \frac{\partial \widetilde{\mathcal{W}}^{\text{full}}}{\partial \boldsymbol{\alpha}}}.
\end{aligned} \tag{5.6.22}$$

This expression exactly matches with (5.5.42) under the identification of parameters,

$$\mathfrak{m}_{-1} = e^{\pi i \frac{a_{0,1}-a_{0,2}-\varepsilon_1}{\varepsilon_1}}, \quad \mathfrak{m}_0 = e^{\pi i \frac{a_{0,1}+a_{0,2}}{\varepsilon_1}}, \quad \mathfrak{m}_1 = e^{-\pi i \frac{a_{3,1}+a_{3,2}}{\varepsilon_1}}, \quad \mathfrak{m}_2 = e^{\pi i \frac{-a_{3,1}+a_{3,2}+\varepsilon_1}{\varepsilon_1}}. \tag{5.6.23}$$

Most importantly, we observe that

$$\boldsymbol{\beta} = \frac{1}{\varepsilon_1} \frac{\partial \widetilde{\mathcal{W}}^{\text{full}}}{\partial \boldsymbol{\alpha}}. \tag{5.6.24}$$

Consequently, the generating function for the variety $\mathcal{O}_2[\mathbb{P}^1 \backslash \{0, \mathfrak{q}, 1, \infty\}]$ of opers is identified with the effective twisted superpotential, namely,

$$\mathcal{S}\left[\mathcal{O}_2[\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}]\right] = \frac{1}{\varepsilon_1}\widetilde{\mathcal{W}}^{\text{full}}\left[\mathcal{T}[A_1,\mathbb{P}^1\backslash\{0,\mathfrak{q},1,\infty\}]\right], \tag{5.6.25}$$

by the relation (5.6.24). This identification verifies the main assertion of [91] for all orders in the gauge coupling $\mathfrak{q}$.

Remarks

- The equivalence (5.6.25) involves an extra term in the effective twisted superpotential, $\widetilde{\mathcal{W}}^{\text{extra}}$. Note that $\widetilde{\mathcal{W}}^{\text{extra}}$ has been completely determined in gauge theoretical terms in (5.6.21d).

- The regularization scheme that has been used to define the 1-loop part $\widetilde{\mathcal{W}}^{\text{1-loop}}$ was the ζ-function regularization, which is natural in the gauge theory context. Note that it is free to choose other schemes to regularize the infinite product, or the IR divergence, in the 1-loop contribution. The other choices would lead to the correction in the effective twisted superpotential of the form,

$$\widetilde{\mathcal{W}}_{\infty} \sim \text{Li}_2\, e^{l(\mathbf{a},\mathbf{a}_0,\mathbf{a}_3)}, \tag{5.6.26}$$

  where l is some linear function of the arguments. Physically, IR regulator corresponds to cutting the cigar $\mathcal{D}^2$ (5.1.16) at the infinity. Thus the correction $\widetilde{\mathcal{W}}_{\infty}$ to the effective twisted superpotential can be interpreted as the contribution from a three-dimensional theory coupled to the four-dimensional bulk theory at the boundary at infinity. Note that $\widetilde{\mathcal{W}}_{\infty}$ is independent of the coupling $\mathfrak{q}$. Hence this correction to the effective twisted superpotential corresponds to a canonical coordinate transformation.

5.6.2 $SL(3)$-oper

Using the critical exponent (5.4.57) of z, we see that under the $z \mapsto z\, e^{2\pi i}$ loop the partition function $\widetilde{\mathfrak{Z}}^{L\to M}$ transforms by

$$\mathbf{M}_A(\hat{\hat{\mathfrak{D}}}_3) = \mathrm{diag}\left(e^{2\pi i\, \frac{-2a_1+a_2+a_3+3\varepsilon}{3\varepsilon_1}},\ e^{2\pi i\, \frac{a_1-2a_2+a_3+3\varepsilon}{3\varepsilon_1}},\ e^{2\pi i\, \frac{a_1+a_2-2a_3+3\varepsilon}{3\varepsilon_1}}\right). \tag{5.6.27}$$

Hence, by taking the limit $\varepsilon_2 \to 0$, we obtain

$$\begin{aligned} M_A(\widehat{\mathfrak{D}}_3) &= \lim_{\varepsilon_2\to 0} \mathbf{M}_A(\hat{\hat{\mathfrak{D}}}_3) \\ &= \mathrm{diag}\left(e^{2\pi i\, \frac{-2a_1+a_2+a_3}{3\varepsilon_1}},\ e^{2\pi i\, \frac{a_1-2a_2+a_3}{3\varepsilon_1}},\ e^{2\pi i\, \frac{a_1+a_2-2a_3}{3\varepsilon_1}}\right). \end{aligned} \tag{5.6.28}$$

Comparing with (5.5.51), we find

$$\boldsymbol{\alpha}_\alpha = \frac{3a_\alpha - \sum_{\gamma=1}^{3} a_\gamma}{3\varepsilon_1}, \quad \alpha = 1, 2, \tag{5.6.29}$$

so that we have

$$\mathrm{Tr}\, M_A(\widehat{\mathfrak{D}}_3)^{\pm 1} = e^{\mp 2\pi i \boldsymbol{\alpha}_1} + e^{\mp 2\pi i \boldsymbol{\alpha}_2} + e^{\pm 2\pi i (\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2)}. \tag{5.6.30}$$

For notational convenience, let us also define $\boldsymbol{\alpha}_3 \equiv -\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2$ as before.

Next, we obtain the expression for the $\boldsymbol{\beta}$ coordinates restricted to the variety of opers by evaluating the B-monodromy matrix. First, we compute the rotation matrices by shifting $z \mapsto z\ e^{-2\pi i}$ and $z \mapsto z\ e^{2\pi i}$ for $\widetilde{\mathfrak{Z}}^{L}$ and $\widetilde{\mathfrak{Z}}^{R}$, respectively. From their critical exponents (5.3.34) and (5.3.51) we get

$$\begin{aligned} \mathbf{R}_\infty &= \mathrm{diag}\left(e^{2\pi i \frac{2a_{0,1}-a_{0,2}-a_{0,3}-3\varepsilon-2\varepsilon_2}{3\varepsilon_1}},\ e^{2\pi i \frac{-a_{0,1}+2a_{0,2}-a_{0,3}-3\varepsilon-2\varepsilon_2}{3\varepsilon_1}},\ e^{2\pi i \frac{-a_{0,1}-a_{0,2}+2a_{0,3}-3\varepsilon-2\varepsilon_2}{3\varepsilon_1}}\right), \\ \mathbf{R}_0 &= \mathrm{diag}\left(e^{2\pi i \frac{-2a_{3,1}+a_{3,2}+a_{3,3}+3\varepsilon}{3\varepsilon_1}},\ e^{2\pi i \frac{a_{3,1}-2a_{3,2}+a_{3,3}+3\varepsilon}{3\varepsilon_1}},\ e^{2\pi i \frac{a_{3,1}+a_{3,2}-2a_{3,3}+3\varepsilon}{3\varepsilon_1}}\right). \end{aligned} \tag{5.6.31}$$

Then the B-monodromy matrix for $\hat{\hat{\mathfrak{D}}}_3$ is obtained by (5.6.6)

$$\mathbf{M}_B(\hat{\hat{\mathfrak{D}}}_3) = \mathbf{R}_\infty \, \mathbf{C}_\infty \, \mathbf{S} \, \mathbf{C}_0^{-1} \, \mathbf{R}_0 \, \mathbf{C}_0 \, \mathbf{S}^{-1} \, \mathbf{C}_\infty^{-1}, \tag{5.6.32}$$

where the connection matrices and the shift matrices are given by (5.4.35) and (5.4.64). The B-monodromy matrix for the oper $\widehat{\mathfrak{D}}_3$ is obtained by taking the limit,

$$M_B(\widehat{\mathfrak{D}}_3) = \lim_{\varepsilon_2 \to 0} \mathbf{R}_\infty \, \mathbf{C}_\infty \, \mathbf{S} \, \mathbf{C}_0^{-1} \, \mathbf{R}_0 \, \mathbf{C}_0 \, \mathbf{S}^{-1} \, \mathbf{C}_\infty^{-1} \, e^{\frac{\widetilde{\mathrm{w}}}{\varepsilon_2}}. \tag{5.6.33}$$

To determine the $\boldsymbol{\beta}$ coordinates, we need to compute the following traces of $M_B(\widehat{\mathfrak{D}}_3)$

$$\begin{aligned} \mathrm{Tr}\, M_B(\widehat{\mathfrak{D}}_3) &= \mathrm{Tr}\left(\lim_{\varepsilon_2 \to 0} \mathbf{R}_\infty \, \mathbf{C}_\infty \, \mathbf{S} \, \mathbf{C}_0^{-1} \, \mathbf{R}_0 \, \mathbf{C}_0 \, \mathbf{S}^{-1} \, \mathbf{C}_\infty^{-1} \, e^{\frac{\widetilde{\mathrm{w}}}{\varepsilon_2}}\right) \\ \mathrm{Tr}\, M_B(\widehat{\mathfrak{D}}_3)^{-1} &= \mathrm{Tr}\left(\lim_{\varepsilon_2 \to 0} \mathbf{C}_\infty \, \mathbf{S} \, \mathbf{C}_0^{-1} \, \mathbf{R}_0^{-1} \, \mathbf{C}_0 \, \mathbf{S}^{-1} \, \mathbf{C}_\infty^{-1} \, \mathbf{R}_\infty^{-1} \, e^{\frac{\widetilde{\mathrm{w}}}{\varepsilon_2}}\right). \end{aligned} \tag{5.6.34}$$

The computation of these traces can be broken in several steps. First note that

$$\mathrm{Tr}\, M_B(\widehat{\mathfrak{D}}_3) = \mathrm{Tr}\left(\lim_{\varepsilon_2 \to 0} (\mathbf{C}_\infty^{-1} \, \mathbf{R}_\infty \, \mathbf{C}_\infty) \, \mathbf{S} \, (\mathbf{C}_0^{-1} \, \mathbf{R}_0 \, \mathbf{C}_0) \, \mathbf{S}^{-1} \, e^{\frac{\widetilde{\mathrm{w}}}{\varepsilon_2}}\right), \tag{5.6.35}$$

due to the limit $\varepsilon_2 \to 0$. Then

$$\mathrm{Tr}\, M_B(\widehat{\mathfrak{D}}_3) = \sum_{\alpha,\beta=1}^{3} (\boldsymbol{\mathcal{C}}_\infty)_{\beta\alpha} \, (\boldsymbol{\mathcal{C}}_0)_{\alpha\beta} \, e^{\left(\frac{\partial}{\partial a_\alpha} - \frac{\partial}{\partial a_\beta}\right)\widetilde{\mathrm{w}}}, \tag{5.6.36}$$

where we have defined

$$\begin{aligned} &(\boldsymbol{\mathcal{C}}_\infty)_{\alpha\beta} \equiv \left(\lim_{\varepsilon_2 \to 0} \mathbf{C}_\infty^{-1} \, \mathbf{R}_\infty \, \mathbf{C}_\infty\right)_{\alpha\beta} \\ &= e^{\frac{i\pi}{\varepsilon_1}(a_\alpha + a_\beta - 2\bar{a}_0)} \\ &\left(\delta_{\alpha,\beta} - 2i \, e^{\frac{3\pi i}{\varepsilon_1}(\bar{a}_0 - \bar{a})} \frac{\prod_{\beta' \neq \beta} \Gamma\left(\frac{a_\beta - a_{\beta'}}{\varepsilon_1}\right)}{\prod_{\alpha' \neq \alpha} \Gamma\left(\frac{a_\alpha - a_{\alpha'}}{\varepsilon_1}\right) \, \sin \pi \frac{a_\alpha - a_{\alpha'}}{\varepsilon_1}} \, \prod_{\alpha'=1}^{3} \frac{\Gamma\left(\frac{a_\alpha - a_{0,\alpha'}}{\varepsilon_1}\right) \, \sin \pi \frac{a_\alpha - a_{0,\alpha'}}{\varepsilon_1}}{\Gamma\left(\frac{a_\beta - a_{0,\alpha'}}{\varepsilon_1}\right)}\right), \end{aligned} \tag{5.6.37}$$

and

$$(\boldsymbol{\mathcal{C}}_0)_{\alpha\beta} \equiv \left(\lim_{\varepsilon_2\to 0} \mathbf{C}_0^{-1}\, \mathbf{R}_0\, \mathbf{C}_0\right)_{\alpha\beta}$$
$$= e^{\frac{i\pi}{\varepsilon_1}(-a_\alpha - a_\beta + 2\bar{a}_3)}$$
$$\left(\delta_{\alpha,\beta} - 2i\, e^{\frac{3\pi i}{\varepsilon_1}(-\bar{a}_3+\bar{a})} \frac{\prod_{\beta'\neq\beta}\Gamma\left(\frac{a_{\beta'}-a_\beta}{\varepsilon_1}\right)}{\prod_{\alpha'\neq\alpha}\Gamma\left(\frac{a_{\alpha'}-a_\alpha}{\varepsilon_1}\right)\,\sin\pi\frac{a_{\alpha'}-a_\alpha}{\varepsilon_1}} \prod_{\alpha'=1}^{3} \frac{\Gamma\left(\frac{a_{3,\alpha'}-a_\alpha}{\varepsilon_1}\right)\,\sin\pi\frac{a_{3,\alpha'}-a_\alpha}{\varepsilon_1}}{\Gamma\left(\frac{a_{3,\alpha'}-a_\beta}{\varepsilon_1}\right)}\right). \tag{5.6.38}$$

Similarly, we can write

$$\begin{aligned} \mathrm{Tr}\, M_B(\widehat{\mathfrak{D}}_3)^{-1} &= \mathrm{Tr}\left(\lim_{\varepsilon_2\to 0} \mathbf{S}\, (\mathbf{C}_0^{-1}\, \mathbf{R}_0^{-1}\, \mathbf{C}_0)\, \mathbf{S}^{-1}\, (\mathbf{C}_\infty^{-1}\, \mathbf{R}_\infty^{-1}\, \mathbf{C}_\infty)\, e^{\frac{\widetilde{\mathcal{W}}}{\varepsilon_2}}\right), \\ &= \sum_{\alpha,\beta=1}^{3} (\boldsymbol{\mathcal{C}}_0^{-1})_{\alpha\beta}\, (\boldsymbol{\mathcal{C}}_\infty^{-1})_{\beta\alpha}\, e^{\left(\frac{\partial}{\partial a_\alpha} - \frac{\partial}{\partial a_\beta}\right)\widetilde{\mathcal{W}}}, \end{aligned} \tag{5.6.39}$$

where we have used

$$(\boldsymbol{\mathcal{C}}_\infty^{-1})_{\alpha\beta} = \left(\lim_{\varepsilon_2\to 0} \mathbf{C}_\infty^{-1}\, \mathbf{R}_\infty^{-1}\, \mathbf{C}_\infty\right)_{\alpha\beta}$$
$$= e^{\frac{i\pi}{\varepsilon_1}(-a_\alpha - a_\beta + 2\bar{a}_0)}$$
$$\left(\delta_{\alpha,\beta} + 2i\, e^{\frac{3\pi i}{\varepsilon_1}(-\bar{a}_0+\bar{a})} \frac{\prod_{\beta'\neq\beta}\Gamma\left(\frac{a_\beta-a_{\beta'}}{\varepsilon_1}\right)}{\prod_{\alpha'\neq\alpha}\Gamma\left(\frac{a_\alpha-a_{\alpha'}}{\varepsilon_1}\right)\,\sin\pi\frac{a_\alpha-a_{\alpha'}}{\varepsilon_1}} \prod_{\alpha'=1}^{3} \frac{\Gamma\left(\frac{a_\alpha-a_{0,\alpha'}}{\varepsilon_1}\right)\,\sin\pi\frac{a_\alpha-a_{0,\alpha'}}{\varepsilon_1}}{\Gamma\left(\frac{a_\beta-a_{0,\alpha'}}{\varepsilon_1}\right)}\right), \tag{5.6.40}$$

and

$$\begin{aligned} (\boldsymbol{\mathcal{C}}_0^{-1})_{\alpha\beta} &= \left(\lim_{\varepsilon_2\to 0} \mathbf{C}_0^{-1}\, \mathbf{R}_0^{-1}\, \mathbf{C}_0\right)_{\alpha\beta} \\ &= e^{\frac{i\pi}{\varepsilon_1}(a_\alpha + a_\beta - 2\bar{a}_3)} \\ &\left(\delta_{\alpha,\beta} + 2i\, e^{\frac{3\pi i}{\varepsilon_1}(\bar{a}_3-\bar{a})} \frac{\prod_{\beta'\neq\beta}\Gamma\left(\frac{a_{\beta'}-a_\beta}{\varepsilon_1}\right)}{\prod_{\alpha'\neq\alpha}\Gamma\left(\frac{a_{\alpha'}-a_\alpha}{\varepsilon_1}\right)\,\sin\pi\frac{a_{\alpha'}-a_\alpha}{\varepsilon_1}} \prod_{\alpha'=1}^{3} \frac{\Gamma\left(\frac{a_{3,\alpha'}-a_\alpha}{\varepsilon_1}\right)\,\sin\pi\frac{a_{3,\alpha'}-a_\alpha}{\varepsilon_1}}{\Gamma\left(\frac{a_{3,\alpha'}-a_\beta}{\varepsilon_1}\right)}\right). \end{aligned} \tag{5.6.41}$$

Therefore, the traces can be expressed as

$$\mathrm{Tr}\, M_B(\widehat{\mathfrak{D}}_3)^{\pm 1} = B_0^{\pm} + \sum_{\alpha \neq \beta} \widetilde{B}^{\pm}_{\alpha\beta}\, e^{\left(\frac{\partial}{\partial a_\alpha} - \frac{\partial}{\partial a_\beta}\right)\widetilde{\mathcal{W}}}, \tag{5.6.42}$$

where we have computed the coefficients as

$$B_0^{\pm} \equiv \sum_{\alpha=1}^{3} \left(\boldsymbol{\mathcal{C}}_0^{\pm 1}\right)_{\alpha\alpha} \left(\boldsymbol{\mathcal{C}}_\infty^{\pm 1}\right)_{\alpha\alpha}, \tag{5.6.43}$$

and

$$\begin{aligned}\widetilde{B}^{\pm}_{\alpha\beta} &\equiv \left(\boldsymbol{\mathcal{C}}_0^{\pm 1}\right)_{\alpha\beta} \left(\boldsymbol{\mathcal{C}}_\infty^{\pm 1}\right)_{\beta\alpha} \\ &= -4 e^{\pm i\pi \frac{\bar{a}_0 - \bar{a}_3}{\varepsilon_1}} \; \frac{\prod_{\gamma=1}^{3} \sin\pi \frac{a_\beta - a_{0,\gamma}}{\varepsilon_1} \; \sin\pi\frac{a_{3,\gamma} - a_\alpha}{\varepsilon_1}}{\prod_{\alpha' \neq \alpha} \sin\pi \frac{a_{\alpha'} - a_\alpha}{\varepsilon_1} \; \prod_{\beta' \neq \beta} \sin\pi \frac{a_\beta - a_{\beta'}}{\varepsilon_1}} \\ &\quad \prod_{\alpha' \neq \alpha} \frac{\Gamma\left(\frac{a_\alpha - a_{\alpha'}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{\alpha'} - a_\alpha}{\varepsilon_1}\right)} \prod_{\beta' \neq \beta} \frac{\Gamma\left(\frac{a_{\beta'} - a_\beta}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_\beta - a_{\beta'}}{\varepsilon_1}\right)} \prod_{\gamma=1}^{3} \frac{\Gamma\left(\frac{a_{3,\gamma} - a_\alpha}{\varepsilon_1}\right) \; \Gamma\left(\frac{a_\beta - a_{0,\gamma}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_\alpha - a_{0,\gamma}}{\varepsilon_1}\right) \; \Gamma\left(\frac{a_{3,\gamma} - a_\beta}{\varepsilon_1}\right)}.\end{aligned} \tag{5.6.44}$$

It is crucial to note that the last line of (5.6.44) is precisely the contribution from 1-loop part of the effective twisted superpotential of the A_1-theory, under the ζ-function regularization (see (2.1.37) and its derivation above),

$$\left(\frac{\partial}{\partial a_\alpha} - \frac{\partial}{\partial a_\beta}\right) \widetilde{\mathcal{W}}^{\text{1-loop}} = \log \prod_{\alpha' \neq \alpha} \frac{\Gamma\left(\frac{a_\alpha - a_{\alpha'}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_{\alpha'} - a_\alpha}{\varepsilon_1}\right)} \prod_{\beta' \neq \beta} \frac{\Gamma\left(\frac{a_{\beta'} - a_\beta}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_\beta - a_{\beta'}}{\varepsilon_1}\right)} \prod_{\gamma=1}^{3} \frac{\Gamma\left(\frac{a_{3,\gamma} - a_\alpha}{\varepsilon_1}\right) \; \Gamma\left(\frac{a_\beta - a_{0,\gamma}}{\varepsilon_1}\right)}{\Gamma\left(\frac{a_\alpha - a_{0,\gamma}}{\varepsilon_1}\right) \; \Gamma\left(\frac{a_{3,\gamma} - a_\beta}{\varepsilon_1}\right)}. \tag{5.6.45}$$

We define the full effective twisted superpotential by

$$\widetilde{\mathcal{W}}^{\text{full}} \equiv \widetilde{\mathcal{W}}^{\text{classical}} + \widetilde{\mathcal{W}}^{\text{1-loop}} + \widetilde{\mathcal{W}}^{\text{inst}} + \widetilde{\mathcal{W}}^{\text{extra}}. \tag{5.6.46}$$

Here, the 1-loop part of the effective twisted superpotential is given in (2.1.33) and the rest was obtained in (5.3.43),

$$\widetilde{\mathcal{W}}^{\text{classical}} = -\frac{(a_1-a_2)^2+(a_1-a_3)^2-(a_1-a_2)(a_1-a_3)}{3\varepsilon_1}\log\mathfrak{q} \tag{5.6.47a}$$

$$\widetilde{\mathcal{W}}^{\text{1-loop}} = \lim_{\varepsilon_2\to 0}\varepsilon_2\log\frac{\prod_{\alpha,\beta=1}^{3}\Gamma_2(a_\alpha-a_\beta;\varepsilon_1,\varepsilon_2)}{\prod_{\alpha,\beta=1}^{3}\Gamma_2(a_\alpha-a_{0,\beta};\varepsilon_1,\varepsilon_2)\Gamma_2(a_{3,\alpha}-a_\beta;\varepsilon_1,\varepsilon_2)} \tag{5.6.47b}$$

$$\widetilde{\mathcal{W}}^{\text{inst}} = \lim_{\varepsilon_2\to 0}\varepsilon_2\log\mathcal{Z}^{\text{inst}}_{A_1} \tag{5.6.47c}$$

$$\widetilde{\mathcal{W}}^{\text{extra}} = \varepsilon_1\left(1-\delta_{\mathfrak{q}}-\delta_0\right)\log\mathfrak{q}+\frac{3(\bar{a}-\bar{a}_3+\varepsilon)(\bar{a}_0-\bar{a})}{\varepsilon_1}\log(1-\mathfrak{q}). \tag{5.6.47d}$$

Again, the expression for the traces of the B-monodromy matrix are simplified with the full effective twisted superpotential $\widetilde{\mathcal{W}}^{\text{full}}$. Let us make an overall shift of the Coulomb moduli and the masses of the hypermultiplets to recover the $SU(3)$ parameters (see section 5.2). Then we get the final expressions for the traces of the B-monodromy:

$$\begin{aligned}
&\operatorname{Tr} M_B(\widehat{\mathfrak{D}}_3)^{\pm 1}\\
&= B_0^{\pm}+B_{12}^{\pm}\,e^{\frac{1}{\varepsilon_1}\left(\frac{\partial}{\partial\boldsymbol{\alpha}_1}-\frac{\partial}{\partial\boldsymbol{\alpha}_2}\right)\widetilde{\mathcal{W}}^{\text{full}}}+B_{13}^{\pm}\,e^{\frac{1}{\varepsilon_1}\frac{\partial\widetilde{\mathcal{W}}^{\text{full}}}{\partial\boldsymbol{\alpha}_1}}+B_{23}^{\pm}\,e^{\frac{1}{\varepsilon_1}\frac{\partial\widetilde{\mathcal{W}}^{\text{full}}}{\partial\boldsymbol{\alpha}_2}}\\
&\qquad+B_{21}^{\pm}\,e^{-\frac{1}{\varepsilon_1}\left(\frac{\partial}{\partial\boldsymbol{\alpha}_1}-\frac{\partial}{\partial\boldsymbol{\alpha}_2}\right)\widetilde{\mathcal{W}}^{\text{full}}}+B_{31}^{\pm}\,e^{-\frac{1}{\varepsilon_1}\frac{\partial\widetilde{\mathcal{W}}^{\text{full}}}{\partial\boldsymbol{\alpha}_1}}+B_{32}^{\pm}\,e^{-\frac{1}{\varepsilon_1}\frac{\partial\widetilde{\mathcal{W}}^{\text{full}}}{\partial\boldsymbol{\alpha}_2}},
\end{aligned} \tag{5.6.48}$$

where the coefficients are computed as

$$\begin{aligned}
B_0^{\pm} &= 3e^{\pm\frac{2\pi i}{\varepsilon_1}(\bar{a}_3-\bar{a}_0)}\pm 2i\,e^{\pm\frac{i\pi}{\varepsilon_1}(2\bar{a}_3+\bar{a}_0)}\sin\pi\frac{3\bar{a}_0}{\varepsilon_1}\mp 2i\,e^{\mp\frac{i\pi}{\varepsilon_1}(\bar{a}_3+2\bar{a}_0)}\sin\pi\frac{3\bar{a}_3}{\varepsilon_1}\\
&\quad-4e^{\pm\frac{i\pi}{\varepsilon_1}(\bar{a}_0-\bar{a}_3)}\sum_{\alpha=1}^{3}\frac{\prod_{\gamma=1}^{3}\sin\pi\left(\boldsymbol{\alpha}_\alpha-\frac{a_{0,\gamma}}{\varepsilon_1}\right)\sin\pi\left(\frac{a_{3,\gamma}}{\varepsilon_1}-\boldsymbol{\alpha}_\alpha\right)}{\prod_{\alpha'\neq\alpha}\sin^2\pi\left(\boldsymbol{\alpha}_\alpha-\boldsymbol{\alpha}_{\alpha'}\right)}
\end{aligned} \tag{5.6.49}$$

and

$$B_{\alpha\beta}^{\pm} = -4e^{\pm i\pi\frac{\bar{a}_0-\bar{a}_3}{\varepsilon_1}}\frac{\prod_{\gamma=1}^{3}\sin\pi\left(\boldsymbol{\alpha}_\beta-\frac{a_{0,\gamma}}{\varepsilon_1}\right)\,\sin\pi\left(\frac{a_{3,\gamma}}{\varepsilon_1}-\boldsymbol{\alpha}_\alpha\right)}{\prod_{\alpha'\neq\alpha}\sin\pi\left(\boldsymbol{\alpha}_{\alpha'}-\boldsymbol{\alpha}_\alpha\right)\,\prod_{\beta'\neq\beta}\sin\pi\left(\boldsymbol{\alpha}_\beta-\boldsymbol{\alpha}_{\beta'}\right)}. \tag{5.6.50}$$

We observe the precise agreement between (5.5.56) and (5.6.48) under the identification of parameters,

$$
\begin{aligned}
&\mathfrak{m}_{-1}^{(\alpha)} = e^{2\pi i \frac{a_{0,\alpha}-\bar{a}_0}{\varepsilon_1}}, \quad \mathfrak{m}_1^{(\alpha)} = e^{2\pi i \frac{a_{3,\alpha}-\bar{a}_3}{\varepsilon_1}}, \quad \alpha = 1,2, \\
&\mathfrak{m}_0 = e^{2\pi i \frac{\bar{a}_0}{\varepsilon_1}}, \quad \mathfrak{m}_1 = e^{-2\pi i \frac{\bar{a}_3}{\varepsilon_1}}.
\end{aligned}
\tag{5.6.51}
$$

Most importantly, we find

$$
\boldsymbol{\beta}_\alpha = \frac{1}{\varepsilon_1} \frac{\partial \widetilde{\mathcal{W}}^{\text{full}}}{\partial \boldsymbol{\alpha}_\alpha}, \quad \alpha = 1,2. \tag{5.6.52}
$$

Therefore, we verify that the generating function for the variety $\mathcal{O}_3[\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}]$ of opers with respect to the generalized NRS coordinate system is identical to the effective twisted superpotential, namely,

$$
\mathcal{S}\left[\mathcal{O}_3[\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}]\right] = \frac{1}{\varepsilon_1} \widetilde{\mathcal{W}}^{\text{full}} \left[\mathcal{T}[A_2, \mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}]\right], \tag{5.6.53}
$$

by the relation (5.6.52).

Remarks

- The validity of the equivalence (5.6.53) at the 1-loop level was checked in [98].[12] The gauge theoretical derivation of (5.6.53) that we have shown guarantees its validity at all orders in the gauge coupling $\mathfrak{q}$.

5.6.3 Higher $SL(N)$-oper

It is straightforward to generalize the procedure to the higher $SL(N)$-opers $\widehat{\mathfrak{D}}_N$ on $\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}$. We schematically describe how we proceed. First, we need to express the traces of the

[12] In [98], a different, the so-called Liouville/Toda regularization scheme was used. Although Liouville/Toda scheme is natural in the context of the AGT correspondence [63], the ζ-function regularization arises more naturally in the gauge theoretical context. Besides, the ζ-function regularization has a notational advantage in that the defining equations for the generalized NRS coordinates (5.5.51), (5.5.56) are written more simply without any Γ-functions or square roots.

holonomies of the flat $SL(N)$-connections in terms of the generalized NRS coordinates, as we did for $N=2$ and $N=3$ in section 5.5.2. It is clear that the holonomy along the A-cycle is still given by

$$M_A = M_0^{-1} = \sum_{\alpha}^{N} \left(\mathfrak{m}_0^{(\alpha)}\right)^{-1} \Pi_0^{(\alpha)}. \tag{5.6.54}$$

Hence we obtain

$$\operatorname{Tr} M_A^k = \sum_{\alpha=1}^{N} \left(\mathfrak{m}_0^{(\alpha)}\right)^{-k}, \quad k = 1, \cdots, N-1. \tag{5.6.55}$$

The holonomy along the B-cycle is written as

$$\begin{aligned} M_B &= g_1^{-1} g_0^{-1} \\ &= \mathfrak{m}_0^{-1}\mathfrak{m}_1^{-1} \left(\mathbb{1}_N + (\mathfrak{m}_1^N - 1)\Pi_1\right) \left(\mathbb{1}_N + (\mathfrak{m}_0^N - 1)\Pi_0\right). \end{aligned} \tag{5.6.56}$$

Due to the properties of the projection operators, we have

$$\operatorname{Tr}\ (\Pi_0\Pi_1)^k = (\operatorname{Tr}\Pi_0\Pi_1)^k, \quad k \in \mathbb{Z}^{>0}. \tag{5.6.57}$$

Thus we can expand the traces of (5.6.56) as a polynomial in $\operatorname{Tr}\Pi_0\Pi_1$,

$$\begin{aligned} \operatorname{Tr} M_B^k &= \mathfrak{m}_0^{-k}\mathfrak{m}_1^{-k}(N-2+\mathfrak{m}_0^{Nk}+\mathfrak{m}_1^{Nk}) \\ &\quad + \cdots + \mathfrak{m}_0^{-k}\mathfrak{m}_1^{-k}(\mathfrak{m}_0^N-1)^k(\mathfrak{m}_1^N-1)^k\left(\operatorname{Tr}\Pi_0\Pi_1\right)^k, \end{aligned} \tag{5.6.58}$$

for any $k = 1, \cdots, N-1$. Since we can express $\operatorname{Tr}\Pi_0\Pi_1$ by the $\boldsymbol{\beta}$ coordinates,

$$
\begin{aligned}
\operatorname{Tr}\Pi_0\Pi_1 &= \sum_{\alpha=1}^{N} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)}\Pi_1 \\
&= \sum_{\alpha=1}^{N} e^{-\tilde{\boldsymbol{\beta}}_0^{(\alpha)}+\tilde{\boldsymbol{\beta}}} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)} \\
&= \sum_{\alpha=1}^{N} \operatorname{Tr}\Pi_0\Pi_0^{(\alpha)} \operatorname{Tr}\Pi_1\Pi_0^{(\alpha)} + \sum_{\alpha\neq\beta} e^{\tilde{\boldsymbol{\beta}}_0^{(\alpha)}-\tilde{\boldsymbol{\beta}}_0^{(\beta)}} \operatorname{Tr}\Pi_0\Pi_0^{(\beta)} \operatorname{Tr}\Pi_1\Pi_0^{(\alpha)},
\end{aligned}
\tag{5.6.59}
$$

we obtain the representation of the traces (5.6.58) in terms of the generalized NRS coordinates $\boldsymbol{\alpha}_\alpha$, $\boldsymbol{\beta}_\alpha \equiv \tilde{\boldsymbol{\beta}}_0^{(\alpha)} - \tilde{\boldsymbol{\beta}}_0^{(N)}$.

Next, we evaluate the monodromies of the oper $\widehat{\mathfrak{D}}_N$. By shifting $z \mapsto z\, e^{2\pi i}$ for $\widetilde{\boldsymbol{\mathfrak{Z}}}^{L\to M}$ we compute $\mathbf{M}_A(\widehat{\widehat{\mathfrak{D}}}_N)$. The A-monodromy matrix for the oper $\widehat{\mathfrak{D}}_N$ is then

$$
M_A(\widehat{\mathfrak{D}}_N) = \lim_{\varepsilon_2\to 0} \mathbf{M}_A(\widehat{\widehat{\mathfrak{D}}}_N), \tag{5.6.60}
$$

which can be expressed in terms of the $\boldsymbol{\alpha}$ coordinates by comparing its traces with (5.6.55). We also compute the B-monodromy for $\widehat{\widehat{\mathfrak{D}}}_N$ by

$$
\mathbf{M}_B(\widehat{\widehat{\mathfrak{D}}}_N) = \mathbf{R}_\infty\, \mathbf{C}_\infty\, \mathbf{S}\, \mathbf{C}_0^{-1}\, \mathbf{R}_0\, \mathbf{C}_0\, \mathbf{S}^{-1}\, \mathbf{C}_\infty^{-1}, \tag{5.6.61}
$$

from which we compute the B-monodromy matrix for the oper $\widehat{\mathfrak{D}}_N$ as

$$
\begin{aligned}
M_B(\widehat{\mathfrak{D}}_3) &= \lim_{\varepsilon_2\to 0} \mathbf{M}_B(\widehat{\widehat{\mathfrak{D}}}_N)\, e^{\frac{\widetilde{\mathrm{w}}}{\varepsilon_2}} \\
&= \lim_{\varepsilon_2\to 0} \mathbf{R}_\infty\, \mathbf{C}_\infty\, \mathbf{S}\, \mathbf{C}_0^{-1}\, \mathbf{R}_0\, \mathbf{C}_0\, \mathbf{S}^{-1}\, \mathbf{C}_\infty^{-1}\, e^{\frac{\widetilde{\mathrm{w}}}{\varepsilon_2}}.
\end{aligned}
\tag{5.6.62}
$$

Then we find the expressions for the traces

$$
\operatorname{Tr} M_B(\widehat{\mathfrak{D}}_N)^k, \qquad k = 1, \cdots, N-1. \tag{5.6.63}
$$

By comparing these expressions with (5.6.58), we find that

$$\boldsymbol{\beta}_\alpha = \frac{1}{\varepsilon_1} \frac{\partial \widetilde{\mathcal{W}}^{\text{full}}}{\partial \boldsymbol{\alpha}_\alpha}, \quad \alpha = 1, \cdots, N-1. \tag{5.6.64}$$

This relation verifies that the generating function for the variety $\mathcal{O}_N[\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}]$ of opers in the generalized NRS coordinate system $\{\boldsymbol{\alpha}_\alpha, \boldsymbol{\beta}_\alpha \mid \alpha = 1, \cdots, N-1\}$ is identical to the effective twisted superpotential:

$$\mathcal{S}\left[\mathcal{O}_N[\mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}]\right] = \frac{1}{\varepsilon_1} \widetilde{\mathcal{W}}^{\text{full}} \left[\mathcal{T}[A_{N-1}, \mathbb{P}^1 \backslash \{0, \underline{\mathfrak{q}}, \underline{1}, \infty\}]\right]. \tag{5.6.65}$$

5.7 Discussion

We have shown that non-perturbative Dyson-Schwinger equations for the class $\mathcal{S}$ theories with the insertion of a surface defect produce the operators $\hat{\hat{\hat{\mathfrak{D}}}}$ annihilating their partition functions. These operators were reduced to the opers $\hat{\mathfrak{D}}$ in the limit $\varepsilon_2 \to 0$, providing an explicit relation between the holomorphic coordinates on the variety of opers and the expectation values of the chiral observables in the limit $\varepsilon_2 \to 0$. The surface defect partition functions, i.e., the solutions to $\hat{\hat{\hat{\mathfrak{D}}}}$, were analytically continued to different convergence domains and glued together in the intermediate domain. This procedure enabled the computation of the monodromies of the solutions to $\hat{\hat{\hat{\mathfrak{D}}}}$, and therefore the monodromies of the opers $\hat{\mathfrak{D}}$ by taking the limit $\varepsilon_2 \to 0$. We constructed a higher-rank generalization of the NRS coordinate system, and represented the monodromies of opers in terms of these coordinates. The effective twisted superpotential arose as the generating function for the variety of opers in the generalized NRS coordinate system by construction.

We believe that the subject deserves more investigations in various aspects. Let us consider the example of $\mathfrak{g} = A_1$. We have constructed the Darboux coordinate system $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in which the generating function for the variety $\mathcal{O}_2[\mathcal{C}]$ oper $\hat{\mathfrak{D}}_2$ is identified with the effective twisted superpotential. Meanwhile, $\mathcal{O}_2[\mathcal{C}]$ is a a Lagrangian submanifold of

$\mathcal{M}_{\text{flat}}(SL(2),\mathcal{C})$, which is spanned by the off-shell spectra $u_2 = \lim_{\varepsilon_2\to 0}\left\langle \mathcal{O}_2 \right\rangle$ for fixed gauge couplings $\mathfrak{q}$. The variation of the gauge couplings, i.e., the elements of the Teichmüller space $\mathbb{T}[\mathcal{C}]$ of $\mathcal{C}$, gives the foliation of the moduli space $\mathcal{M}_{\text{flat}}(SL(2),\mathcal{C})$ by the leaves of the varieties of opers with varying gauge couplings. Thus there exists another Darboux coordinate system $(\tau_2 = \log\mathfrak{q}, u_2)$ on $\mathcal{M}_{\text{flat}}(SL(2),\mathcal{C})$ induced from the identification $\mathcal{M}_{\text{flat}}(SL(2),\mathcal{C}) \simeq T^*\mathbb{T}[\mathcal{C}]$. We observe that the relations

$$\boldsymbol{\beta} = \frac{1}{\varepsilon_1}\frac{\partial\widetilde{\mathcal{W}}}{\partial\boldsymbol{\alpha}}, \quad u_2 = \frac{\partial\widetilde{\mathcal{W}}}{\partial\tau_2}, \tag{5.7.1}$$

identify the effective twisted superpotential with the generating function for the canonical transformation of the Darboux coordinate systems,

$$(\tau_2, u_2) \overset{\widetilde{\mathcal{W}}}{\longleftrightarrow} (\boldsymbol{\alpha},\boldsymbol{\beta}). \tag{5.7.2}$$

Let us consider generalizing this relation to the higher rank $\mathfrak{g} = A_2$, for a fixed Riemann surface, say, $\mathbb{P}^1_{2,\underline{r+1}}$. We still have the generalized NRS coordinate system $\{\boldsymbol{\alpha}_i^{(\alpha)}, \boldsymbol{\beta}_i^{(\alpha)} \mid i = 0,1,\cdots,r-1,\ \alpha = 1,\cdots,N-1\}$ on one hand, but it is apparent that the variation on the Teichmüller space $\mathbb{T}[\mathbb{P}^1_{2,\underline{r+1}}]$ does not saturate the half of the dimension of the moduli space, since the dimension of the moduli space increases as the rank increases, $\dim\mathcal{M}_{\text{flat}}(SL(3),\mathbb{P}^1_{2,\underline{r+1}}) = 2r(N-1) = 4r$, while the dimension of the Teichmüller space is independent of the rank, $\dim\mathbb{T}[\mathbb{P}^1_{2,\underline{r+1}}] = r$. In other words, we need r more parameters $\tau_{\mathbf{i},3}$ to form a Darboux coordinate system,

$$\{\tau_{\mathbf{i},2}, \tau_{\mathbf{i},3}, u_{\mathbf{i},2}, u_{\mathbf{i},3} \mid \mathbf{i} = 1,\cdots,r\}, \tag{5.7.3}$$

in which the effective twisted superpotential produces the spectrum $u_{\mathbf{i},3} = \lim_{\varepsilon_2\to 0}\left\langle \mathcal{O}_{\mathbf{i},3} \right\rangle$ of the higher Hamiltonian $\mathcal{O}_{\mathbf{i},3} = \text{Tr}\phi_{\mathbf{i}}^3$ under the differentiation with respect to $\tau_{\mathbf{i},3}$. Then the effective twisted superpotential becomes the generating function for the canonical transformation between Darboux coordinate systems, through the relations

$$\boldsymbol{\beta}_i^{(\alpha)} = \frac{1}{\varepsilon_1} \frac{\partial \widetilde{\mathcal{W}}}{\partial \boldsymbol{\alpha}_i^{(\alpha)}}, \qquad i = 0, 1, \cdots, r-1, \ \alpha = 1, \cdots, N-1, \tag{5.7.4}$$

$$u_{\mathbf{i},2} = \frac{\partial \widetilde{\mathcal{W}}}{\partial \tau_{\mathbf{i},2}}, \ u_{\mathbf{i},3} \stackrel{?}{=} \frac{\partial \widetilde{\mathcal{W}}}{\partial \tau_{\mathbf{i},3}}, \qquad \mathbf{i} = 1, \cdots, r. \tag{5.7.5}$$

But what is the meaning of the parameters $\tau_{\mathbf{i},3}$?

In the gauge theory side, the meaning of $\tau_{\mathbf{i},3}$ is clear. As investigated in [118], we may extend the theory by manually adding *the higher times* to the microscopic action,

$$\mathcal{L} = \sum_{\mathbf{i}=1}^{r} \tau_{\mathbf{i},2} \int d^4\theta \ \mathrm{Tr} \ \mathbf{\Phi}_{\mathbf{i}}^2 + \tau_{\mathbf{i},3} \int d^4\theta \ \mathrm{Tr} \ \mathbf{\Phi}_{\mathbf{i}}^3, \tag{5.7.6}$$

whose partition function can still be computed by equivariant localization as, schematically,

$$\mathcal{Z}^{\mathrm{inst}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \tau_2, \tau_3) = \sum_{\boldsymbol{\lambda}} \prod_{\mathbf{i}=1}^{r} \mathfrak{q}_{\mathbf{i}}^{|\boldsymbol{\lambda}^{(\mathbf{i})}|} \exp\left[\sum_{\mathbf{i}=1}^{r} \tau_{\mathbf{i},3} \, \mathcal{O}_{\mathbf{i},3}[\boldsymbol{\lambda}]\right] \boldsymbol{\mu}_{\boldsymbol{\lambda}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2). \tag{5.7.7}$$

Under the limit $\varepsilon_2 \to 0$, the partition function shows the asymptotic behavior,

$$\mathcal{Z}^{\mathrm{inst}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \tau_2, \tau_3) = e^{\frac{\widetilde{\mathcal{W}}(\mathbf{a}, \mathbf{m}, \varepsilon_1; \tau_2, \tau_3)}{\varepsilon_2}} \left(1 + \mathcal{O}(\varepsilon_2)\right). \tag{5.7.8}$$

Then it is straightforward that we produce the relation

$$u_{\mathbf{i},3} = \lim_{\varepsilon_2 \to 0} \left\langle \mathcal{O}_{\mathbf{i},3} \right\rangle = \frac{\partial \widetilde{\mathcal{W}}}{\partial \tau_{\mathbf{i},3}}. \tag{5.7.9}$$

Therefore, the extra parameters that foliate the remaining orthogonal directions to the varieties of opers are the higher times of the extended theory. The varieties $\mathcal{O}_3[\mathcal{C}]$ of opers such as (5.3.46) are located at $\tau_{\mathbf{i},3} = 0$ and only probe the $\tau_{\mathbf{i},2}$-variations.

The question is, then, what *the extended opers* are, which rise under the flow along the directions of the higher times. When re-phrased in terms of the qq-characters, the problem is to derive proper extended operators $\hat{\hat{\hat{\mathfrak{D}}}}$ from the non-perturbative Dyson-Schwinger equations of the extended theories with an insertion of a surface defect. The limit $\varepsilon_2 \to 0$ of these objects would yield the desired extended opers. Note that the expectation values of $\mathcal{O}_{\mathbf{i},3}$ would be compensated by the derivatives with respect to $\tau_{\mathbf{i},3}$, so that the issue of equating the analytically continued expectation values in the intermediate domain would also be resolved with this enhancement. It is not clear, however, how to derive meaningful expressions for these extended quantized opers $\hat{\hat{\hat{\mathfrak{D}}}}$ as of yet, so we leave this to future work. The variation along the higher times has many different manifestations. It corresponds to varying the higher Teichmüller structures studied in [105, 106], flowing along the higher Hamiltonians in the isomonodromic deformation of Fuchsian systems, and properly extending the Hamilton-Jacobi formulation of the Painlevé equations discussed in [77] to the higher order Painlevé-type equations. It would be interesting to see how the extended gauge theory ties up these different realms of mathematical physics.

In the context of the BPS/CFT correspondence, the subject reveals still another feature along the line of [63]. The well-established relation between the partition functions of $\mathcal{T}[A_{N-1}, \mathcal{C}]$ and the correlation functions of A_{N-1}-Toda CFTs has to be extended when we deal with the higher ranks $N \geq 3$. Namely, we start to face the expectation values of higher chiral observables in the gauge theory side, which cannot be compensated by the derivatives of gauge couplings, and they are supposed to correspond to the correlation functions with inclusion of $\mathcal{W}$-descendant fields in the CFT side. The precise dictionary between the two objects are yet to be accomplished. The realization of the higher times of the extended theories in the CFT side is even more unclear. The free field representation of the monodromies of degenerate fields studied in [119] can be relavant for this study.

Another problem related to this work is the generalized NRS coordinate systems corresponding to the non-Lagrangian theories. It is well-known that the higher rank class $\mathcal{S}$ theories do not always admit Lagrangian descriptions. Our computation of monodromy data of opers heavily utilized the availability of the exact computations of the partition functions and the expectation values of the chiral observables. For the non-Lagrangian theories, it is not even clear what *the instanton counting* means. Nevertheless, the Fuchsian systems with the prescribed monodromies around the punctures are still well-defined, and we may wonder if it is possible to explicily link the accessory parameters of the corresponding opers and the expectation values of the chiral observables in the non-Lagrangian theories. In the case when the non-Lagrangian theory is S-dual to a Lagrangian theory, it is desirable to explicitly construct the coordinate transformation [91] between the relavant generalized NRS coordinate systems and investigate their field theoretical meaning.

Part II

Quantum Toroidal Algebras

Chapter 6

New quantum toroidal algebras from 5d $\mathcal{N} = 1$ instantons on orbifolds

6.1 Introduction

As we observed in the Part I to some extent, the Nekrasov partition function has been a powerful tool to investigate the correspondences of four-dimensional $\mathcal{N} = 2$ supersymmetric quiver gauge theories with various objects in theoretical physics, i.e., quantum integrable systems [29, 30, 28], two-dimensional CFTs [63, 64, 10, 14, 15, 21, 32], flat connections on Riemann surfaces [91, 1], and isomonodromic deformations of Fuchsian systems [77, 120, 121].

Very rich algebraic structures lie at the heart of these correspondences [9]. For instance, the AGT correspondence [63, 64] between Nekrasov partition functions and conformal blocks of Liouville/Toda 2D CFTs can be understood algebraically as the action of W-algebras on the cohomology of instantons moduli space [122, 123, 124]. In this context, the W-algebra currents are coupled to an infinite Heisenberg algebra, and the total action is formulated in terms of a quantum algebra, namely the Spherical Hecke central algebra [124] (isomorphic to the affine Yangian of $\mathfrak{gl}(1)$ [125, 126]). The coupling to an Heisenberg algebra is essential for the definition of a coalgebraic structure, thus emphasizing the underlying quantum integrability since the coproduct provides the R-matrix satisfying the celebrated quantum Yang-Baxter equation.

A closely related but different connection with quantum algebras arises from the type IIB strings theory realization of the five-dimensional uplifts of 4d $\mathcal{N} = 2$ gauge theories, that is the 5d $\mathcal{N} = 1$ quiver gauge theories compactified on S^1. In this construction, $\mathcal{N} = 1$ gauge theories emerge as the low-energy description of the dynamics of 5-branes webs [127, 128]. Here, each brane carries the charges (p, q), generalizing D5-branes (charge $(1, 0)$) and NS5-branes (charge $(0, 1)$). Their world-volume include the five-dimensional gauge theory spacetime, together with an extra line segment in the 56-plane. Individual branes' line segments are joined by trivalent vertices, and form the (p, q)-branes web. Alternatively, the (p, q)-brane web can be seen as the toric diagram of a Calabi-Yau threefold on which topological strings can be compactified [129]. The trivalent vertices are then identified with the (refined) topological vertex, thereby leading to a very efficient method of computing 5d Nekrasov partition functions as topological strings amplitudes [130, 131].

Awata, Feigin and Shiraishi observed in [132] that a specific representation of the quantum toroidal $\mathfrak{gl}(1)$ algebra (or Ding-Iohara-Miki algebra [133, 134]) can be associated to each edge of the (p, q)-branes web. The charges (p, q) are identified with the values of the two central charges while the brane position define the weight of the representation. As such, the D5-branes correspond to the so-called *vertical representation* while an *horizontal representation* is associated to NS5-branes (possibly dressed by extra D5-branes).[1] The refined topological vertex is then identified with an intertwiner between vertical and horizontal representations, that is in fact the toroidal version of the vertex operator introduced in [137] for the quantum group $U_q(\widehat{\mathfrak{sl}(2)})$. In this way, the Nekrasov partition function is written as a purely algebraic object using the quantum toroidal algebra, just like conformal blocks with W-algebras [138,

[1]In fact, the vertical representation is simply the q-deformation of the affine Yangian action mentioned previously, it is expected to describe a quantum toroidal action on the K-equivariant cohomology of the quiver variety describing the instanton moduli space. The equivalent of the horizontal representation can also be defined for 4d $\mathcal{N} = 2$ theories, thus extending the whole algebraic construction of the Nekrasov partition functions [135]. However, for this purpose, it is necessary to consider the central extension of the Drinfeld double of the affine Yangian following from the construction given in [136].

139]. This algebraic construction turns out to be useful in probing various properties of the partition function, e.g. in addressing the (q-deformed) AGT correspondence [140, 141], or in studying strings' S-duality [142, 143].

In [10], an important class of half-BPS observables, called *qq-characters*, were defined, whose characteristic property is the regularity of their gauge theory expectation values. This regularity property encodes efficiently an infinite set of constraints on the partition function called *non-perturbative Dyson-Schwinger equations* [10]. The algebraic nature of these constraints was observed in [144, 145]. Actually, the constraints take an even more elegant form in the algebraic construction described above as they express the invariance of an operator $\mathcal{T}$ under the adjoint action of the quantum toroidal algebra [51]. This operator is obtained by gluing intertwiners along the edges of the (p, q)-branes web, and its vacuum expectation value reproduces the 5d Nekrasov instanton partition.

A natural question is how to generalize the algebraic construction to gauge theories on more complicated manifolds. Among other manifolds, the $\mathbb{Z}_p$-orbifolded $\mathbb{C}^2$ are of a particular interest, since the partition functions on these spaces can be computed by simply projecting out the contributions which are not invariant under the $\mathbb{Z}_p$-action [20, 146, 15]. The generalization of the algebraic construction is not entirely straightforward since it is necessary to introduce the information of the *coloring* corresponding to the $\mathbb{Z}_p$-action of the orbifolding. In this scope, deformations of the quantum toroidal $\mathfrak{gl}(1)$ algebra must be considered. A special case of the $\mathbb{Z}_p$-orbifolded $\mathbb{C}^2$ is the (un-resolved) A_p-type ALE spaces. The ALE instantons were introduced by Kronheimer in [147], and the ALE instanton moduli spaces were constructed as quiver varieties in [148, 149]. The algebraic construction of the corresponding 5d Nekrasov partition functions has been realized recently using an underlying quantum toroidal $\mathfrak{gl}(p)$ algebra. There, the index carried by the Drinfeld currents renders the $\mathbb{Z}_p$-coloring due to the orbifolding. Incidentally, the vertical representation of this quantum toroidal algebra should coincide with the q-deformation of the affine Yangian of $\mathfrak{gl}(p)$ acting on the cohomology of the moduli space of ALE instantons, extending by further affinization the algebraic actions discovered in [149, 150].

In this work, we extend the algebraic construction of 5d Nekrasov partition functions to a more general $\mathbb{Z}_p$-orbifolding depending on two integer parameters (ν_1, ν_2). We propose an extended quantum toroidal algebra relevant to the construction, and prove its Hopf algebra structure. We define both horizontal and vertical representations, and derive the vertex operator which intertwines between these representations. Finally, using these ingredients, we give an algebraic construction of Nekrasov partition functions and qq-characters. The orbifolds considered in this work incorporate the case of codimension-two defect insertion, whose applications to BPS/CFT correspondence, Bethe/gauge correspondence, and Nekrasov-Rosly-Shatashvili correspondence have been largely investigated [21, 32, 2, 3, 1].

This chapter is written in such a way that mathematicians interested only in the formulation of the extended algebra can focus on the reading of section three, together with the appendices G (quantum toroidal $\mathfrak{gl}(p)$), I (representations) and J (automorphisms and gradings) for more details. Instead, the section two provides a brief description of the physical context in which the algebra emerges, i.e. instantons of 5d $\mathcal{N} = 1$ gauge theories on the spacetime $\mathbb{C}^2/\mathbb{Z}_p$. Finally, the section four is dedicated to the algebraic construction of gauge theories observables, giving the expression of the (ν_1, ν_2)-colored refined topological vertex and a few examples of application.

6.2 Instantons on orbifolds

6.2.1 Action of the abelian group $\mathbb{Z}_p$ on the ADHM data

In order to derive the group action on the instanton moduli space, we focus first on the case of a pure $U(m)$ gauge theory. In this case, the ADHM construction of the moduli space [151] involves only two vector spaces M and K of dimension m and k respectively, where k is the instanton number. Introducing the four matrices $B_1, B_2 : K \to K$ $I : M \to K$ and $J : K \to M$, the instanton moduli space is identified with the quiver variety (see, for instance, [152])

$$\mathcal{M}_k = \{B_1, B_2, I, J \big/ [B_1, B_2] + IJ = 0, \mathbb{C}[B_1; B_2]I(M) = K\} \big/ \mathrm{GL}(K). \tag{6.2.1}$$

The complexified global symmetry group $\mathrm{GL}(M) \times \mathrm{SL}(2,\mathbb{C})^2$ acts on the ADHM matrices, preserving the quiver variety $\mathcal{M}_k$. It contains an $(m+2)$-dimensional torus that acts follows,

$$(B_1, B_2, I, J) \to (t_1 B_1, t_2 B_2, It, t^{-1} J t_1 t_2), \quad (t, t_1, t_2) \in (\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^2. \tag{6.2.2}$$

The fixed points of this action parameterize the configurations of instantons with total charge k. They are in one-to-one correspondence with the m-tuple partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots \lambda^{(m)})$ of the integer k, here identified with the m-tuple Young diagrams with $|\boldsymbol{\lambda}| = k$ boxes. At the fixed point, the vector space K is decomposed into

$$K = \bigoplus_{\alpha=1}^{m} \bigoplus_{(i,j)\in\lambda^{(\alpha)}} B_1^{i-1} B_2^{j-1} I(M_\alpha), \tag{6.2.3}$$

where M_α denotes the one-dimensional vector spaces generated by the basis vectors of M. Thus, each box $\square = (\alpha, i, j)$ of the m-tuple partition $\boldsymbol{\lambda}$ with coordinate $(i,j) \in \boldsymbol{\lambda}^{(\alpha)}$ corresponds to a one-dimensional vector space $B_1^{i-1} B_2^{j-1} I(M_\alpha)$. We further associate to the box $\square$ the complex variable $\phi_\square = a_\alpha + (i-1)\varepsilon_1 + (j-1)\varepsilon_2$ called *instanton position* or, sometimes, the *box content of* $\square$. The parameters $a_1, \cdots, a_m$ are the *Coulomb branch vevs* of the gauge theory. We also define the exponentiated quantities $v_\alpha = e^{Ra_\alpha}$, $(q_1, q_2) = (e^{R\varepsilon_1}, e^{R\varepsilon_2})$ and $\chi_\square = e^{R\phi_\square} = v_\alpha q_1^{i-1} q_2^{j-1}$.

In this chapter, gauge theories are considered on the 5d orbifolded Ω-background $S_R^1 \times (\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\varepsilon_2})/\mathbb{Z}_p$ where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is a subgroup of the torus $U(1)^2 \subset SO(4)$. The action of the group $\mathbb{Z}_p$ on the spacetime is parameterized by two integers ν_1, ν_2,

$$(\theta, z_1, z_2) \in S_R^1 \times \mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\varepsilon_2} \to (\theta, e^{2i\pi\nu_1/p} z_1, e^{2i\pi\nu_2/p} z_2), \quad \text{with} \quad (\nu_1, \nu_2) \in \mathbb{Z}_p \times \mathbb{Z}_p. \tag{6.2.4}$$

Furthermore, it is possible to combine it with a global gauge transformation in the subgroup $U(1)^m \subset U(m)$. As a result, we obtain an action of $\mathbb{Z}_p$ on the ADHM data by specialization of the $(m+2)$-torus action 6.2.2, taking

$$t = \mathrm{diag}(e^{2i\pi c_\alpha/p})_{\alpha=1,\cdots,m}, \quad t_1 = e^{2i\pi\nu_1/p}, \quad t_2 = e^{2i\pi\nu_2/p}. \tag{6.2.5}$$

This action of the abelian group $\mathbb{Z}_p$ is parameterized by the $m+2$ integers (c_α, ν_1, ν_2) considered modulo p. The transformation of the vector spaces in the decomposition 6.2.3 of K leads to associate to each box $\square = (\alpha, i, j) \in \boldsymbol{\lambda}$, in addition to the complex variables $\phi_\square$ and $\chi_\square$, the integer $c(\square)$ such that

$$B_1^{i-1}B_2^{j-1}I(M_\alpha) \to e^{2i\pi c(\square)/p}B_1^{i-1}B_2^{j-1}I(M_\alpha), \quad \text{with} \quad c(\square) = c_\alpha + (i-1)\nu_1 + (j-1)\nu_2 \in \mathbb{Z}_p. \tag{6.2.6}$$

We call *color* any integer parameter defined modulo p. For short, we also say that c_α and ν_1, ν_2 are respectively color of the Coulomb branch vevs, and of the parameters q_1, q_2. The map $c : \boldsymbol{\lambda} \to \mathbb{Z}_p$ defines a coloring of the m-tuple partitions $\boldsymbol{\lambda}$, and K has a natural decomposition into sectors of a given color $c(\square) = \omega$,

$$K = \bigoplus_{\omega\in\mathbb{Z}_p} K_\omega(\boldsymbol{\lambda}). \tag{6.2.7}$$

Notations We denote $C_\omega(m)$ the subset of $[\![1, m]\!]$ such that the Coulomb branch vevs a_α (or v_α) with $\alpha \in C_\omega(m)$ have color $c_\alpha = \omega$ (or, equivalently, that the box $(1,1) \in \lambda^{(\alpha)}$ with $\alpha \in C_\omega(m)$ is of color $c(\alpha, 1, 1) = c_\alpha = \omega$). Similarly, $K_\omega(\boldsymbol{\lambda})$ denotes the set of boxes $\square \in \boldsymbol{\lambda}$ of the m-tuple colored partition $\boldsymbol{\lambda}$ that carry the color $c(\square) = \omega$. Besides, in the generic case $\nu_1 + \nu_2 \neq 0$, the shift of color indices ω by the quantity $\nu_1 + \nu_2$ appears in many formulas. To simplify these expressions, we introduce the notation $\bar{\omega} = \omega + \nu_1 + \nu_2$ for the shifted indices, along with the map $\bar{c}(\square) = c(\square) + \nu_1 + \nu_2$. Finally, we also introduce the extra variables q_3 and ν_3 such that $q_1q_2q_3 = 1$ and $\nu_1 + \nu_2 + \nu_3 = 0$. Due to the fact that the $\mathbb{Z}_p$-action coincides with a subgroup of the torus action, in all formulas the shift of color indices $\omega+\nu_i$ coincide with a factor q_i multiplying the parameters associated to instanton positions in the moduli space.

McKay subgroups in $SO(4)$ Although we are considering here a different problem, it is interesting to make a short parallel with the action of $SU(2)_L \times SU(2)_R \subset SO(4)$ on the Ω-background (see, for instance, [153]). This action takes a simpler form if we employ the quaternionic coordinates

$$Z = \left(\begin{smallmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{smallmatrix}\right), \quad (z_1, z_2) \in \mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\varepsilon_2}. \tag{6.2.8}$$

Then the 2×2 matrices $(G_L, G_R) \in SU(2)_L \times SU(2)_R$ act on the quaternions as $Z \to G_L Z G_R$. The McKay subgroups of $SU(2)$ possess an ADE-classification. For instance, the A_{p-1} series corresponds to the action of $\mathbb{Z}_p$, it is generated (multiplicatively) by the diagonal matrices

$$G = \left(\begin{smallmatrix} e^{2i\pi/p} & 0 \\ 0 & e^{-2i\pi/p} \end{smallmatrix}\right). \tag{6.2.9}$$

Considering only the action of the A_{p-1} subgroup on the left, the background coordinates transform as $(z_1, z_2) \to (e^{2i\pi/p} z_1, e^{-2i\pi/p} z_2)$. This transformation can be recovered from the action 6.2.4 of $\mathbb{Z}_p$ by choosing $\nu_1 = -\nu_2 = 1$. The orbifold of the spacetime under this action of $\mathbb{Z}_p$ reproduces the ALE space constructed in [147]. Instantons of $\mathcal{N}=1$ gauge theories defined on ALE spacetimes have been extensively studied [147, 148, 149, 150]. In [154], their contributions to the gauge theory partition functions have been reproduced using algebraic techniques based on the quantum toroidal algebra of $\mathfrak{gl}(p)$. The generalization to DE-type McKay subgroups with only left action is expected to involve quantum toroidal algebras based on either $\mathfrak{so}(p)$ or $\mathfrak{sp}(p)$ Lie algebras [150].

It is also possible to consider simultaneously the action of two McKay subgroups A_{p_1-1} and A_{p_2-1}, with one acting on the left, the other on the right. As a result, coordinates now transform as

$$(z_1, z_2) \to (e^{2i\pi(p_1+p_2)/(p_1p_2)} z_1, e^{2i\pi(p_1-p_2)/(p_1p_2)} z_2). \tag{6.2.10}$$

We recognize here another particular case of the $\mathbb{Z}_p$-action defined in 6.2.4, albeit more general than before. It is simply obtained by the specialization $\nu_1 = p_1 + p_2$, $\nu_2 = p_1 - p_2$ and $p = p_1p_2$. Thus, the action 6.2.4 leads to a particularly rich context. Moreover, taking $\nu_1 = 0$, the first coordinate z_1 is invariant and the orbifolded spacetime can be reinterpreted as the insertion of a codimension-two defect in a 5d Ω-background with no orbifold [21, 20]. We build here a general algebraic framework to address this kind of problems. It may be possible to further generalize our approach to the action of DE-type McKay subgroups with both left and right actions, but this is beyond the scope of this work.

6.2.2 Instantons partition function

The computation of the Nekrasov instanton partition function on such $\mathbb{Z}_p$-orbifolds has been performed in [20, 146, 15].[2] For simplicity, we do not introduce fundamental matter multiplets, those being obtained in the limit $\mathfrak{q} \to 0$ of the gauge coupling parameters. Furthermore, we only discuss the case of linear quiver gauge theories A_r, with $U(m^{(i)})$ gauge groups at each node $i = 1 \cdots r$. Thus, the node i carries the following parameters:

- a set of colored exponentiated gauge couplings $\mathfrak{q}_{\omega,i}$,
- a p-vector of colored Chern-Simons levels $\boldsymbol{\kappa}^{(i)} = (\kappa_\omega^{(i)})_{\omega\in\mathbb{Z}_p}$,
- an $m^{(i)}$-vector of Coulomb branch vevs $\boldsymbol{a}^{(i)} = (a_\alpha^{(i)})_{\alpha=1}^{m^{(i)}}$ defining the exponentiated parameters $\boldsymbol{v}^{(i)} = (v_\alpha^{(i)})_{\alpha=1}^{m^{(i)}}$ with $v_\alpha^{(i)} = e^{Ra_\alpha^{(i)}}$,
- an associated vector of colors $\boldsymbol{c}^{(i)} = (c_\alpha^{(i)})_{\alpha=1}^{m^{(i)}}$.

[2]Strictly speaking, in [146], the authors consider the instantons on the minimal resolution of the orbifold. Instead, here, following [20, 15], we simply consider the $\mathbb{Z}_p$-invariant part of the instanton moduli space $\mathcal{M}_k$. Both approaches should provide the same result [155].

In addition, each link $i \to j$ between two nodes i and j represent a chiral multiplet of matter fields in the bifundamental representation of the gauge group $U(m^{(i)}) \times U(m^{(j)})$, with mass $\mu_{ij} \in \mathbb{C}$. For linear quivers, all bifundamental masses can be set to q_3^{-1} by a rescaling of the Coulomb branch vevs.

The instantons contribution to the gauge theories partition function is expressed as a sum over the content of r $m^{(i)}$-tuple Young diagrams $\boldsymbol{\lambda}^{(i)}$ describing the configuration of instantons at the ith node. Each term can be further decomposed into the contributions of vector (gauge) multiplets, bifundamental chiral (matter) multiplets, and Chern-Simons factors:

$$\mathcal{Z}_{\text{inst.}} = \sum_{\boldsymbol{\lambda}^{(i)}} \prod_{i=1}^{r} \left(\prod_{\omega \in \mathbb{Z}_p} \mathfrak{q}_{\omega,i}^{|K_\omega(\boldsymbol{\lambda}^{(i)})|} \mathcal{Z}_{\text{vect.}}(\boldsymbol{v}^{(i)}, \boldsymbol{\lambda}^{(i)}) \mathcal{Z}_{\text{CS}}(\boldsymbol{\kappa}^{(i)}, \boldsymbol{\lambda}^{(i)}) \right) \prod_{i \to j} \mathcal{Z}_{\text{bfd.}}(\boldsymbol{v}^{(i)}, \boldsymbol{\lambda}^{(i)}, \boldsymbol{v}^{(j)}, \boldsymbol{\lambda}^{(j)} | \mu_{ij}),$$
$$\text{with} \quad \mathcal{Z}_{\text{vect.}}(\boldsymbol{v}, \boldsymbol{\lambda}) = N(\boldsymbol{v}, \boldsymbol{\lambda} | \boldsymbol{v}, \boldsymbol{\lambda})^{-1}, \quad \mathcal{Z}_{\text{bfd.}}(\boldsymbol{v}, \boldsymbol{\lambda}, \boldsymbol{v}', \boldsymbol{\lambda}' | \mu) = N(\boldsymbol{v}, \boldsymbol{\lambda} | \mu \boldsymbol{v}', \boldsymbol{\lambda}'), \quad \mathcal{Z}_{\text{CS}}(\boldsymbol{\kappa}, \boldsymbol{\lambda}) = \prod_{\square \in \boldsymbol{\lambda}} \chi_{\square}^{\kappa_{c(\square)}}. \tag{6.2.11}$$

Vector and bifundamental contributions are written in terms of the Nekrasov factor $N(\boldsymbol{v}, \boldsymbol{\lambda} | \mu \boldsymbol{v}', \boldsymbol{\lambda}')$. For a better readability, we drop the node indices in the following, and simply distinguish the two nodes involved in the definition of the Nekrasov factor with a prime. In order to write down the expression of $N(\boldsymbol{v}, \boldsymbol{\lambda} | \mu \boldsymbol{v}', \boldsymbol{\lambda}')$ given in [15], we need to introduce the equivariant character $M_{\boldsymbol{v}}$ and $K_{\boldsymbol{\lambda}}$ of the vector spaces M and K associated to each node,

$$M_{\boldsymbol{v}} = \sum_{\alpha=1}^{m} e^{Ra_\alpha}, \quad K_{\boldsymbol{\lambda}} = \sum_{\square \in \boldsymbol{\lambda}} e^{R\phi_\square}, \tag{6.2.12}$$

A linear involutive operation $*$ acts on such characters by flipping the sign of R: $(e^{Ra_\alpha})^* = e^{-Ra_\alpha}$, $(q_1^*, q_2^*) = (q_1^{-1}, q_2^{-1})$ and thus $(e^{R\phi_\square})^* = e^{-R\phi_\square}$ (see [10, 14, 15] for more details on these notations). Introducing $S_{\boldsymbol{\lambda}} = M - P_{12} K_{\boldsymbol{\lambda}}$ with $P_{12} = (1 - q_1)(1 - q_2)$, the Nekrasov factor writes

$$N(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}')=\mathbb{I}\left[\frac{M_{\boldsymbol{v}}M_{\boldsymbol{v}'}^*-S_{\boldsymbol{\lambda}}S_{\boldsymbol{\lambda}'}^*}{P_{12}^*}\right]^{\mathbb{Z}_p}=\mathbb{I}\left[M_{\boldsymbol{v}}K_{\boldsymbol{\lambda}'}^*+q_3^{-1}M_{\boldsymbol{v}'}^*K_{\boldsymbol{\lambda}}-P_{12}K_{\boldsymbol{\lambda}}K_{\boldsymbol{\lambda}'}^*\right]^{\mathbb{Z}_p}, \quad (6.2.13)$$

where the $\mathbb{I}$-symbol is the equivariant index functor,

$$\mathbb{I}\left[\sum_{i\in I_+}e^{Rw_i}-\sum_{i\in I_-}e^{Rw_i}\right]=\frac{\prod_{i\in I_+}1-e^{Rw_i}}{\prod_{i\in I_-}1-e^{Rw_i}}, \qquad (6.2.14)$$

and $[\cdots]^{\mathbb{Z}_p}$ denotes the operation of keeping only the $\mathbb{Z}_p$-invariant parts. In particular, the RHS of (6.2.13) involves a coloring function $c:\mathbb{Z}[a_\alpha,\varepsilon_1,\varepsilon_2]\to\mathbb{Z}_p$ defined on weights w_i as the linear map taking the values $c(a_\alpha)=c_\alpha$, $c(\varepsilon_1)=\nu_1$ and $c(\varepsilon_2)=\nu_2$ so that $c(\phi_\square)=c(\square)$ (justifying our slight abuse of notations). The $[\cdots]^{\mathbb{Z}_p}$ projects on $\mathbb{Z}_p$-invariant factors.

Replacing the equivariant characters by their expressions 6.2.12, the Nekrasov factor can be written in a more explicit form,

$$N(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}')=\prod_{\substack{\square\in\boldsymbol{\lambda}\\ \blacksquare\in\boldsymbol{\lambda}'}}S_{c(\square)c(\blacksquare)}(\chi_\square/\chi_\blacksquare)\times\prod_{\square\in\boldsymbol{\lambda}}\prod_{\alpha\in C_{\bar{c}(\square)}(m')}\left(1-\frac{\chi_\square}{q_3v'_\alpha}\right)\times\prod_{\square\in\boldsymbol{\lambda}'}\prod_{\alpha\in C_{c(\square)}(m)}\left(1-\frac{v_\alpha}{\chi_\square}\right). \qquad (6.2.15)$$

The function $S_{\omega\omega'}(z)$ is sometimes called the *scattering function*, it carries two color indices ω,ω':

$$S_{\omega\omega'}(z)=\frac{(1-q_1z)^{\delta_{\omega,\omega'-\nu_1}}(1-q_2z)^{\delta_{\omega,\omega'-\nu_2}}}{(1-z)^{\delta_{\omega,\omega'}}(1-q_1q_2z)^{\delta_{\omega,\omega'-\nu_1-\nu_2}}}. \qquad (6.2.16)$$

In this expression, the non-zero matrix elements have been expressed in a compact way using the delta function $\delta_{\omega,\omega'}$ defined modulo p (i.e. $\delta_{\omega,\omega'}=1$ iff $\omega=\omega'$ modulo p, zero otherwise). In fact, $S_{\omega\omega'}(z)$, and more generally all the matrices of size $p\times p$ with indices ω,ω' appearing in thischapter, are circulant matrices: their matrix elements only depend on the difference $\omega-\omega'$ of row and column indices. In particular, $S_{\omega+\nu\ \omega'}(z)=S_{\omega\ \omega'-\nu}(z)$ for all $\nu\in\mathbb{Z}_p$.

Finally, the function $S_{\omega\omega'}(z)$ satisfies a sort of *crossing symmetry*,

$$S_{\omega\bar{\omega}'}(q_3/z) = f_{\omega\omega'}(z) S_{\omega'\omega}(z), \tag{6.2.17}$$

with the function $f_{\omega\omega'}(z) = F_{\omega\omega'} z^{\beta_{\omega\omega'}}$ defined by[3]

$$\beta_{\omega\omega'} = \delta_{\omega\omega'} + \delta_{\omega\omega'+\nu_1+\nu_2} - \delta_{\omega\omega'+\nu_1} - \delta_{\omega\omega'+\nu_2}, \quad F_{\omega\omega'} = (-1)^{\delta_{\omega\omega'}} (-q_3)^{-\delta_{\omega,\omega'-\nu_3}} (-q_1)^{-\delta_{\omega\omega'+\nu_1}} (-q_2)^{-\delta_{\omega\omega'+\nu_2}}. \tag{6.2.18}$$

6.2.3 $\mathcal{Y}$-observables

A new class of BPS-observables for supersymmetric gauge theories was introduced in [10], they are called *qq-characters*. As the name suggests, they correspond to a natural deformation of the q-characters of Frenkel-Reshetikhin [12] from the gauge theory point of view [9]. They were defined in [10] as particular combinations of chiral ring observables in such a way that their expectation values exhibit an important regularity property [10, 14]. This regularity property encodes an infinite set of constraints called *non-perturbative Dyson-Schwinger equations*. From a different viewpoint, qq-characters in 5d gauge theories can also be defined in terms of Wilson loops [156] (see also [157] for a string theory perspective).[4]

The qq-characters are half-BPS observables written as combinations of $\mathcal{Y}$*-observables*.[5] In the case of a $\mathbb{Z}_p$-orbifold, it is natural to introduce two inequivalent $\mathcal{Y}$-observables $\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z) = \mathbb{I}\left[e^{-R\zeta} S_{\boldsymbol{\lambda}}\right]^{\mathbb{Z}_p}$ and $\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]*}(z) = \mathbb{I}\left[e^{R\zeta} S_{\boldsymbol{\lambda}}^*\right]^{\mathbb{Z}_p}$ where $z = e^{R\zeta}$ and $c(\zeta) = \omega$. These two observables

[3]The function $f_{\omega\omega'}(z)$ also controls the asymptotics of the scattering function since $S_{\omega\omega'}(z) \underset{0}{\sim} 1$ and $S_{\omega\omega'}(z) \underset{\infty}{\sim} f_{\omega'\omega}(z)^{-1}$. It obeys an important reflection symmetry $f_{\bar{\omega}\omega'}(q_3/z) = f_{\omega'\omega}(z)^{-1}$ coming from $F_{\omega\omega'} F_{\bar{\omega}'\omega} = q_3^{-\beta_{\omega\omega'}}$ and $\beta_{\bar{\omega}'\omega} = \beta_{\omega\omega'}$.

[4]In [21], the qq-characters of 4d $\mathcal{N} = 2$ gauge theories with the insertion of surface defects were considered. In this case, the non-perturbative Dyson-Schwinger equations produce either Knizhnik - Zamolodchikov equations or BPZ equations that are satisfied by the surface defect partition functions [32, 3]. These surface defect partition functions were investigated in the context of Bethe/gauge correspondence in [2], and in their relation to the oper submanifold of the moduli space of flat connections on Riemann surfaces in [1].

[5]Note that we use slightly different notations for the observables in the five-dimensional theories, compared to the four-dimensional counterparts which appeared in the Part I. Namely, the Y-observable is denoted as $\mathscr{Y}(x)$ in 4d and $\mathcal{Y}(z)$ in 5d. In the same way, the qq-character is denoted as $\mathscr{X}(x)$ in 4d and $\mathcal{X}(z)$ in 5d.

encode the recursion relations satisfied by Nekrasov factors,

$$\frac{N(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}'+\square)}{N(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}')}=\mathcal{Y}_{c(\square)}^{[\boldsymbol{\lambda}]}(\chi_\square)\quad\frac{N(\boldsymbol{v},\boldsymbol{\lambda}+\square|\boldsymbol{v}',\boldsymbol{\lambda}')}{N(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}')}=\mathcal{Y}_{\bar{c}(\square)}^{[\boldsymbol{\lambda}']*}(q_3^{-1}\chi_\square),\tag{6.2.19}$$

Replacing the equivariant characters with the expressions 6.2.12, we find the explicit formulas

$$\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)=\prod_{\alpha\in C_\omega(m)}(1-v_\alpha/z)\times\prod_{\square\in\boldsymbol{\lambda}}S_{c(\square)\omega}(\chi_\square/z),\quad\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]*}(z)=\prod_{\alpha\in C_\omega(m)}(1-z/v_\alpha)\prod_{\square\in\boldsymbol{\lambda}}S_{\omega\bar{c}(\square)}(q_3z/\chi_\square).\tag{6.2.20}$$

Due to the crossing symmetry 6.2.17, these two $\mathcal{Y}$-observables satisfy the relation

$$\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]*}(z)=f_\omega^{[\boldsymbol{\lambda}]}(z)\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z),\tag{6.2.21}$$

with

$$f_\omega^{[\boldsymbol{\lambda}]}(z)=\prod_{\alpha\in C_\omega(m)}(-z/v_\alpha)\prod_{\square\in\boldsymbol{\lambda}}f_{\omega c(\square)}(\chi_\square/z).\tag{6.2.22}$$

It will allow us to express all the equations below in terms of $\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)$ only.[6] Furthermore, the $\mathcal{Y}$-observables possess an alternative expression following from the *shell formula* derived in appendix H,

$$\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)=\frac{\prod_{\square\in A_\omega(\boldsymbol{\lambda})}(1-\chi_\square/z)}{\prod_{\square\in R_{\omega-\nu_1-\nu_2}(\boldsymbol{\lambda})}(1-\chi_\square/(q_3z))},\quad f_\omega^{[\boldsymbol{\lambda}]}(z)=\frac{\prod_{\square\in R_{\omega-\nu_1-\nu_2}(\boldsymbol{\lambda})}(-\chi_\square/(q_3z))}{\prod_{\square\in A_\omega(\boldsymbol{\lambda})}(-\chi_\square/z)}.\tag{6.2.23}$$

Here, the sets $A_\omega(\boldsymbol{\lambda})$ and $R_\omega(\boldsymbol{\lambda})$ denote respectively the set of boxes of color ω that can be added to or removed from the m-tuple Young diagram $\boldsymbol{\lambda}$. This expression arises from the cancellations of contributions by neighboring boxes, it plays an essential role in the definition of the vertical representation of the algebra.

[6] The presence of the function $f_\omega^{[\boldsymbol{\lambda}]}(z)$ can be interpreted as follows. Note that $\mathbb{I}(X^*)=(-1)^{\mathrm{rk}X^*}\det X^*\ \mathbb{I}(X)$, for $X=\sum_{i\in I_+}e^{Rw_i}-\sum_{i\in I_-}e^{Rw_i}$, $\mathrm{rk}X=|I_+|-|I_-|$, and $\det X=\prod_{i\in I_+}e^{Rw_i}/\prod_{i\in I_-}e^{Rw_i}$. Applying this reflection relation to $X=\left[e^{-R\zeta}S_{\boldsymbol{\lambda}}\right]^{\mathbb{Z}_p}$, we recover the relation (6.2.21) with $f_\omega^{[\boldsymbol{\lambda}]}(z)$ given in (6.2.22) identified with $(-1)^{\mathrm{rk}X^*}\det X^*$.

6.3 New quantum toroidal algebras

In order to reconstruct the instanton partition functions on the general orbifold 6.2.4, the definition of a new quantum toroidal algebra is necessary. In addition to the complex parameters q_1, q_2 and the rank $p \in \mathbb{Z}^{>0}$, this algebra will depend on the integers (ν_1, ν_2) modulo $\mathbb{Z}_p$. Taking $\nu_1 = -\nu_2 = 1$, the $\mathbb{Z}_p$-action 6.2.4 reduces to the standard action defining ALE spaces. Thus, in this limit the (ν_1, ν_2)-deformed algebra should reduce to the quantum toroidal algebra of $\mathfrak{gl}(p)$. In fact, this is true only up to a twist in the definition of the Drinfeld currents (see the subsection G.4 of the appendix). A brief reminder on the quantum toroidal algebra of $\mathfrak{gl}(p)$ is given in appendix G, it includes its two main representations called, in the gauge theory context, *vertical* and *horizontal* representations.

The key ingredient to define the deformation of the quantum toroidal algebra of $\mathfrak{gl}(p)$ is the scattering function $S_{\omega\omega'}(z)$ defined in 6.2.16. Indeed, this function plays an essential role in the two elementary representations involved in the algebraic engineering of partition functions and qq-characters. In the vertical representation, it enters through the definition 6.2.20 of the $\mathcal{Y}$-observables that describe the recursion relations among Nekrasov factors. Instead, in the horizontal representation, it expresses the normal-ordering relations between vertex operators. Thus, from the physics perspective, the scattering function is the natural object to consider for the deformation of the algebra. Moreover, through the crossing symmetry relation 6.2.17, this function defines the $p \times p$ matrix $\beta_{\omega\omega'}$ that could be identified with the underlying Cartan matrix of the deformed quantum toroidal algebra (see the subsection G.4). Note that the matrix $\beta_{\omega\omega'}$ naturally reduces to the generalized Cartan matrix of the Kac-Moody algebra $\widehat{\mathfrak{gl}}_p$ when $\nu_1 = -\nu_2 = 1$. In general, it is non-symmetrizable, yet, like in the case of $\widehat{\mathfrak{gl}}_p$, it is a circulant matrix. Its eigenvectors $v_j = (1, \Omega_j, \Omega_j^2, \cdots, \Omega_j^{p-1})$ are written in terms of the pth root of unity $\Omega_j = e^{2i\pi j/p}$, and the corresponding eigenvalues read

$$\lambda_j = -4e^{i\pi\nu_3 j/p} \sin(\pi\nu_1 j/p) \sin(\pi\nu_2 j/p). \tag{6.3.1}$$

In particular, the eigenvector $v_0 = (1, 1, \cdots, 1)$ has the eigenvalue zero which relates $\beta_{\omega\omega'}$ to the Cartan matrix of affine Lie algebras, and thus justifies the designation *toroidal* of the deformed algebra.

6.3.1 Definition of the algebra

Like in the case of $\mathfrak{gl}(p)$, the (ν_1, ν_2)-deformed quantum toroidal algebra is defined in terms of a central element c and $4p$ Drinfeld currents, denoted $x^\pm_\omega(z)$ and $\psi^\pm_\omega(z)$, with $\omega \in \mathbb{Z}_p$. The currents $\psi^\pm_\omega(z)$ (together with c) generate the Cartan subalgebra, while the currents $x^\pm_\omega(z)$ deform the notion of Chevalley generators e_ω, f_ω. The algebraic relations obeyed by the currents resemble those defining the quantum toroidal algebra of $\mathfrak{gl}(p)$ in G.1.3, the main difference being the presence of shifts in the indices ω by the product $\nu_3 c$:[7]

$$\begin{aligned}
&x^\pm_\omega(z)x^\pm_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm1}x^\pm_{\omega'}(w)x^\pm_\omega(z), \quad \psi^+_\omega(z)x^\pm_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm1}x^\pm_{\omega'}(w)\psi^+_\omega(z),\\
&\psi^-_\omega(z)x^+_{\omega'}(w) = g_{\omega-\nu_3c\ \omega'}(q_3^{-c}z/w)x^+_{\omega'}(w)\psi^-_\omega(z), \quad \psi^-_\omega(z)x^-_{\omega'}(w) = g_{\omega\omega'}(z/w)^{-1}x^-_{\omega'}(w)\psi^-_\omega(z),\\
&\psi^+_\omega(z)\psi^-_{\omega'}(w) = \frac{g_{\omega\omega'-\nu_3c}(q_3^c z/w)}{g_{\omega\omega'}(z/w)}\psi^-_{\omega'}(w)\psi^+_\omega(z), \quad [\psi^\pm_\omega(z), \psi^\pm_{\omega'}(w)] = 0,\\
&[x^+_\omega(z), x^-_{\omega'}(w)] = \Omega\left[\delta_{\omega,\omega'}\delta(z/w)\psi^+_\omega(z) - \delta_{\omega,\omega'-\nu_3c}\delta(q_3^c z/w)\psi^-_{\omega+\nu_3c}(q_3^c z)\right].
\end{aligned} \tag{6.3.2}$$

In the last relation $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$ denotes the multiplicative Dirac delta function and we introduced the complex parameter

$$\Omega = \frac{(1-q_1)^{\delta_{\nu_1,0}}(1-q_2)^{\delta_{\nu_2,0}}}{(1-q_1q_2)^{\delta_{\nu_1+\nu_2,0}}}F^{1/2}, \quad F = F_{\omega\omega} = -\prod_i(-q_i)^{-\delta_{\nu_i,0}}. \tag{6.3.3}$$

[7] Comparing with the standard definition of quantum toroidal algebras, the Drinfeld currents have been redefined as follows: $x^\pm(z) \to x^\pm(q_3^{\pm c/4}z)$, $\psi^+_\omega(z) \to \psi^+_\omega(z)$ and $\psi^-_\omega(z) \to \psi^-_\omega(q_3^{-c/2}z)$. This redefinition makes the coincidence between shifts of indices $\omega \pm \nu_3 c$ and spectral parameters $zq_3^{\pm c}$ manifest. In fact, this asymmetric form of the algebraic relations appears naturally in the construction of a central extension of the Yangian double [136].

The other relations in 6.3.2 involve the *structure function* $g_{\omega\omega'}(z)$ defined as a ratio of two scattering functions. This function depends on the variables $(q_1, q_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$ and the integers $(\nu_1, \nu_2) \in \mathbb{Z}_p \times \mathbb{Z}_p$:

$$g_{\omega\omega'}(z) = \frac{S_{\omega\omega'}(z)}{S_{\omega'\omega}(z^{-1})} = f_{\omega\omega'}(z^{-1}) \prod_{i=1,2,3} \frac{(1-q_i z)^{\delta_{\omega,\omega'-\nu_i}}}{(1-q_i^{-1}z)^{\delta_{\omega,\omega'+\nu_i}}}, \tag{6.3.4}$$

where the extra variables q_3 and ν_3 obey $q_1 q_2 q_3 = 1$ and $\nu_1 + \nu_2 + \nu_3 = 0$. Note that the invariance under the S_3-permutation of indices (ν_i, q_i) is broken to S_2 corresponding to exchange (ν_1, q_1) and (ν_2, q_2). The structure function satisfies the property $g_{\omega\omega'}(z) g_{\omega'\omega}(z^{-1}) = 1$ necessary for the definiteness of the algebraic relations.

The algebraic relations 6.3.2 are expected to include additional Serre relations. However, the Drinfeld currents employed here are a twisted version of those used in the formulation of the quantum toroidal algebra of $\mathfrak{gl}(p)$. This explains why the function $g_{\omega\omega'}(z)$ defined in 6.3.4 does not quite reproduce the $\mathfrak{gl}(p)$ structure function G.1.6 as we set $\nu_1 = -\nu_2 = 1$. Even in the case of $\mathfrak{gl}(p)$, the twist of the currents make the derivation of Serre relations difficult. We hope to come back to this question in the near future.

Due to the non-trivial power of z in the asymptotics of the functions $g_{\omega\omega'}(z)$, namely

$$g_{\omega\omega'}(z) \underset{0}{\sim} f_{\omega\omega'}(z^{-1}), \quad g_{\omega\omega'}(z) \underset{\infty}{\sim} f_{\omega'\omega}(z)^{-1}, \tag{6.3.5}$$

the Cartan currents $\psi^\pm_\omega(z)$ cannot be expanded in powers of $z^{\mp k}$ with $k > 0$ as it is usually the case for quantum groups. Instead, it is necessary to introduce a *zero modes* part using extra operators $a^\pm_{\omega,0}$:

$$x^\pm_\omega(z) = \sum_{k\in\mathbb{Z}} z^{-k} x^\pm_{\omega,k}, \quad \psi^\pm_\omega(z) = \psi^\pm_{\omega,0} z^{\mp a^\pm_{\omega,0}} \exp\left(\pm \sum_{k>0} z^{\mp k} a_{\omega,\pm k}\right). \tag{6.3.6}$$

In the appendix J, the operators $\psi_{\omega,0} z^{\mp a^\pm_{\omega,0}}$ are constructed as a specific combination of grading operators. The Cartan zero modes $\psi^\pm_{\omega,0}$ are invertible, they can be used to define another central element $\bar{c}$ setting

$$q_3^{-\bar{c}} = \left(\prod_{\omega\in\mathbb{Z}_p} \psi_{\omega,0}^{+}\right)\left(\prod_{\omega\in\mathbb{Z}_p} \psi_{\omega,0}^{-}\right)^{-1} = \prod_{\substack{\omega,\omega'=0\\ \omega\leq\omega'}}^{p-1} \frac{F_{\omega'\omega}}{F_{\omega'\omega+\nu_3 c}} \prod_{\omega\in\mathbb{Z}_p} \psi_{\omega,0}^{+}(\psi_{\omega,0}^{-})^{-1}. \tag{6.3.7}$$

Note that the ordering of the zero modes is important since they do not commute. It is chosen here such that the expression of the coproduct defined below simplifies.

Coalgebraic structure A Hopf algebra $\mathcal{A}$ over the field $\mathbb{C}$ is a $\mathbb{C}$-module equipped with a unit $1_{\mathcal{A}}$, a product ∇, a counit ε, a coproduct Δ and an antipode S satisfying the following properties [158].

- $\mathcal{A}$ is both an algebra and a coalgebra. This implies the property $\nabla(1\otimes\varepsilon)\Delta = \nabla(\varepsilon\otimes 1)\Delta = 1$ and the coassociativity of the coproduct $(1\otimes\Delta)\Delta = (\Delta\otimes 1)\Delta$.
- The counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ and the coproduct $\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ are homomorphisms of algebras. The compatibility with the scalar multiplication and the addition are trivially satisfied. On the other hand, the compatibility with the product requires to verify $\varepsilon(ee') = \varepsilon(e)\varepsilon(e')$ and $\Delta(e)\Delta(e') = \Delta(ee')$ for any two elements $e, e' \in \mathcal{A}$.
- The unit $1_{\mathcal{A}} : \mathbb{C} \to \mathcal{A}$ and product $\nabla : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ are homomorphisms of coalgebras. This means $\Delta(1_{\mathcal{A}}) = 1_{\mathcal{A}}\otimes 1_{\mathcal{A}}$, $\varepsilon(1_{\mathcal{A}}) = 1$, and, once again, $\Delta(e)\Delta(e') = \Delta(ee')$.
- The antipode $S : \mathcal{A} \to \mathcal{A}$ is a bijective $\mathbb{C}$-module map satisfying $\nabla(S\otimes 1)\Delta = \varepsilon = \nabla(1\otimes S)\Delta$.

The algebra 6.3.2 is a Hopf algebra with the coproduct, counit and antipode given by

$$
\begin{aligned}
&\Delta(x_\omega^+(z)) = x_\omega^+(z)\otimes 1 + \psi^-_{\omega+\nu_3 c_{(1)}}(q_3^{c_{(1)}}z)\otimes x_\omega^+(z),\\
&\Delta(x_\omega^-(z)) = x_\omega^-(z)\otimes \psi^+_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z) + 1\otimes x^-_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z),\\
&\Delta(\psi_\omega^+(z)) = \psi_\omega^+(z)\otimes \psi^+_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z),\quad \Delta(\psi_\omega^-(z)) = \psi^-_{\omega-\nu_3 c_{(2)}}(q_3^{-c_{(2)}}z)\otimes \psi^-_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z),\\
&S(x_\omega^+(z)) = -\psi^-_{\omega+\nu_3 c}(q_3^c z)^{-1}x_\omega^+(z),\quad S(x_\omega^-(z)) = -x^-_{\omega+\nu_3 c}(q_3^c z)\psi^+_{\omega+\nu_3 c}(q_3^c z)^{-1},\quad \varepsilon(x_\omega^\pm(z))=0,\\
&S(\psi_\omega^+(z)) = \psi^+_{\omega+\nu_3 c}(q_3^c z)^{-1},\quad S(\psi_\omega^-(z)) = \psi^-_{\omega+2\nu_3 c}(q_3^{2c} z)^{-1},\quad \varepsilon(\psi_\omega^\pm(z))=1,
\end{aligned}
\tag{6.3.8}
$$

with the standard notation $c_{(1)} = c\otimes 1$, $c_{(2)} = 1\otimes c$. The central element c obeys $\Delta(c) = c_{(1)} + c_{(2)}$, $S(c) = -c$ and $\varepsilon(c) = 0$. The proof is a tedious but straightforward calculation that the axioms defining a Hopf algebra hold for any pair of currents. The antipode is an anti-homomorphism of algebra, it satisfies $S^2 = (-1)^{1+\varepsilon}\mathrm{Id}$. Using the coproduct of the Cartan zero modes $\psi^\pm_{\omega,0}$, it is possible to compute the coproduct of the central charge $\bar{c}$ defined in 6.3.7, we find[8]

$$
\Delta(q_3^{-\bar{c}}) = (q_3^{-\bar{c}}\otimes q_3^{-\bar{c}})\left(q_3^{c_{(2)}\sum_{\omega\in\mathbb{Z}_p} a^-_{\omega,0}}\otimes q_3^{c_{(1)}\sum_{\omega\in\mathbb{Z}_p} a^+_{\omega,0}}q_3^{c_{(1)}\sum_{\omega\in\mathbb{Z}_p} a^-_{\omega,0}}\right). \tag{6.3.10}
$$

In order to reconstruct the instanton partition functions, we need to introduce two types of representations: a vertical representation $\rho^{(V)}$ with level $c=0$ and a horizontal representation $\rho^{(H)}$ with level $c=1$. Such representations are already known in the case of quantum toroidal algebras of $\mathfrak{gl}(p)$ (see [159, 160], or the brief summary presented in appendix G), but also for the quantum toroidal $\mathfrak{gl}(1)$ algebra (or Ding-Iohara-Miki algebra [133, 134]) [161, 162]. In fact, there are two different point of view concerning these representations.

[8] The extra factor in the RHS comes from the shifts of the currents' arguments in the coproduct that brings

$$
\Delta(\psi^+_{\omega,0}) = \psi^+_{\omega,0}\otimes q_3^{2c_{(1)}a^+_{\omega-\nu_3 c_{(1)}}}\psi^+_{\omega-\nu_3 c_{(1)},0},\quad \Delta(\psi^-_{\omega,0}) = \psi^-_{\omega-\nu_3 c_{(2)},0}q_3^{-2c_{(2)}a^-_{\omega-\nu_3 c_{(2)}}}\otimes \psi^-_{\omega-\nu_3 c_{(1)},0}q_3^{-2c_{(1)}a^-_{\omega-\nu_3 c_{(1)}}}. \tag{6.3.9}
$$

In the mathematics literature [161, 162, 159], one often considers a single module, the *Fock module*, and present the action of two subalgebras called horizontal and vertical. Miki's automorphism $\mathcal{S}$ [163, 134] exchanges the two subalgebras, allowing us to define (for instance) $\rho^{(H)} = \rho^{(V)} \circ \mathcal{S}$. On the opposite, physicists usually introduce two different types of modules referred as vertical and horizontal modules, somehow fixing the choice of subalgebra. Of course, the modules are isomorphic thanks to Miki's automorphism and the two point of views are equivalent [143]. However, no analogue of Miki's automorphism is known yet for the (ν_1, ν_2)-deformed algebra. Thus, at this stage, we have no choice but to follow the second approach and define two distinct representations. This will be done in the next two subsections.

6.3.2 Vertical representation

The vertical representation presented here is a deformation of the Fock representation for the quantum toroidal algebra of $\mathfrak{gl}(p)$ [159] (see appendix G.3). This representation is similar to the usual finite dimensional representations of quantum groups. Indeed, the Cartan currents $\psi_\omega^\pm(z)$ are diagonal on a set of weight vectors. The currents $x_\omega^-(z)$ annihilates the highest weight (or vacuum) $|\varnothing\rangle\rangle$, and $x_\omega^+(z)$ creates excitations. However, the weight vectors are labeled here by the box configurations of an m-tuple Young diagrams $\boldsymbol{\lambda}$. Thus, this representation is infinite dimensional, yet it is graded by the total number of boxes $|\boldsymbol{\lambda}|$.

From the gauge theory perspective, the vertical representation of the algebra 6.3.2 describes the relation between sectors of different instanton numbers. Thus, vertical modules are characterized by a basis of states $|\boldsymbol{\lambda}\rangle\rangle$ labeled by instanton configurations. Accordingly, the representation depend on a set of m (highest) weights $\boldsymbol{v} = (v_\alpha)_{\alpha=1\cdots m}$ and a choice of color c_α for each weight. This coloring defines the integers $m_\omega = |C_\omega(m)|$ corresponding to the number of weights v_α of color ω. The integers m_ω provide the levels of the vertical representation: $\rho^{(V)}(c) = 0$ and $\rho^{(V)}(\bar{c}) = m$ with $m = \sum_{\omega\in\mathbb{Z}_p} m_\omega$. As mentioned previously, the Cartan currents $\psi_\omega^\pm(z)$ are diagonal on the basis $|\boldsymbol{\lambda}\rangle\rangle$. On the other hand, the operators

$x_\omega^\pm(z)$ relate the sectors of instanton charge $|\boldsymbol{\lambda}|$ and $|\boldsymbol{\lambda}|\pm 1$ by adding/removing a box to the m-tuple Young diagram $\boldsymbol{\lambda}$. Their action encodes the recursion relation 6.2.19 obeyed by Nekrasov factors [164]. The action of the Drinfeld currents on the states $|\boldsymbol{\lambda}\rangle\!\rangle$ is derived in appendix I.1, it reads[9]

$$\begin{aligned}
&\rho^{(V)}(x_\omega^+(z))\,|\boldsymbol{\lambda}\rangle\!\rangle = F^{1/2}\sum_{\square\in A_\omega(\boldsymbol{\lambda})}\delta(z/\chi_\square)\operatorname*{Res}_{z=\chi_\square} z^{-1}\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)^{-1}\,|\boldsymbol{\lambda}+\square\rangle\!\rangle,\\
&\rho^{(V)}(x_\omega^-(z))\,|\boldsymbol{\lambda}\rangle\!\rangle = \mathring{f}_{\bar\omega}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)\sum_{\square\in R_\omega(\boldsymbol{\lambda})}\delta(z/\chi_\square)\operatorname*{Res}_{z=\chi_\square} z^{-1}\mathcal{Y}_{\bar\omega}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)\,|\boldsymbol{\lambda}-\square\rangle\!\rangle,\\
&\rho^{(V)}(\psi_\omega^\pm(z))\,|\boldsymbol{\lambda}\rangle\!\rangle = \left[\Psi_\omega^{[\boldsymbol{\lambda}]}(z)\right]_\pm |\boldsymbol{\lambda}\rangle\!\rangle.
\end{aligned}\tag{6.3.12}$$

In the first two lines, $A_\omega(\boldsymbol{\lambda})$ and $R_\omega(\boldsymbol{\lambda})$ correspond respectively to the set of boxes of color ω that can be added to or removed from $\boldsymbol{\lambda}$. In the last line, the subscript $\pm$ denotes the expansion of the function $\Psi_\omega^{[\boldsymbol{\lambda}]}(z)$ for $|z|^{\pm1}\to\infty$. This function is written as a ratio of the $\mathcal{Y}$-observables defined in 6.2.20,

$$\Psi_\omega^{[\boldsymbol{\lambda}]}(z) = \mathring{f}_{\bar\omega}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)\frac{\mathcal{Y}_{\bar\omega}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)}{\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)},\quad \text{with}\quad \mathring{f}_\omega^{[\boldsymbol{\lambda}]}(z) = f_\omega^{[\boldsymbol{\lambda}]}(z)\prod_{\alpha\in C_\omega(m)}(-v_\alpha/z) = \prod_{\square\in\boldsymbol{\lambda}} f_{\omega c(\square)}(\chi_\square/z).\tag{6.3.13}$$

We notice that the highest weights are still encoded in the form of a Drinfeld polynomial $p_\omega(z)$:

$$\Psi_\omega^{[\varnothing]}(z) = z^{m_\omega-m_{\bar\omega}}\frac{\prod_{\alpha\in C_{\bar\omega}(m)}(-q_3v_\alpha)}{\prod_{\alpha\in C_\omega(m)}(-v_\alpha)}\frac{p_{\bar\omega}(q_3^{-1/2}z)}{p_\omega(q_3^{1/2}z)},\quad \text{with}\quad p_\omega(z) = \prod_{\alpha\in C_\omega(m)}(1-q_3^{-1/2}z/v_\alpha).\tag{6.3.14}$$

When $\nu_3=0$, we have $\bar\omega=\omega$ and the prefactor reduces to the usual expression $q_3^{m_\omega}$ where $m_\omega=\deg p_\omega(z)$.

The functions $f_\omega^{[\boldsymbol{\lambda}]}(z)$ and $\mathring{f}_\omega^{[\boldsymbol{\lambda}]}(z)$ controls the asymptotics of the functions $\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)$ and

---

[9] The definition of the vertical representation is not unique due, for instance, to the following invariance of Drinfeld currents at $c=0$:

$$\psi_\omega^\pm(z)\to C_\omega z^{\alpha_\omega}\psi_\omega^\pm(z),\quad x_\omega^+(z)\to x_\omega^+(z),\quad x_\omega^-(z)\to C_\omega z^{\alpha_\omega}x_\omega^-(z).\tag{6.3.11}$$

Here a particular choice is made to simplify the derivation of intertwiners in section 4 below.

$\Psi_\omega^{[\boldsymbol{\lambda}]}(z)$,

$$\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z) \underset{\infty}{\sim} 1, \quad \mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z) \underset{0}{\sim} f_\omega^{[\boldsymbol{\lambda}]}(z)^{-1} \quad \Rightarrow \quad \Psi_\omega^{[\boldsymbol{\lambda}]}(z) \underset{0}{\sim} f_\omega^{[\boldsymbol{\lambda}]}(z) \frac{\mathring{f}_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)}{f_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)}, \quad \Psi_\omega^{[\boldsymbol{\lambda}]}(z) \underset{\infty}{\sim} \mathring{f}_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z). \tag{6.3.15}$$

As a result, the action of the zero-modes of the Cartan currents read

$$\begin{aligned}
&\rho^{(V)}(\psi_{\omega,0}^+)\,|\boldsymbol{\lambda}\rangle\!\rangle = \mathring{f}_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1})\,|\boldsymbol{\lambda}\rangle\!\rangle, \quad \rho^{(V)}(\psi_{\omega,0}^-)\,|\boldsymbol{\lambda}\rangle\!\rangle = \mathring{f}_\omega^{[\boldsymbol{\lambda}]}(1)\frac{\prod_{\alpha\in C_{\bar{\omega}}(m)}(-q_3 v_\alpha)}{\prod_{\alpha\in C_\omega(m)}(-v_\alpha)}\,|\boldsymbol{\lambda}\rangle\!\rangle, \\
&\rho^{(V)}(a_{\omega,0}^+)\,|\boldsymbol{\lambda}\rangle\!\rangle = \left(\sum_{\square\in\boldsymbol{\lambda}}\beta_{\bar{\omega}c(\square)}\right)|\boldsymbol{\lambda}\rangle\!\rangle, \quad \rho^{(V)}(a_{\omega,0}^-)\,|\boldsymbol{\lambda}\rangle\!\rangle = \left(m_\omega - m_{\bar{\omega}} - \sum_{\square\in\boldsymbol{\lambda}}\beta_{\omega c(\square)}\right)|\boldsymbol{\lambda}\rangle\!\rangle
\end{aligned} \tag{6.3.16}$$

The value of the second central charge is obtained by taking the product over ω, we recover $\rho^{(V)}(\bar{c}) = m$.

Contragredient representation The definition of intertwiners in the next section requires the introduction of the dual basis $\langle\!\langle\boldsymbol{\lambda}|$. The algebra 6.3.2 acts on the dual basis with the contragredient representation $\rho^{(V)*}$, defined such that

$$\langle\!\langle\boldsymbol{\lambda}|\left(\rho^{(V)}(e)\,|\boldsymbol{\lambda}'\rangle\!\rangle\right) = \left(\langle\!\langle\boldsymbol{\lambda}|\,\rho^{(V)*}(e)\right)|\boldsymbol{\lambda}'\rangle\!\rangle, \tag{6.3.17}$$

for any element e of the algebra. Thus, the action of the contragredient representation depends on the choice of a scalar product for the vertical states. It turns out that the analysis of intertwining relations simplifies for a particular choice of scalar product for which states are orthogonal but not orthonormal,

$$\langle\!\langle\boldsymbol{\lambda}|\,\boldsymbol{\lambda}'\rangle\!\rangle = a_{\boldsymbol{\lambda}}(\boldsymbol{v})^{-1}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}. \tag{6.3.18}$$

The norms $a_{\boldsymbol{\lambda}}(\boldsymbol{v})^{-1}$ are chosen so that the contragredient representation of $x_\omega^\pm(z)$ acts on $\langle\!\langle\boldsymbol{\lambda}|$ in the same way as the original representation $\rho^{(V)}(x_\omega^\mp(z))$ acts on $|\boldsymbol{\lambda}\rangle\!\rangle$ (note that $x_\omega^\pm$ becomes $x_\omega^\mp$). As a result, the norms have to obey the two following recursion relations for a box x of color $c(\square) = \omega$:

$$\begin{aligned}
\frac{a_{\boldsymbol{\lambda}-\square}(\boldsymbol{v})}{a_{\boldsymbol{\lambda}}(\boldsymbol{v})} &= \Omega^{-1} f_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}\chi_\square) \operatorname*{Res}_{z=\chi_\square} z^{-1}\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)\mathcal{Y}_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z), \\
\frac{a_{\boldsymbol{\lambda}+\square}(\boldsymbol{v})}{a_{\boldsymbol{\lambda}}(\boldsymbol{v})} &= -\Omega^{-1} F f_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}\chi_\square)^{-1} \operatorname*{Res}_{z=\chi_\square} z^{-1}\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)^{-1}\mathcal{Y}_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)^{-1}.
\end{aligned} \tag{6.3.19}$$

The solution is expressed in terms of the vector contribution $\mathcal{Z}_{\text{vect.}}(\boldsymbol{v}, \boldsymbol{\lambda})$ defined in 6.2.11,

$$a_{\boldsymbol{\lambda}}(\boldsymbol{v}) = (-F^{1/2})^{|\boldsymbol{\lambda}|}\mathcal{Z}_{\text{vect.}}(\boldsymbol{v}, \boldsymbol{\lambda}) \prod_{\square\in\boldsymbol{\lambda}} \prod_{\alpha\in C_{\bar{c}(\square)}(m)} (-\chi_\square/(q_3 v_\alpha)). \tag{6.3.20}$$

6.3.3 Horizontal representation

The horizontal representation of the algebra 6.3.2 is the equivalent of the vertex representations constructed by Saito in [160] for quantum toroidal algebras of $\mathfrak{gl}(p)$. It has level $\rho^{(H)}(c) = 1$ and depends on p weights $u_\omega \in \mathbb{C}^\times$ and p integers $n_\omega \in \mathbb{Z}$. In this representation, Drinfeld currents are constructed as a direct product of two (commuting) algebras. The first algebra is called here the *zero modes* factor, it is defined in terms the two operators $Q_\omega(z), P_\omega(z)$ satisfying the exchange relation

$$P_\omega(z)Q_{\omega'}(w) = f_{\omega\omega'}(w/z)Q_{\omega'}(w)P_\omega(z). \tag{6.3.21}$$

In appendix I.2, these operators are constructed explicitly in terms of $2p$ Heisenberg algebras. As a result, the operator $P_\omega(z)$ acts on the vacuum state $|\varnothing\rangle$ as $P_\omega(z)\,|\varnothing\rangle = |\varnothing\rangle$, and $Q_\omega(z)$ acts on the dual vacuum $\langle\varnothing|$ as $\langle\varnothing|\,Q_\omega(z) = \langle\varnothing|$. Accordingly, we define the normal ordering of these operators by writing the $Q_\omega(z)$-dependence on the left.

The second algebra involved in the horizontal representation is defined upon the modes

$\alpha_{\omega,k}$ of p coupled free bosons ($\omega \in \mathbb{Z}_p$ and $k \in \mathbb{Z}^\times$) satisfying the commutation relations,[10]

$$[\alpha_{\omega,k}, \alpha_{\omega',l}] = k\delta_{k+l} q_3^{k/2} \left[\delta_{\omega\omega'} + q_3^{-k}\delta_{\omega\bar{\omega}'} - q_1^k \delta_{\omega\ \omega'+\nu_1} - q_2^k \delta_{\omega\ \omega'+\nu_2}\right], \quad (k > 0). \tag{6.3.23}$$

As usual, the vacuum state $|\varnothing\rangle$ is annihilated by the positive modes ($k > 0$), while negative modes create excitations. The dual state $\langle\varnothing|$ is annihilated by negative modes. Thus, these modes are normal ordered by moving the positive modes to the right. The representation of the Drinfeld currents $x_\omega^\pm$ and $\psi_\omega^\pm$ is given in terms of the vertex operators

$$\begin{aligned}
&\eta_\omega^+(z) = \exp\left(\sum_{k>0} \frac{z^k}{k}\alpha_{\omega,-k}\right) \exp\left(-\sum_{k>0} \frac{z^{-k}}{k} q_3^{-k/2}\alpha_{\omega,k}\right), \quad \eta_\omega^-(z) = \exp\left(-\sum_{k>0} \frac{z^k}{k}\alpha_{\omega,-k}\right) \exp\left(\sum_{k>0} \frac{z^{-k}}{k} q_3^{k/2}\alpha_{\omega,k}\right)\\
&\varphi_\omega^+(z) = \exp\left(-\sum_{k>0} \frac{z^{-k}}{k}(q_3^{-k/2}\alpha_{\omega,k} - q_3^{k/2}\alpha_{\bar{\omega},k})\right), \quad \varphi_\omega^-(z) = \exp\left(\sum_{k>0} \frac{z^k}{k}(q_3^{-k}\alpha_{\bar{\omega},-k} - \alpha_{\omega,-k})\right).
\end{aligned} \tag{6.3.24}$$

Combining the zero modes and vertex operators, the horizontal representation writes

$$\begin{aligned}
&\rho^{(H)}(x_\omega^+(z)) = u_\omega z^{-n_\omega} Q_\omega(z)\eta_\omega^+(z), \quad \rho_u^{(1,n)}(x_\omega^-(z)) = u_\omega^{-1} z^{n_\omega} Q_\omega(z)^{-1} P_{\bar{\omega}}(q_3^{-1}z)\eta_\omega^-(z),\\
&\rho^{(H)}(\psi_\omega^+(z)) = F^{-1/2} P_{\bar{\omega}}(q_3^{-1}z)\varphi_\omega^+(z),\\
&\rho^{(H)}(\psi_\omega^-(z)) = F^{1/2} \frac{u_{\bar{\omega}}}{u_\omega} q_3^{n_{\bar{\omega}}} z^{n_\omega - n_{\bar{\omega}}} \frac{Q_{\bar{\omega}}(q_3^{-1}z)}{Q_\omega(z)} P_{\bar{\omega}}(q_3^{-1}z)\varphi_\omega^-(z).
\end{aligned} \tag{6.3.25}$$

It is shown in appendix I.2 that the expressions in the RHS obey the algebraic relations 6.3.2 at the levels $\rho^{(H)}(c) = 1$ and $\rho^{(H)}(\bar{c}) = n + p$ if $\nu_1 + \nu_2 < p$ and $\rho^{(H)}(\bar{c}) = n$ otherwise, where $n = \sum_{\omega\in\mathbb{Z}_p} n_\omega$. Note that even in the ALE case $\nu_1 = -\nu_2 = 1$, the horizontal representation

[10] The RHS of these commutation relations involves the coefficients $\sigma_{\omega\omega'}^{(k)} = -\sigma_{\bar{\omega}'\omega}^{(-k)} = kq_3^{k/2}\left[\delta_{\omega\omega'} + q_3^{-k}\delta_{\omega\bar{\omega}'} - q_1^k\delta_{\omega\ \omega'+\nu_1} - q_2^k\delta_{\omega\ \omega'+\nu_2}\right]$ appearing in the expansion of the scattering function 6.2.16,

$$[S_{\omega\omega'}(z)]_- = \exp\left(\sum_{k>0} \frac{z^k}{k^2} q_3^{-k/2}\sigma_{\omega'\omega}^{(k)}\right), \quad [S_{\omega\omega'}(z)]_+ = f_{\omega'\omega}(z)^{-1} \exp\left(-\sum_{k>0} \frac{z^{-k}}{k^2} q_3^{k/2}\sigma_{\omega'\omega}^{(-k)}\right). \tag{6.3.22}$$

given here is slightly more general than the one proposed in [165]. Indeed, in the latter the $\mathbb{Z}_p$-symmetry is broken by a choice of color ω_0, setting $u_\omega = u\delta_{\omega,\omega_0}$ and $n_\omega = n\delta_{\omega,\omega_0}$. Instead, in our construction of the gauge theory partition functions, it is necessary to keep u_ω and n_ω arbitrary in order to be able to assign a different gauge coupling $\mathfrak{q}_\omega$ and Chern-Simons level κ_ω for each color ω.

6.4 Algebraic engineering

The algebraic engineering of 5d $\mathcal{N} = 1$ quiver gauge theories on $\mathbb{C}_{\varepsilon_1} \times \mathbb{C}_{\varepsilon_2} \times S^1_R$ (without orbifold) follows from their correspondence with topological string theories in which the Nekrasov instanton partition function is obtained as a topological strings amplitude [130]. Indeed, these amplitudes are computed using the (refined) topological vertex [131, 166, 167] that was identified in [141] with an intertwiner between certain modules of the Ding-Iohara-Miki algebra [133, 134], also known as the quantum toroidal algebra of $\mathfrak{gl}(1)$. This intertwiner is in fact the toroidal analogue of the vertex operators introduced in [137] to compute the form factors of the XXZ Heisenberg spin chain. As result, the powerful topological strings computational methods for supersymmetric gauge theories can be reformulated in the language of quantum integrability.

The correspondence between 5d $\mathcal{N} = 1$ gauge theories and quantum toroidal algebras is better formulated using the (p, q)-brane realization of the gauge theories in type IIB string theory [127, 128]. In this realization, quiver gauge theories are reproduced by the low energy dynamics of a network of 5-branes with charges (p, q). These branes generalize both NS5-branes $(0, 1)$ and D5-branes $(1, 0)$. They wrap the 5-dimensional spacetime, and define a line segment in the 56-plane of the ten dimensional strings spacetime. These segments meet at trivalent vertices and form a web called the (p, q)-*branes web*. For instance, in the case of linear quivers, a set of m-D5 branes is associated to each node bearing a $U(m)$ gauge group. These D5-branes are suspended between dressed NS5-branes (i.e. branes of charge $(n, 1)$).

In this context, the relevant quantum toroidal algebra is determined by the spacetime of the gauge theory. Then, each brane of the (p,q)-branes web is associated to a representation of the algebra, identifying the levels with the charges $\rho(c) = q$, $\rho(\bar{c}) = p$ and the weights with the (exponentiated) position of the branes [154, 139, 138]. Thus, to a D5-brane corresponds a vertical representation with $m = 1$, while horizontal representations are associated to dressed NS-branes of charge $(n,1)$. It was further noticed in [51] that the set of m D5-branes of a single node (with a $U(m)$ gauge group) can be directly described by a vertical representation with $\rho^{(V)}(\bar{c}) = m$. Following the identification of the (p,q)-branes web with the toric diagram of the Calabi-Yau in topological strings [129], the trivalent junctions of branes coincide with the vertex operator of the algebra acting on the modules determined by the branes charge. Finally, the automorphisms of the algebra renders the various geometrical operations (translations, rotations) applied to the branes web [143].

For each (p,q)-branes web it is possible to write down an operator $\mathcal{T}$ constructed by 'gluing' the vertex operators of nodes connected by an edge. The gluing procedure is done by a product of operators in horizontal representations (NS5), and a scalar product in vertical ones (D5). The $\mathcal{T}$-operator obtained in this way acts on the tensor product of representations corresponding to the external branes of the web (i.e. the semi-infinite line segments). These representations are in fact horizontal modules, and the vacuum expectation value of the $\mathcal{T}$-operator reproduces the instantons partition function. The qq-characters are further obtained by introducing algebra elements (in the proper representation) within the vacuum expectation value [51]. We will give several examples below.

This algebraic construction of gauge theories BPS-observables has been generalized in a several directions: D-type quivers [168], 6D spacetime and elliptic algebras [169], 4d $\mathcal{N} = 2$ gauge theories and the affine Yangian of $\mathfrak{gl}(1)$ [135], 5d $\mathcal{N} = 1$ gauge theories on ALE spaces [154], and 3D $\mathcal{N} = 2^*$ gauge theories [165]. In this section, we present yet another generalization corresponding to deformed ALE spaces with the $\mathbb{Z}_p$-action described in section two. However, we do not wish to reproduce the whole construction here as it is a straightforward application of the methods developed earlier [154, 139, 138, 51]. Instead, we will only provide the main ingredient, namely the expression of the vertex operator, and a few selected examples to illustrate the construction.

6.4.1 Vertex operators

We consider two types of vertex operators, denoted Φ and Φ^*, and obtained, up to a normalization factor, by solving the following equations

$$\rho^{(H')}(e)\Phi = \Phi\ \left(\rho^{(V)} \otimes \rho^{(H)}\ \Delta(e)\right), \quad \left(\rho^{(V)} \otimes \rho^{(H)}\ \Delta'(e)\right)\ \Phi^* = \Phi^*\rho^{(H')}(e), \tag{6.4.1}$$

where e is any of the currents $x^{\pm}_{\omega}(z), \psi^{\pm}_{\omega}(z)$ or the central charge c.[11] Here Δ' denotes the opposite coproduct obtained by permutation $\Delta' = \mathcal{P}\Delta\mathcal{P}$. In order to distinguish the two horizontal representations, we denoted them $\rho^{(H)}$ and $\rho^{(H')}$, they depend on the parameters u_ω, n_ω and u'_ω, n'_ω respectively. Thus, the vertex operator Φ (and also Φ^*) depend on the set of weights $u_\omega, u'_\omega, v_\omega$ and integers $n_\omega, n'_\omega, m_\omega$. A solution to the equations 6.4.1 is found only if these parameters satisfy the two constraints

$$u'_\omega = u_\omega \prod_{\alpha\in C_{\bar\omega}} (-q_3 v_\alpha), \quad n'_\omega = n_\omega + m_{\bar\omega}. \tag{6.4.2}$$

The first relation expresses a constraint among the position of the branes in the 56-planes. The second equation is the charge conservation at the vertex. Due to the spacetime orbifold, the branes charges p in (p, q) degenerates into charges p_ω with $\omega \in \mathbb{Z}_p$ identified with the integers n_ω and $m_{\bar\omega}$ of horizontal/vertical representations.[12] Summing over ω, these constraints reproduce the conservation of the levels $n' = n+m$ that follows from the application of the intertwining relations 6.4.1 to the element $e = \bar c$ with the coproduct 6.3.10. Due to the presence of an algebra automorphism exchanging c and $\bar c$ in the $\mathfrak{gl}(p)$-case [163], we expect

[11] In fact, these relations are also satisfied for the grading operator $\xi_\omega(z)$ (see appendix J).

[12] There is an unfortunate conflict of notations here since the integer p labeling the $\mathbb{Z}_p$-orbifold is unrelated to the charge $p = \sum_\omega p_\omega$ of the branes.

a similar degeneration of the charge q into q_ω. It is not observed here because only a single charge $q=1$ flow through the topological vertex.

By definition, the vertex operator Φ^* is a vector in the vertical module while Φ is a dual vector,

$$\Phi = \sum_{\boldsymbol{\lambda}} \Phi_{\boldsymbol{\lambda}} \, \langle\!\langle \boldsymbol{\lambda} | \,, \quad \Phi^* = \sum_{\boldsymbol{\lambda}} \Phi^*_{\boldsymbol{\lambda}} \, | \boldsymbol{\lambda} \rangle\!\rangle. \tag{6.4.3}$$

Each vertical component $\Phi_{\boldsymbol{\lambda}}$ (or $\Phi^*_{\boldsymbol{\lambda}}$) is a Fock vertex operator acting on the horizontal module,

$$\begin{aligned} &\Phi_{\boldsymbol{\lambda}} = t_{\boldsymbol{\lambda}} : \Phi_\varnothing \prod_{\square\in\boldsymbol{\lambda}} \eta^+_{c(\square)}(\chi_\square) :, \quad \Phi^*_{\boldsymbol{\lambda}} = t^*_{\boldsymbol{\lambda}} : \Phi^*_\varnothing \prod_{\square\in\boldsymbol{\lambda}} \eta^-_{c(\square)-\nu_1-\nu_2}(q_3\chi_\square) :, \\ &t_{\boldsymbol{\lambda}} = F^{-|\boldsymbol{\lambda}|/2} \prod_{\square\in\boldsymbol{\lambda}} u'_{c(\square)} \chi_\square^{-n'_{c(\square)}} \prod_{\square\in\boldsymbol{\lambda}} Q_{c(\square)}(\chi_\square), \\ &t^*_{\boldsymbol{\lambda}} = F^{-|\boldsymbol{\lambda}|/2} \prod_{\square\in\boldsymbol{\lambda}} (-u_{c(\square)-\nu_1-\nu_2})^{-1} (q_3\chi_\square)^{n_{c(\square)-\nu_1-\nu_2}} \prod_{\square\in\boldsymbol{\lambda}} : Q_{c(\square)-\nu_1-\nu_2}(q_3\chi_\square)^{-1} P_{c(\square)}(\chi_\square) : . \end{aligned} \tag{6.4.4}$$

A sketch of the derivation can be found in the appendix K, together with the (rather lengthy) expressions of the vacuum components $\Phi_\varnothing$ and $\Phi^*_\varnothing$. The vertex operators Φ and Φ^* given here are a generalization of the colored refined topological vertex derived in [154, 170] with extra parameters (ν_1, ν_2).

The vertical components 6.4.4 of the vertex operators obey important normal ordering relations, from which we recover the vector and bifundamental contributions to the partition functions [51],[13]

$$
\begin{aligned}
\Phi_{\boldsymbol{\lambda}}\Phi_{\boldsymbol{\lambda}'} &= \mathcal{G}(\boldsymbol{v}'|\boldsymbol{v})^{-1}N(\boldsymbol{v}',\boldsymbol{\lambda}'|\boldsymbol{v},\boldsymbol{\lambda})^{-1} :\Phi_{\boldsymbol{\lambda}}\Phi_{\boldsymbol{\lambda}'}: \\
\Phi_{\boldsymbol{\lambda}}\Phi^*_{\boldsymbol{\lambda}'} &= \mathcal{G}(\boldsymbol{v}'|q_3^{-1}\boldsymbol{v})N(\boldsymbol{v}',\boldsymbol{\lambda}'|q_3^{-1}\boldsymbol{v},\boldsymbol{\lambda}) :\Phi_{\boldsymbol{\lambda}}\Phi^*_{\boldsymbol{\lambda}'}: \\
\Phi^*_{\boldsymbol{\lambda}}\Phi_{\boldsymbol{\lambda}'} &= \mathcal{G}(\boldsymbol{v}'|\boldsymbol{v})N(\boldsymbol{v}',\boldsymbol{\lambda}'|\boldsymbol{v},\boldsymbol{\lambda}) :\Phi^*_{\boldsymbol{\lambda}}\Phi_{\boldsymbol{\lambda}'}: \\
\Phi^*_{\boldsymbol{\lambda}}\Phi^*_{\boldsymbol{\lambda}'} &= \mathcal{G}(\boldsymbol{v}'|q_3^{-1}\boldsymbol{v})^{-1}N(\boldsymbol{v}',\boldsymbol{\lambda}'|q_3^{-1}\boldsymbol{v},\boldsymbol{\lambda})^{-1} :\Phi^*_{\boldsymbol{\lambda}}\Phi^*_{\boldsymbol{\lambda}'}:,
\end{aligned}
\tag{6.4.6}
$$

The expression of the one-loop factors $\mathcal{G}(\boldsymbol{v}|\boldsymbol{v}')$ can be found in appendix K, formula K.1.11. Note also that, following the method presented in [138, 154], it is a priori possible to show that $\Phi_{\boldsymbol{\lambda}}$ and $\Phi^*_{\boldsymbol{\lambda}}$ are solutions of the double deformed Knizhnik - Zamolodchikov (or (q,t)-KZ) equations.

6.4.2 Partition functions and qq-characters

The simplest example of algebraic engineering is given by the pure $U(m)$ gauge theory with quiver A_1. In this case, the (p,q)-brane web can be described roughly as a set of m D5-branes suspended between two (dressed) NS5-branes. The corresponding $\mathcal{T}$-operator is obtained as a product of vertex operators Φ and Φ^* in the vertical channel [51], it acts on the tensor product of two horizontal modules,

$$
\mathcal{T}[U(m)] = \Phi\cdot\Phi^* = \sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}(\boldsymbol{v})\ \Phi_{\boldsymbol{\lambda}}\otimes\Phi^*_{\boldsymbol{\lambda}}\ :\ H\otimes H'_*\to H'\otimes H_*. \tag{6.4.7}
$$

[13] To simplify the notations, we have omitted the dependence of the Nekrasov factors in the vectors of colors $\boldsymbol{c}=(c_\alpha)_{\alpha=1}^m$ and $\boldsymbol{c}'$. The shortcut notation $q_3^{-1}\boldsymbol{v}'$ in $N(\boldsymbol{v},\boldsymbol{\lambda}|q_3^{-1}\boldsymbol{v}',\boldsymbol{\lambda}')$ should be understood as a shift of the weights $q_3^{-1}v'_\alpha$ together with the corresponding shift of indices $c'_\alpha-\nu_3=\bar{c}'_\alpha$. Thus, we have the important relation

$$
\begin{aligned}
&N(\boldsymbol{v},\boldsymbol{\lambda}|q_3^{-1}\boldsymbol{v}',\boldsymbol{\lambda}') = N(\boldsymbol{v}',\boldsymbol{\lambda}'|\boldsymbol{v},\boldsymbol{\lambda})f(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}'),\quad \text{with:}\\
&f(\boldsymbol{v},\boldsymbol{\lambda}|\boldsymbol{v}',\boldsymbol{\lambda}') = \prod_{\substack{\square\in\boldsymbol{\lambda}\\ \blacksquare\in\boldsymbol{\lambda}'}} f_{c(\square)c(\blacksquare)}(\chi_\blacksquare/\chi_\square)\times\prod_{\square\in\boldsymbol{\lambda}}\prod_{\alpha\in C_{c(\square)}(m')}\left(-\frac{\chi_\square}{v'_\alpha}\right)\times\prod_{\square\in\boldsymbol{\lambda}'}\prod_{\alpha\in C_{\bar{c}(\square)}(m)}\left(-\frac{v_\alpha}{q_3\chi_\square}\right).
\end{aligned}
\tag{6.4.5}
$$

In order to distinguish the horizontal modules, we added the subscript $*$ to the ones on which Φ^* act. Accordingly, we denote the parameters of these representations (n_ω^*, u_ω^*) and $(n_\omega^{*\prime}, u_\omega^{*\prime})$. Evaluating the vacuum expectation value of this operator, we recover the instanton partition function of the underlying gauge theory:

$$\mathcal{Z}_{\text{inst}} = \langle\varnothing| \otimes \langle\varnothing|\ \mathcal{T}[U(m)]\, |\varnothing\rangle \otimes |\varnothing\rangle = \sum_{\boldsymbol{\lambda}} \prod_{\omega\in\mathbb{Z}_p} \mathfrak{q}_\omega^{K_\omega(\boldsymbol{\lambda})} \mathcal{Z}_{\text{vect}}(\boldsymbol{v}, \boldsymbol{\lambda}) \mathcal{Z}_{\text{CS}}(\boldsymbol{\kappa}, \boldsymbol{\lambda}) \tag{6.4.8}$$

where we have identified the colored gauge coupling $\mathfrak{q}_\omega$ and Chern-Simons level κ_ω with

$$\mathfrak{q}_\omega = F^{-1/2} \frac{u_\omega}{u^*_{\omega+\nu_3}} q_3^{n^*_{\omega+\nu_3}}, \quad \kappa_\omega = n^*_{\omega+\nu_3} - n_\omega. \tag{6.4.9}$$

By construction, the operator $\mathcal{T}[U(m)]$ commutes with the action of the algebra defined by the opposite coproduct Δ' [51], namely,

$$\left(\rho^{(H')} \otimes \rho^{(H_*)}\right) \Delta'(e)\, \mathcal{T}[U(m)] = \mathcal{T}[U(m)]\, \left(\rho^{(H_*)} \otimes \rho^{(H')}\right) \Delta'(e), \quad e \in \mathcal{A}. \tag{6.4.10}$$

For this reason, $\mathcal{T}[U(m)]$ plays the role of the *screening operator* in [138]. The gauge theory expectation value of the fundamental qq-characters is obtained by insertion of $\Delta'(x^-_{\omega+\nu_3}(q_3 z))$ in the horizontal vacuum expectation value,

$$\left\langle \mathcal{X}_{\bar{\omega}}^{[\boldsymbol{\lambda}]*}(q_3^{-1}z)\right\rangle_{\text{gauge}} = u_\omega^{-1} z^{n_\omega} \frac{\langle\varnothing| \otimes \langle\varnothing|\ \left(\rho^{(H')} \otimes \rho^{(H_*)}\ \Delta'(x^-_{\omega+\nu_3}(q_3 z))\right) \mathcal{T}[U(m)]\, |\varnothing\rangle \otimes |\varnothing\rangle}{\langle\varnothing| \otimes \langle\varnothing|\ \mathcal{T}[U(m)]\, |\varnothing\rangle \otimes |\varnothing\rangle} \tag{6.4.11}$$

where the gauge averaging of a chiral ring observable $\mathcal{O}^{[\boldsymbol{\lambda}]}$ is performed over the instanton configurations weighted by the vector (and Chern-Simons) contributions to the partition function,

$$\left\langle \mathcal{O}^{[\boldsymbol{\lambda}]}\right\rangle_{\text{gauge}} = \frac{1}{\mathcal{Z}_{\text{inst.}}} \sum_{\boldsymbol{\lambda}} \prod_{\omega\in\mathbb{Z}_p} \mathfrak{q}_\omega^{K_\omega(\boldsymbol{\lambda})} \mathcal{Z}_{\text{vect}}(\boldsymbol{v}, \boldsymbol{\lambda}) \mathcal{Z}_{\text{CS}}(\boldsymbol{\kappa}, \boldsymbol{\lambda}) \mathcal{O}^{[\boldsymbol{\lambda}]}, \tag{6.4.12}$$

and the qq-character writes

$$\mathcal{X}_{\omega}^{[\boldsymbol{\lambda}]*}(z) = \mathcal{Y}_{\omega}^{[\boldsymbol{\lambda}]*}(z) + \mathfrak{q}_{\omega+\nu_3}\frac{(q_3z)^{\kappa_{\omega+\nu_3}}}{\mathcal{Y}_{\omega+\nu_3}^{[\boldsymbol{\lambda}]}(q_3z)}. \tag{6.4.13}$$

Note that the first term involve the $\mathcal{Y}$-observable $\mathcal{Y}_{\omega}^{[\boldsymbol{\lambda}]*}(z) = f_{\omega}^{[\boldsymbol{\lambda}]}(z)\mathcal{Y}_{\omega}^{[\boldsymbol{\lambda}]}(z)$. As shown in [51], it follows from the commutation relations (6.4.10) that the quantity $\left\langle \mathcal{X}_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)\right\rangle_{\text{gauge}}$ is a finite Laurent series in z (i.e. a polynomial upon multiplication by a positive power of z). This is in fact due to the radial ordering of operators in the horizontal Fock spaces. Indeed, when $x_{\omega}^{-}(z)$ is inserted on the left of $\mathcal{T}$, the correlator as a well-defined expansion around $z=\infty$. On the other hand, when $x_{\omega}^{-}(z)$ in inserted on the right, the expansion around $z=0$ is now well-defined. The non-trivial equality between the two expansions (6.4.10) implies that both series are finite, and thus that the correlator is a finite Laurent series in z. Asymptotically, the $\mathcal{Y}$-observables behave as $\mathcal{Y}_{\omega}^{[\boldsymbol{\lambda}]} \underset{0}{\sim} z^{-\beta_{\omega}^{[\boldsymbol{\lambda}]}}$, $\mathcal{Y}_{\omega}^{[\boldsymbol{\lambda}]*} \underset{0}{\sim} 1$ and $\mathcal{Y}_{\omega}^{[\boldsymbol{\lambda}]} \underset{\infty}{\sim} 1$, $\mathcal{Y}_{\omega}^{[\boldsymbol{\lambda}]*} \underset{\infty}{\sim} z^{\beta_{\omega}^{[\boldsymbol{\lambda}]}}$ with $\beta_{\omega}^{[\boldsymbol{\lambda}]} = |A_{\omega}(\boldsymbol{\lambda})| - |R_{\omega+\nu_3}(\boldsymbol{\lambda})|$. When $\nu_3 = 0$, the exponent $\beta_{\omega}^{[\boldsymbol{\lambda}]}$ becomes independent of $\boldsymbol{\lambda}$, $\beta_{\omega}^{[\boldsymbol{\lambda}]} = m_{\omega}$. As a result, the gauge average of the qq-character $\mathcal{X}_{\omega}(z)$ is a polynomial of degree m_{ω} when $|\kappa_{\omega}| < m_{\omega}$. Unfortunately, when $\nu_3 \neq 0$ not much can be said.

Another fundamental qq-character can be obtained using the generator $x_{\omega}^{+}(z)$ instead,

$$\begin{aligned}
&\left\langle \mathcal{X}_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)\right\rangle_{\text{gauge}} = (u_{\omega}^{*})^{-1}z^{n_{\omega}^{*}}\frac{\langle\varnothing|\otimes\langle\varnothing|\ \left(\rho^{(H')}\otimes\rho^{(H_*)}\ \Delta'(x_{\omega}^{+}(z))\right)\mathcal{T}[U(m)]\,|\varnothing\rangle\otimes|\varnothing\rangle}{\langle\varnothing|\otimes\langle\varnothing|\ \ \mathcal{T}[U(m)]\,|\varnothing\rangle\otimes|\varnothing\rangle},\\
&\text{with}\quad \mathcal{X}_{\omega}^{[\boldsymbol{\lambda}]}(z) = \mathcal{Y}_{\omega}^{[\boldsymbol{\lambda}]}(z) + \mathfrak{q}_{\omega+\nu_3}F\frac{(q_3z)^{\kappa_{\omega+\nu_3}}}{f_{\omega}^{[\boldsymbol{\lambda}]}(z)\mathcal{Y}_{\omega+\nu_3}^{[\boldsymbol{\lambda}]}(q_3z)}.
\end{aligned} \tag{6.4.14}$$

The presence of two different fundamental qq-characters is a specificity of 5d $\mathcal{N}=1$ gauge theories on orbifolds: when $p=1$, the two qq-characters are equivalent (they only differ by multiplication of a constant times a power of z). Further, as we shall see below, in the 4d limit $R\to 0$, the qq-characters $\mathcal{X}_{\omega}^{[\boldsymbol{\lambda}]}(z)$ and $\mathcal{X}_{\omega}^{[\boldsymbol{\lambda}]*}(z)$ reduce to the same expression. The gauge averages (6.4.11) and (6.4.14) for the qq-characters have been computed at the first few orders in the gauge couplings $\mathfrak{q}_\omega$ for the gauge groups $U(1)$ and $U(2)$ and various orbifold parameters. In all cases, it has been observed that these quantities are indeed finite Laurent series in the argument z. Finally, it is worth mentioning that higher qq-characters can be obtained by multiple insertions of the coproducts $\Delta'(x_\omega^\pm(z))$. We refer to [51] for more details on the computation of qq-characters.

4d limit When the radius R of the background circle S^1_R is sent to zero, the gauge theory reduces to a 4d $\mathcal{N}=2$ gauge theory. This limit can be performed directly on the partition functions and qq-characters, re-introducing the radius dependences in the parameters $(q_1,q_2)=(e^{R\varepsilon_1},e^{R\varepsilon_2})$, $v_\alpha=e^{Ra_\alpha}$, $\chi_\square=e^{R\phi_\square}$,... Sending $R\to 0$ in the expression 6.2.11 of the instanton partition function, we observe that the Chern-Simons contribution is subdominant while, after setting the spectral variable to $z=e^{R\zeta}$, the scattering function 6.2.16 becomes

$$S^{(4\mathrm{d})}_{\omega\omega'}(\zeta)=\frac{(\zeta+\varepsilon_1)^{\delta_{\omega,\omega'-\nu_1}}(\zeta+\varepsilon_2)^{\delta_{\omega,\omega'-\nu_2}}}{\zeta^{\delta_{\omega,\omega'}}(\zeta+\varepsilon_1+\varepsilon_2)^{\delta_{\omega,\omega'-\nu_1-\nu_2}}}. \tag{6.4.15}$$

This function satisfies a simpler crossing symmetry $S^{(4\mathrm{d})}_{\omega\bar{\omega}'}(-\zeta-\varepsilon_1-\varepsilon_2)=f^{(4\mathrm{d})}_{\omega\omega'}S^{(4\mathrm{d})}_{\omega'\omega}(\zeta)$ where $f^{(4\mathrm{d})}_{\omega\omega'}=(-1)^{\beta_{\omega\omega'}}$ in now independent of the spectral variable ζ. As a result, the function $f^{[\boldsymbol{\lambda}]}_\omega(z)$ reduces to a sign. When $\nu_3=0$, this sign is simply $(-1)^{m_\omega}$, it can be absorbed in the definition of $\mathfrak{q}_\omega$. In this way, both $\mathcal{X}^{[\boldsymbol{\lambda}]}_\omega(z)$ and $\mathcal{X}^{[\boldsymbol{\lambda}]*}_\omega(z)$ reproduce the expression of the 4d fundamental qq-character given in [15, 21].

A_2 quiver Linear quiver gauge theories can be treated along the same lines. For instance, the A_2 quiver gauge theory with gauge group $U(m_1)\times U(m_2)$ is obtained by considering two sets of m_1 and m_2 D5-branes suspended between three dressed NS5-branes. The $\mathcal{T}$-operator is simply the product of the single nodes operators $\mathcal{T}[U(m_1)]$ and $\mathcal{T}[U(m_2)]$ in a common horizontal representation,

$$\mathcal{T}[U(m_1)\times U(m_2)]=\Phi_1\cdot\Phi_2\Phi_1^*\cdot\Phi_2^*=\sum_{\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)}}a_{\boldsymbol{\lambda}^{(1)}}(\boldsymbol{v}^{(1)})a_{\boldsymbol{\lambda}^{(2)}}(\boldsymbol{v}^{(2)})\ \Phi^{(1)}_{\boldsymbol{\lambda}^{(1)}}\otimes\Phi^{(2)}_{\boldsymbol{\lambda}^{(2)}}\Phi^{(1)*}_{\boldsymbol{\lambda}^{(1)}}\otimes\Phi^{(2)*}_{\boldsymbol{\lambda}^{(2)}}. \tag{6.4.16}$$

The vacuum expectation value is computed using the normal ordering relation 6.4.6 for the product $\Phi^{(2)}_{\boldsymbol{\lambda}^{(2)}}\Phi^{(1)*}_{\boldsymbol{\lambda}^{(1)}}$,

$$\mathcal{Z}_{\text{inst.}}=\frac{1}{\mathcal{G}(\boldsymbol{v}^{(1)}|q_3^{-1}\boldsymbol{v}^{(2)})}\left\langle\varnothing\right|\otimes\left\langle\varnothing\right|\otimes\left\langle\varnothing\right|\ \mathcal{T}[U(m_1)\times U(m_2)]\left|\varnothing\right\rangle\otimes\left|\varnothing\right\rangle\otimes\left|\varnothing\right\rangle. \tag{6.4.17}$$

It reproduces the instanton partition function 6.2.11 for the A_2 quiver gauge theory, with the identification 6.4.9 of the parameters at each node $i=1,2$. The qq-characters can also be constructed along the lines of [51].

6.5 Discussion

In this chapter, we have reconstructed algebraically the instanton partition functions for $\mathcal{N}=1$ linear quiver gauge theories with unitary gauge groups on the five dimensional background $S^1_R\times(\mathbb{C}_{\varepsilon_1}\times\mathbb{C}_{\varepsilon_2})/\mathbb{Z}_p$. The action of the abelian group considered here is a generalization by two integers (ν_1,ν_2) of the standard action defining ALE spaces. These extra parameters led us to introduce a deformation of the quantum toroidal algebra of $\mathfrak{gl}(p)$. This new quantum toroidal algebra appears to be defined upon a non-symmetrizable Cartan matrix $\beta_{\omega\omega'}$. Yet, we have shown that it still possesses the structure of a Hopf algebra with the deformed Drinfeld coproduct given in 6.3.8. We have also presented two different representations, called *vertical* and *horizontal*, that are respectively the deformation of the Fock module [159] and the vertex representation [160] of the quantum toroidal algebra of $\mathfrak{gl}(p)$. Other types of representations should exist, like the Macmahon representation obtained for $\mathfrak{gl}(p)$ as a tensor products of Fock modules in [159]. Although the definition of this new algebra may appear intricate, the physical context in which it emerges is very natural, and its representations are simple generalizations of the usual ones.

Quantum toroidal algebras extend the definition of quantum affine algebras (or quantum groups) by an extra affinization. In fact, the quantum toroidal algebra of $\mathfrak{gl}(p)$ is generated by two orthogonal quantum affine subalgebra $U_q(\widehat{\mathfrak{sl}(p)})$ [171, 172]. Then, one may wonder if the (ν_1, ν_2)-deformed algebra possesses a similar property. Of course, it is assuming that a quantum affine algebra built upon the Cartan matrix $\beta_{\omega\omega'}$ can be defined properly. In fact, we expect that this is indeed the case, and that such quantum affine algebra retains a quasitriangular Hopf algebra structure, making it suitable for the construction of new quantum integrable systems.

On the gauge theory side, several generalizations of our approach could be implemented. For instance, the abelian group $\mathbb{Z}_p$ could be replaced by a Mckay subgroup of $SU(2)$ of type DE, with either left, right, or both left-right action. As shown by Nakajima in [149, 150], in the first two cases a quantum affine algebra of type $\mathfrak{so}/\mathfrak{sp}$ acts on the cohomology of the instanton moduli space. This action is expected to be lifted to a quantum toroidal algebra in K-theory. Accordingly, the algebraic engineering should involve the quantum toroidal $\mathfrak{so}/\mathfrak{sp}$ algebras. However, the effective construction requires some new developments in the representation theory of these algebras.

When $\nu_2 = 0$, the orbifold can be interpreted as the presence of a surface defect [21]. In this case, the Cartan matrix $\beta_{\omega\omega'}$ appears to vanish but the algebra remains non-trivial,

$$\begin{aligned} S_{\omega\omega'}(z) &= \left(\frac{1-q_2 z}{1-z}\right)^{\delta_{\omega,\omega'}} \left(\frac{1-q_1 z}{1-q_1 q_2 z}\right)^{\delta_{\omega,\omega'-\nu_1}}, \\ g_{\omega\omega'}(z) &= \left(q_2^{-1}\frac{1-q_2 z}{1-q_2^{-1}z}\right)^{\delta_{\omega\omega'}} \left(\frac{1-q_1 z}{1-q_3^{-1}z}\right)^{\delta_{\omega,\omega'-\nu_1}} \left(q_2\frac{1-q_3 z}{1-q_1^{-1}z}\right)^{\delta_{\omega,\omega'+\nu_1}}. \end{aligned} \tag{6.5.1}$$

When $\nu_1 = 1$, the structure function $g_{\omega\omega'}(z)$ reproduces the one that defines the quantum toroidal algebra of $\mathfrak{gl}(p)$ with q_2 and q_3 exchanged (up to a factor $q_3^{m_{\omega\omega'}/2}$). However, the function $S_{\omega\omega'}(z)$ is different from the one appearing in G.2.6, and thus horizontal and vertical representations of the (ν_1, ν_2)-deformed algebra degenerate into new representations for the quantum toroidal algebra of $\mathfrak{gl}(p)$. We hope to come back to the study of this problem in a future publication.

Finally, an important question was left behind in our study, namely the correspondence with (q-deformed) W-algebras. This type of correspondences is now well-understood in the case of quantum toroidal $\mathfrak{gl}(1)$. There, the q-W-algebras appearing in horizontal or vertical representations play different roles. In the horizontal case, a representation of level $c = m$ can be built by tensoring m level one representations. It is thus expressed in terms of m sets of bosonic modes that are coupled through their commutation relations. Diagonalizing these relations, the Drinfeld currents can be expressed in terms of q-W_m currents coupled to an infinite Heisenberg algebra. This dual q-W-algebra corresponds to the quiver W-algebra of Kimura and Pestun [173]. Using Miki's automorphism [163, 134], vertical representations of level $\bar{c} = m$ can be mapped on horizontal ones, and thus expressed in terms of q-W_m currents coupled to the Heisenberg algebra. In the vertical case, the dual W-algebra is responsible for the AGT-like correspondence with q-deformed conformal blocks [140]. Alternatively, the AGT correspondence can also be seen directly in the degenerate limit $R \to 0$ in which the vertical representation of the toroidal algebra reduces to a representation of the affine Yangian of $\mathfrak{gl}(1)$ that is known to contain the action of W_m-currents [124, 125, 126].

A similar type of duality is believed to hold between the degenerate limit of the quantum toroidal algebra of $\mathfrak{gl}(p)$ and the coset[14] $\widehat{\mathfrak{gl}(\alpha)}_m/\widehat{\mathfrak{gl}(\alpha - p)}_m$, leading to an AGT correspondence between instantons on ALE spaces and parafermionic conformal field theories [174, 155, 175]. This conjecture has been verified for small values of p and m by comparing the conformal blocks of the coset theory with the gauge theories instanton partition functions [174, 176, 177, 178, 179, 180], or the limit $R \to 0$ of 5d topological strings amplitudes [170]. There are two main strategies to extend this duality to the (ν_1, ν_2)-deformed algebra. One possibility is again to compare instanton partition functions with conformal blocks. This approach was taken in [146] where the gauge theory calculations led to conjectural expressions

[14] The parameter α is determined by the Ω-background parameters $(\varepsilon_1, \varepsilon_2)$.

for these conformal blocks. But, unfortunately, the corresponding conformal field theory appears to be unknown. Another possible approach consists in identifying directly the (q-deformed) coset algebra generators acting on the vertical modules of the quantum toroidal algebra. For this purpose, one could diagonalize the commutation relations for the modes $\alpha_{\omega,k}$ in the horizontal representations, and then define the analogue of Miki's automorphism to map the horizontal representations to the vertical ones. From the strings theory perspective, the latter is expected to exist since it should describe the fiber-base duality of the topological strings (or, the S-duality in Type IIB string theory) [142, 143]. This approach appears very promising and we hope to be able to report soon on this problem.

Part III

Vertex Operator Algebras

Chapter 7

SCFT/VOA correspondence via Ω-deformation

7.1 Introduction

Superconformal field theories (SCFTs) exhibit interesting aspects and rich structures due to their large symmetry group. A striking feature revealed in [181] is that any superconformal field theory with an $\mathfrak{su}(1,1|2)$ superconformal subalgebra which acts as anti-holomorphic Möbius transformations on a two-dimensional plane possesses a protected sector isomorphic to a two-dimensional vertex operator algebra (VOA).[1] The protected sector is formed as a certain $(\mathcal{Q}+\mathcal{S})$-cohomology, spanned by twisted-translations of Schur operators with their operator product expansions (OPEs) in the cohomology.

For Lagrangian four-dimensional $\mathcal{N}=2$ superconformal theories, the procedure of obtaining this chiral algebra can be briefly described as follows. It can be shown that chiral algebras produced by free hypermultiplet and free vector multiplet are those of symplectic bosons (also known as $\beta\gamma$ system) and bc ghosts, respectively. When they are coupled to produce an interacting SCFT, the prescription is first to take the naive tensor product

[1] We use the terms *vertex operator algebra* and *chiral algebra* interchangeably.

of those two-dimensional chiral algebras with the gauge-invariance constraint and then to pass to the cohomology of the nilpotent BRST operator. Such a procedure led to many conjectural relations in [181] between $\mathcal{N} = 2$ superconformal QCDs and $\mathcal{W}$-algebras, which were checked at the level of the equivalence of the superconformal indices and the vacuum characters. For related works, see also [182, 183, 184, 185].

The protected chiral algebra is particularly interesting since it is a non-commutative algebra of local operators in two dimensions, which is not easily expected for theories in higher dimensions. It turns out that the *non-commutative deformation* parameter $\hbar$, which appears in the numerators of the OPEs of chiral algebra, is given by the relative coefficient of the combination $\mathcal{Q} + \mathcal{S}$. Even though this is a direct consequence of OPE computation, it seems that an intuitive understanding of the appearance of the non-commutative deformation parameter is still absent. Therefore, it could be useful to approach the mentioned chiral algebra in an alternative framework where the origin of the non-commutative deformation parameter is well understood. The main goal of this chapter is to make such an attempt.

The framework that we are referring to is the Ω-deformation of supersymmetric gauge theories [6]. It was firstly introduced in [6] to regularize the partition function of $\mathcal{N} = 2$ gauge theories on the non-compact $\mathbb{C}^2$. Essentially, the Ω-deformation is implemented by modifying the theory as a cohomological field theory with respect to the supersymmetry which squares to an isometry of the underlying manifold. It effectively turns on a potential along the direction orthogonal to the isometry, and thus *localizes* the theory on the fixed points of the isometry. A remarkable discovery made in [28] was that the two-dimensional Ω-deformation on $\mathcal{N} = 2$ gauge theories can be used to quantize the classical integrable system whose Hamiltonians are given by the $\mathcal{N} = 2$ chiral operators. One may regard this quantization at the level of the representations of the non-commutative deformation of the algebra of holomorphic functions on the phase space of the integrable system, where the non-commutative deformation paramter is identified with the Ω-deformation parameter $\varepsilon = \hbar$. A similar feature is also present in other contexts: in three-dimensional $\mathcal{N} = 4$ theories, for example, the Ω-deformation on the Rozansky-Witten theory leads to a non-commutative deformation of the Higgs branch chiral ring [186, 187, 188].

For which theory should we implement the Ω-deformation to recover the chiral algebra? In [189], Kapustin discussed the holomorphic-topological twist of $\mathcal{N} = 2$ gauge theories on a product manifold $\mathcal{C} \times \mathcal{C}^{\perp}$, in which the theory is topological along, say, $\mathcal{C}^{\perp}$ and holomorphic along $\mathcal{C}$ (see also [190, 191, 192] for earlier works on partially holomorphic and partially topological theories). The cohomology of local operators, therefore, forms a chiral algebra on $\mathcal{C}$, albeit a *commutative* one since local operators can commute with each other by escaping to the direction of $\mathcal{C}^{\perp}$. Now we can imagine implementing the Ω-deformation with respect to the isometry on $\mathcal{C}^{\perp}$, effectively creating a potential along the direction of $\mathcal{C}^{\perp}$. As local operators are now trapped on $\mathcal{C}$ due to the potential, it is natural to expect that we obtain a non-commutative deformation of the chiral algebra. The *height* of the potential would be controlled by none other than the Ω-deformation parameter, and we expect the identification of the non-commutative deformation parameter with the Ω-deformation parameter. We will see that this is indeed the case.

To obtain the two-dimensional chiral algebra, we have to perform supersymmetric localization of the Ω-deformed holomorphic-topological theory to produce a chiral CFT on $\mathcal{C}$. The algebra of local operators of this CFT would provide our desired chiral algebra. It turns out that the localization procedure can be conducted in a very similar manner with [193], where the localization of the Ω-deformed two-dimensional Landau-Ginzburg model was discussed. In fact, our localization can be viewed as the gauge theory analogue of [193] on $\mathcal{C}^{\perp}$, which was discussed in [194] in its application of recovering four-dimensional Chern-Simons theory from six-dimensional supersymmetric gauge theory (see also [186, 195] for the discussion of B-models on the compact disk where the localization locus was chosen to be constant maps), occuring at each point of $\mathcal{C}$. The localization locus is given by solutions to certain gradient flow equations (emanating from the critical point of the superpotential as we take $\mathcal{C}^{\perp} = \mathbb{R}^2$). To obtain the action of the localized theory on $\mathcal{C}$, we have to evaluate the action on this localization locus. This can be accomplished with the help of the equivariant integration, in a similar manner that [8] applies an equivariant integration on $\mathbb{C}^2$ to yield the representations of $\mathcal{N} = 2$ chiral operators on the instanton moduli space. For the case at hand, it turns out that there is no non-trivial topological sector of gauge field configurations in the localization locus, so that the further integration on the instanton moduli space would not take place.

This chapter is organized as follows. In section 7.2, we briefly review the Donaldson-Witten twist and the holomorphic-topological twist of Kapustin for four-dimensional $\mathcal{N} = 2$ theories. In section 7.3, we perform the supersymmetric localization of the Ω-deformed holomorphic-topological theory to obtain the two-dimensional chiral CFT. In section 7.4, we discuss the identification of $S^3 \times S^1$ partition function of $\mathcal{N} = 2$ SCFT and torus partition function of chiral CFT, which lead to the equivalence of the Schur index and the vaccum character. We conclude in section 7.5 with discussions.

7.2 Holomorphic-topological twist of $\mathcal{N} = 2$ theories

Let us consider a $\mathcal{N} = 2$ supersymmetric theory on a four-dimensional Euclidean manifold, $X = \mathcal{C} \times \mathcal{C}^{\perp}$, where $\mathcal{C}$ and $\mathcal{C}^{\perp}$ are Riemann surfaces. A curved background on X would generically break all the supersymmetries. To preserve some supersymmetries, we need to twist the holonomy group with the R-symmetry group, for which the supercharges with charge 0 under the twisted holonomy group would remain preserved.

The holonomy group of X is $U(1)_{\mathcal{C}} \times U(1)_{\mathcal{C}^{\perp}}$ and the R-symmetry group of a $\mathcal{N} = 2$ supersymmetric theory is $SU(2)_R \times U(1)_r$. The $\mathcal{N} = 2$ superalgebra contains the following supercharges

$$\mathcal{Q}^A_\alpha, \quad \widetilde{\mathcal{Q}}^A_{\dot\alpha}, \quad A = 1, 2,\ \alpha = \pm,\ \dot\alpha = \pm, \tag{7.2.1}$$

where A is the $SU(2)_R$ R-symmetry index and $\alpha, \dot\alpha$ are un-dotted and dotted spinor indices.

We choose the conventions for the generators of the holonomy as

$$\mathcal{M}_{\mathcal{C}} = \mathcal{M}_{+}{}^{+} + \mathcal{M}^{\dot{+}}{}_{\dot{+}}, \quad \mathcal{M}_{\mathcal{C}^{\perp}} = \mathcal{M}_{+}{}^{+} - \mathcal{M}^{\dot{+}}{}_{\dot{+}}. \tag{7.2.2}$$

The table 7.1 shows the supercharges and their quantum numbers. Note that $U(1)_R \subset SU(2)_R$ is the maximal torus.

	$\mathcal{Q}^1_+$	$\mathcal{Q}^1_-$	$\mathcal{Q}^2_+$	$\mathcal{Q}^2_-$	$\widetilde{\mathcal{Q}}^1_{\dot{+}}$	$\widetilde{\mathcal{Q}}^1_{\dot{-}}$	$\widetilde{\mathcal{Q}}^2_{\dot{+}}$	$\widetilde{\mathcal{Q}}^2_{\dot{-}}$
$U(1)_{\mathcal{C}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$U(1)_{\mathcal{C}^{\perp}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$U(1)_R$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$U(1)_r$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Table 7.1: $\mathcal{N} = 2$ supercharges and quantum numbers

7.2.1 Donaldson-Witten twist

Let us first review how the Donaldson-Witten twist comes about. For a curved metric on $\mathcal{C}^{\perp}$, we twist the holonomy $U(1)_{\mathcal{C}^{\perp}}$ by taking the diagonal subgroup

$$U(1)'_{\mathcal{C}^{\perp}} \hookrightarrow U(1)_{\mathcal{C}} \times U(1)_R. \tag{7.2.3}$$

Under the twist, we preserve the $\mathcal{N} = (2, 2)$ supersymmetry on $\mathcal{C}$ whose fermionic generators are

$$\mathcal{Q}^1_-, \ \mathcal{Q}^2_+, \ \widetilde{\mathcal{Q}}^1_{\dot{+}}, \ \widetilde{\mathcal{Q}}^2_{\dot{-}}. \tag{7.2.4}$$

When $\mathcal{C}$ is also curved, we can make a further twist

$$U(1)'_{\mathcal{C}} \hookrightarrow U(1)_{\mathcal{C}} \times U(1)_R \tag{7.2.5}$$

to preserve $\widetilde{\mathcal{Q}}^1_{\dot{+}}$, $\widetilde{\mathcal{Q}}^2_{\dot{-}}$. The Donaldson-Witten supercharge is precisely the linear combination of these supercharges,

$$\mathcal{Q}_{\mathrm{DW}} = \widetilde{\mathcal{Q}}^1_{\dot{+}} + \widetilde{\mathcal{Q}}^2_{\dot{-}}. \tag{7.2.6}$$

Here, $\widetilde{\mathcal{Q}}^1_{\dot{+}}$ and $\widetilde{\mathcal{Q}}^2_{\dot{-}}$ are preserved independently but $\mathcal{Q}_{\mathrm{DW}}$ is the one which is preserved for any curved background on X, not necessarily a product metric.

To describe the Ω-deformation made upon the twist, let us suppose $\mathcal{C}^{\perp} = \mathbb{R}^2$ for a moment. One may take a specific combination of supercharges

$$\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Q}}^1_{\dot{+}} + \widetilde{\mathcal{Q}}^2_{\dot{-}} + \varepsilon(w\mathcal{Q}^1_{+} - \bar{w}\mathcal{Q}^2_{-}), \tag{7.2.7}$$

where $w = x^1 + ix^2$ and $\bar{w} = x^1 - ix^2$ are the coordinates on $\mathcal{C}^{\perp}$. This supercharge squares to the isometry of $\mathcal{C}^{\perp}$ generated by $V = w\partial_w - \bar{w}\partial_{\bar{w}}$. In general background on $\mathcal{C}^{\perp}$ the deformed supercharge would not be preserved since the last two supercharges are not preserved as we have seen above. However, one can still construct a deformation of the theory which has a supercharge which squares to the isometry on $\mathcal{C}^{\perp}$. In practice, we can start from the theory on $\mathbb{R}^4$, write the variations of component fields with respect to the naive supercharge (7.2.7), and then seek a way of re-writing them in metric-independent fashion so that deformed supersymmetry variations are consistently defined on arbitrary product manifold $\mathcal{C} \times \mathcal{C}^{\perp}$. The action of the theory has to be modified correspondingly to ensure the invariance under the deformed supersymmetry.

	$\mathcal{Q}^1_{+}$	$\mathcal{Q}^1_{-}$	$\mathcal{Q}^2_{+}$	$\mathcal{Q}^2_{-}$	$\widetilde{\mathcal{Q}}^1_{\dot{+}}$	$\widetilde{\mathcal{Q}}^1_{\dot{-}}$	$\widetilde{\mathcal{Q}}^2_{\dot{+}}$	$\widetilde{\mathcal{Q}}^2_{\dot{-}}$
$U(1)'_{\mathcal{C}}$	0	-1	1	0	0	-1	1	0
$U(1)'_{\mathcal{C}^{\perp}}$	1	0	0	-1	0	1	-1	0

Table 7.2: Donaldson-Witten twist

7.2.2 Holomorphic-topological twist

Now we apply a similar procedure to our main subject: the holomorphic-topological twist of four-dimensional $\mathcal{N}=2$ supersymmetry introduced in [189]. Let us first breifly review the holomorphic-topological twist. For a curved metric on $\mathcal{C}^{\perp}$, we twist the holonomy $U(1)_{\mathcal{C}^{\perp}}$ by taking the diagonal subgroup

$$U(1)'_{\mathcal{C}^{\perp}} \hookrightarrow U(1)_{\mathcal{C}^{\perp}} \times U(1)_r. \tag{7.2.8}$$

Under the twist, we preserve the $\mathcal{N}=(0,4)$ supersymmetry on $\mathcal{C}$ whose fermionic generators are

$$\mathcal{Q}^A_-,\ \widetilde{\mathcal{Q}}^A_{\dot{-}}, \quad A=1,2. \tag{7.2.9}$$

When $\mathcal{C}$ is also curved, we can make a further twist

$$U(1)'_{\mathcal{C}} \hookrightarrow U(1)_{\mathcal{C}} \times U(1)_R \tag{7.2.10}$$

to preserve $\mathcal{Q}^1_-,\ \widetilde{\mathcal{Q}}^1_{\dot{-}}$. The holomorphic-twist supercharge is the following linear combination of supercharges,

$$\mathcal{Q} = \mathcal{Q}^1_- + \widetilde{\mathcal{Q}}^1_{\dot{-}}. \tag{7.2.11}$$

Note that the translations along $\mathcal{C}^{\perp}$ and the anti-holomorphic translation along $\mathcal{C}$ are actually $\mathcal{Q}$-exact:

$$\begin{aligned} \{\mathcal{Q}, \mathcal{Q}^2_+\} &= -\mathcal{P}_{+\dot{-}}, \\ \{\mathcal{Q}, \widetilde{\mathcal{Q}}^2_{\dot{+}}\} &= \mathcal{P}_{-\dot{+}} \\ \{\mathcal{Q}, \mathcal{Q}^2_-\} &= -\{\mathcal{Q}, \widetilde{\mathcal{Q}}^2_{\dot{-}}\} = -\mathcal{P}_{-\dot{-}}, \end{aligned} \tag{7.2.12}$$

hence it gets the name *holomorphic-topological* twist. Let us suppose $\mathcal{C}^\perp = \mathbb{R}^2$ for a moment. Then we would preserve

$$\mathbb{Q}_\varepsilon = \mathcal{Q}^1_- + \widetilde{\mathcal{Q}}^1_{\dot{-}} + \varepsilon(w\mathcal{Q}^2_+ + \bar{w}\widetilde{\mathcal{Q}}^2_{\dot{+}}), \tag{7.2.13}$$

which squares to the isometry on $\mathcal{C}^\perp$:

$$\mathbb{Q}_\varepsilon^2 = \varepsilon(w\{\widetilde{\mathcal{Q}}^1_{\dot{-}}, \mathcal{Q}^2_+\} + \bar{w}\{\mathcal{Q}^1_-, \widetilde{\mathcal{Q}}^2_{\dot{+}}\}) = -2\varepsilon(w\mathcal{P}_w - \bar{w}\mathcal{P}_{\bar{w}}). \tag{7.2.14}$$

In general background on $\mathcal{C}^\perp$, the deformed supercharge would not be preserved since the last two supercharges are not preserved as we have seen. However, just as the case of the Donaldson-Witten twist, it is still possible to implement the Ω-deformation of the holomorphic-topological theory by consistently deforming the supersymmetry variations and the action. We will see in the following section how this is actually accomplished.

It is crucial to note that, unlike the Donaldson-Witten case, we make use of the $U(1)_r$ R-symmetry to make a twist with the isometry on $\mathcal{C}^\perp$. Recalling that the deformed supercharge squares to the isometry on $\mathcal{C}^\perp$, we see that the localization with respect to this supercharge would not work if the $U(1)_r$ R-symmetry is anomalous. This is precisely the case when the theory is not superconformal. Thus we restrict our attention to $\mathcal{N} = 2$ superconformal theories in relating their Ω-deformation on holomorphic-topological twist with two-dimensional chiral algebras. It is interesting to see that the superconformality is required in a slightly different manner compared to the $(\mathcal{Q} + \mathcal{S})$-cohomology story in [181], where the superconformal supercharge $\mathcal{S}$ explicitly appears in defining the cohomology of local operators in the chiral algebra.

	$\mathcal{Q}^1_+$	$\mathcal{Q}^1_-$	$\mathcal{Q}^2_+$	$\mathcal{Q}^2_-$	$\widetilde{\mathcal{Q}}^1_{\dot{+}}$	$\widetilde{\mathcal{Q}}^1_{\dot{-}}$	$\widetilde{\mathcal{Q}}^2_{\dot{+}}$	$\widetilde{\mathcal{Q}}^2_{\dot{-}}$
$U(1)'_{\mathcal{C}}$	1	0	0	-1	1	0	0	-1
$U(1)'_{\mathcal{C}^\perp}$	1	0	1	0	-1	0	-1	0

Table 7.3: Holomorphic-topological twist

7.3 Chiral CFT from Ω-deformation and localization

The general analysis of the previous section can be applied to $\mathcal{N} = 2$ gauge theories, on which we focus from now on. We perform supersymmetric localization on the Ω-deformed holomorphic-topological theory, to produce a two-dimensional chiral CFT. The desired chiral algebra is obtained as the algebra of local operators of this two-dimensional CFT.

7.3.1 Holomorphic-topological twist of $\mathcal{N} = 2$ gauge theory

Let us start from the $\mathcal{N} = 2$ vector multiplet. The vector multiplet contains a gauge connection A, gaugini λ^A_α and $\tilde{\lambda}^A_{\dot\alpha}$, a complex scalar ϕ, and an auxiliary field D_{AB}, where $A = 1, 2$ is the $SU(2)_R$ R-symmetry index. Following the analysis of the previous section, the holomorphic-topological twist changes the quantum numbers of these component fields as in the table 7.4.

	λ^1_+	λ^1_-	λ^2_+	λ^2_-	$\tilde{\lambda}^1_{\dot+}$	$\tilde{\lambda}^1_{\dot-}$	$\tilde{\lambda}^2_{\dot+}$	$\tilde{\lambda}^2_{\dot-}$	ϕ	$\tilde{\phi}$	${D^2}_2$	${D^1}_2$	${D^2}_1$
$U(1)_{\mathcal{C}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0
$U(1)_{\mathcal{C}^\perp}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0
$U(1)_R$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	1	-1
$U(1)_r$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	1	0	0	0
$U(1)'_{\mathcal{C}}$	1	0	0	-1	1	0	0	-1	0	0	0	1	-1
$U(1)'_{\mathcal{C}^\perp}$	0	-1	0	-1	0	1	0	1	-1	1	0	0	0

Table 7.4: $\mathcal{N} = 2$ vector multiplet; gaugini, scalars, and auxiliary field

Correspondingly, we change the notation for the component fields by their representations under the Lorentz group after the twist,

$$
\begin{aligned}
&\lambda^1_+ = \lambda_z, \quad \lambda^1_- = \lambda_{\bar{w}}, \quad \lambda^2_+ = \lambda, \quad \lambda^2_- = \lambda_{\bar{z}\bar{w}}, \\
&\tilde{\lambda}^1_{\dot+} = \tilde{\lambda}_z, \quad \tilde{\lambda}^1_{\dot-} = \tilde{\lambda}_w, \quad \tilde{\lambda}^2_{\dot+} = \tilde{\lambda}, \quad \tilde{\lambda}^2_{\dot-} = \tilde{\lambda}_{\bar{z}w} \\
&\phi = \phi_{\bar{w}}, \quad \tilde{\phi} = \tilde{\phi}_w, \quad {D^2}_2 = D, \quad {D^1}_2 = D_z, \quad {D^2}_1 = D_{\bar{z}}.
\end{aligned}
\tag{7.3.1}
$$

The $\mathcal{N} = 2$ supersymmetry variations can be written as

$$\begin{aligned}
\delta A_\mu &= i\zeta^A \sigma_\mu \tilde{\lambda}_A - i\tilde{\zeta}^A \tilde{\sigma}_\mu \lambda_A \\
\delta\phi &= -i\zeta^A \lambda_A \\
\delta\tilde{\phi} &= i\tilde{\zeta}^A \tilde{\lambda}_A \\
\delta\lambda_A &= \frac{1}{2} F_{\mu\nu}\sigma^{\mu\nu}\zeta_A + 2D_\mu\phi\sigma^\mu\tilde{\zeta}_A + \phi\sigma^\mu D_\mu\tilde{\zeta}_A + 2i\zeta_A[\phi, \tilde{\phi}] + D_{AB}\zeta^B \\
\delta\tilde{\lambda}_A &= \frac{1}{2} F_{\mu\nu}\tilde{\sigma}^{\mu\nu}\tilde{\zeta}_A + 2D_\mu\tilde{\phi}\tilde{\sigma}^\mu\zeta_A + \tilde{\phi}\tilde{\sigma}^\mu D_\mu\zeta_A - 2i\tilde{\zeta}_A[\phi, \tilde{\phi}] + D_{AB}\tilde{\zeta}^B \\
\delta D_{AB} &= -i\tilde{\zeta}_A\tilde{\sigma}^\mu D_\mu\lambda_B + i\zeta_A\sigma^\mu D_\mu\tilde{\lambda}_B - 2[\phi, \tilde{\zeta}_A\tilde{\lambda}_B] + 2[\tilde{\phi}, \zeta_A\lambda_B] + (A \leftrightarrow B),
\end{aligned} \tag{7.3.2}$$

with the fermionic parameters ζ^A and $\tilde{\zeta}^A$. In a general metric background, the supersymmetry would be preserved only if ζ^A and $\tilde{\zeta}^A$ are Killing spinors. Let us first place the theory on the flat $\mathbb{R}^4$. Since the holomorphic-topological supercharge is $\mathcal{Q} = \mathcal{Q}^1_- + \tilde{\mathcal{Q}}^1_{\dot{-}}$, it is straightforward to write out the variations of the component fields with respect to the holomorphic-topological supercharge, using the notation (7.3.1), as

$$\begin{aligned}
&\mathcal{Q}A_z = \tilde{\lambda}_z - \lambda_z, \quad \mathcal{Q}A_{\bar{z}} = 0, \quad \mathcal{Q}A_w = \tilde{\lambda}_w, \quad \mathcal{Q}A_{\bar{w}} = -\lambda_{\bar{w}}, \\
&\mathcal{Q}\phi_{\bar{w}} = i\lambda_{\bar{w}}, \quad \mathcal{Q}\tilde{\phi}_w = i\tilde{\lambda}_w, \\
&\mathcal{Q}\lambda_z = D_z, \quad \mathcal{Q}\lambda_{\bar{w}} = 0, \quad \mathcal{Q}\lambda_{\bar{z}\bar{w}} = -4F_{\bar{z}\bar{w}} + 4iD_{\bar{z}}\phi_{\bar{w}}, \\
&\mathcal{Q}\lambda = 2F_{z\bar{z}} + 2F_{w\bar{w}} - 4iD_w\phi_{\bar{w}} + 2i[\phi_{\bar{w}}, \tilde{\phi}_w] + D, \\
&\mathcal{Q}\tilde{\lambda}_z = D_z, \quad \mathcal{Q}\tilde{\lambda}_w = 0, \quad \mathcal{Q}\tilde{\lambda}_{\bar{z}w} = -4F_{\bar{z}w} - 4iD_{\bar{z}}\tilde{\phi}_w, \\
&\mathcal{Q}\tilde{\lambda} = 2F_{z\bar{z}} - 2F_{w\bar{w}} + 4iD_{\bar{w}}\tilde{\phi}_w - 2i[\phi_{\bar{w}}, \tilde{\phi}_w] + D, \\
&\mathcal{Q}D_z = 0, \quad \mathcal{Q}D_{\bar{z}} = 4d_{\bar{z}}(\lambda - \tilde{\lambda}) + 4d_w\lambda_{\bar{z}\bar{w}} - 4d_{\bar{w}}\tilde{\lambda}_{\bar{z}w} + 4[\phi_{\bar{w}}, \tilde{\lambda}_{\bar{z}w}] + 4[\tilde{\phi}_w, \lambda_{\bar{z}\bar{w}}], \\
&\mathcal{Q}D = 2D_{\bar{z}}(\tilde{\lambda}_z - \lambda_z) - 2D_w\lambda_{\bar{w}} + 2D_{\bar{w}}\tilde{\lambda}_w - 2[\phi_{\bar{w}}, \tilde{\lambda}_w] - 2[\tilde{\phi}_w, \lambda_{\bar{w}}].
\end{aligned} \tag{7.3.3}$$

Now we turn to the action for the vector multiplet. It is given by

$$
\begin{aligned}
S_{\text{top}} &= -\frac{i\vartheta}{8\pi^2}\int \operatorname{Tr} F\wedge F \\
S_{\text{vec}} &= \frac{1}{g^2}\int d^4x\ \operatorname{Tr}\ \Big[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D^{AB}D_{AB} - 4d_\mu\tilde{\phi}D^\mu\phi + 4[\phi,\tilde{\phi}]^2 \\
&\qquad\qquad -2i\lambda^A\sigma^\mu D_\mu\tilde{\lambda}_A - 2\lambda^A[\tilde{\phi},\lambda_A] + 2\tilde{\lambda}^A[\phi,\tilde{\lambda}_A]\Big]\,,
\end{aligned}
\tag{7.3.4}
$$

where g is the gauge coupling. As we will see momentarily, the topological term does not affect the theory and there would be no dependence on ϑ. Thus we drop the topological term from now on. Then a computation shows that the rest of the action turns out to be $\mathcal{Q}$-exact:

$$
\begin{aligned}
S_{\text{vec}} = \mathcal{Q}\Bigg[\frac{1}{g^2}\int d^4x\ \operatorname{Tr}\ \Big[&2\lambda_{\bar{z}\bar{w}}(-F_{zw} + iD_z\tilde{\phi}_w) - 2\tilde{\lambda}_{\bar{z}w}(F_{z\bar{w}} + iD_z\phi_{\bar{w}}) + \frac{1}{2}(\lambda_z + \tilde{\lambda}_z)D_{\bar{z}} \\
&+ (\lambda + \tilde{\lambda})\left(-F_{z\bar{z}} + \frac{1}{2}D - iD_{\bar{w}}\tilde{\phi}_w + iD_w\phi_{\bar{w}}\right) \\
&+(\lambda - \tilde{\lambda})\left(-F_{w\bar{w}} - iD_{\bar{w}}\tilde{\phi}_w - iD_w\phi_{\bar{w}} - i[\phi_{\bar{w}},\tilde{\phi}_w]\right)\Big]\Bigg]\,.
\end{aligned}
\tag{7.3.5}
$$

To ensure the positive-definiteness of the action, we impose the reality properties to the bosonic fields,

$$
\bar{A}_\mu = A_\mu, \quad \bar{\phi} = -\tilde{\phi}, \quad \bar{D}_{AB} = -D^{AB},
\tag{7.3.6}
$$

while requiring the symplectic-Majorana conditions to the gaugini,

$$
\overline{(\lambda_{A\alpha})} = \epsilon^{AB}\epsilon^{\alpha\beta}\lambda_{B\beta}, \quad \overline{(\tilde{\lambda}_{A\dot{\alpha}})} = \epsilon^{AB}\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{B\dot{\beta}}.
\tag{7.3.7}
$$

As mentioned in the previous section, the holomorphic-topological supercharge $\mathcal{Q} = \mathcal{Q}^1_- + \widetilde{\mathcal{Q}}^1_{\dot{-}}$ is in fact preserved in any product metric background as long as we make the proper twist of the isometry with the R-symmetry group. Hence we would like to write the supersymmetry variation rules to make sense in a general metric background. This requires a bunch of re-definition of component fields,

$$
\begin{aligned}
&\phi = \tilde{\phi}_w dw - \phi_{\bar{w}} d\bar{w}, \quad \mathcal{A} = A + i\phi, \quad \bar{\mathcal{A}} = A - i\phi, \quad \lambda = 2\lambda_w dw - 2\tilde{\lambda}_{\bar{w}} d\bar{w} \\
&\mu_z = \frac{\lambda_z + \tilde{\lambda}_z}{2} dw \wedge d\bar{w}, \quad \mathsf{D}_z = D_z dw \wedge d\bar{w}, \quad \alpha = \frac{\lambda + \tilde{\lambda}}{2}, \quad \nu = \frac{\lambda - \tilde{\lambda}}{4} dw \wedge d\bar{w} \\
&\theta_z = \tilde{\lambda}_z - \lambda_z, \quad \bar{\rho}_{\bar{z}} = \frac{\tilde{\lambda}_{\bar{z}w} dw + \lambda_{\bar{z}\bar{w}} d\bar{w}}{4}, \quad \mathsf{D}_{\bar{z}} = \frac{1}{16} D_{\bar{z}} dw \wedge d\bar{w} \\
&\mathsf{D} = D + 2F_{z\bar{z}} - 2iD_w \phi_{\bar{w}} + 2iD_{\bar{w}} \tilde{\phi}_w.
\end{aligned}
\tag{7.3.8}
$$

For convenience, let us also denote the curvature of the *complexified* connection $\mathcal{A}$, $\bar{\mathcal{A}}$ by

$$
\begin{aligned}
&\mathcal{F} = \partial_w \mathcal{A}_{\bar{w}} - \partial_{\bar{w}} \mathcal{A}_w - i[\mathcal{A}_w, \mathcal{A}_{\bar{w}}], \quad \bar{\mathcal{F}} = \partial_w \bar{\mathcal{A}}_{\bar{w}} - \partial_{\bar{w}} \bar{\mathcal{A}}_w - i[\bar{\mathcal{A}}_w, \bar{\mathcal{A}}_{\bar{w}}], \\
&\mathcal{F}_{w\bar{z}} = \partial_w A_{\bar{z}} - \partial_{\bar{z}} \mathcal{A}_w - i[\mathcal{A}_w, A_{\bar{z}}], \quad \mathcal{F}_{\bar{w}\bar{z}} = \partial_{\bar{w}} A_{\bar{z}} - \partial_{\bar{z}} \mathcal{A}_{\bar{w}} - i[\mathcal{A}_{\bar{w}}, A_{\bar{z}}], \\
&\bar{\mathcal{F}}_{wz} = \partial_w A_z - \partial_z \mathcal{A}_z - i[\mathcal{A}_w, A_z], \quad \bar{\mathcal{F}}_{\bar{w}z} = \partial_{\bar{w}} A_z - \partial_z \mathcal{A}_{\bar{w}} - i[\mathcal{A}_{\bar{w}}, A_z], \\
&\mathcal{F}_{\bar{z}} = \mathcal{F}_{w\bar{z}} dw + \mathcal{F}_{\bar{w}\bar{z}} d\bar{w}, \quad \bar{\mathcal{F}}_z = \bar{\mathcal{F}}_{wz} dw + \bar{\mathcal{F}}_{\bar{w}z} d\bar{w}.
\end{aligned}
\tag{7.3.9}
$$

Then the supersymmetry variations are significantly simplified in terms of these new fields,

$$
\begin{aligned}
&\mathcal{Q}\mathcal{A} = 0, \quad \mathcal{Q}\bar{\mathcal{A}} = \lambda, \\
&\mathcal{Q}\lambda = 0, \quad \mathcal{Q}\nu = \mathcal{F}, \\
&\mathcal{Q}\alpha = \mathsf{D}, \quad \mathcal{Q}\mathsf{D} = 0, \\
&\mathcal{Q}A_{\bar{z}} = 0, \quad \mathcal{Q}A_z = \theta_z, \\
&\mathcal{Q}\rho_{\bar{z}} = \mathcal{F}_{\bar{z}}, \quad \mathcal{Q}\theta_z = 0, \\
&\mathcal{Q}\mathsf{D}_{\bar{z}} = \mathcal{D}_{\mathcal{C}^\perp} \rho_{\bar{z}} + D_{\bar{z}} \nu, \quad \mathcal{Q}\mu_z = \mathsf{D}_z, \quad \mathcal{Q}\mathsf{D}_z = 0,
\end{aligned}
\tag{7.3.10}
$$

where we have used the new covariant derivative $\mathcal{D}_{\mathcal{C}^\perp} = d_{\mathcal{C}^\perp} - i\mathcal{A}$ (we also denote $\bar{\mathcal{D}}_{\mathcal{C}^\perp} =$

$d_{\mathcal{C}^\perp} - i\bar{\mathcal{A}})$. The action (7.3.5) for the vector multiplet can be also written in these fields as

$$S_{\text{vec}} = \mathcal{Q}\left\{\frac{1}{g^2}\int_{\mathcal{C}} d^2z \int_{\mathcal{C}^\perp} \text{Tr}\left[-\bar{\mathcal{F}} \star_{\mathcal{C}^\perp} \nu - \alpha\left(\star_{\mathcal{C}^\perp}\mathsf{D} - 2iD_{\mathcal{C}^\perp}\star_{\mathcal{C}^\perp}\phi - 4\star_{\mathcal{C}^\perp} F_{z\bar{z}}\right)\right.\right.$$
$$\left.\left. +4\bar{\mathcal{F}}_z \wedge \star_{\mathcal{C}^\perp}\rho_{\bar{z}} + 4\mu_z \star_{\mathcal{C}^\perp} \mathsf{D}_{\bar{z}}\right]\right\}. \tag{7.3.11}$$

To make a $\mathcal{N}=2$ gauge theory superconformal, we in general need to couple hypermultiplets to the vector multiplet. Let us consider r hypermultiplets which consist of scalars q_{AI}, fermions $\psi_I, \tilde{\psi}_I$, and auxiliary fields $F_{\check{A}I}$, where $I = 1, \cdots, 2r$ is the $Sp(r)$ flavor index. The auxiliary $SU(2)$ $\check{A} = 1, 2$ is introduced to achieve an off-shell description of the hypermultiplet. We will only use $Sp(1)^r \subset Sp(r)$ subgroup of the flavor symmetry, so let us restrict a single free hypermultiplet $(r = 1)$ for a moment.

Recall that $U(1)_{\mathcal{C}}$ is twisted with the maximal torus of the $SU(2)_R$ R-symmetry group, $U(1)_R \subset SU(2)_R$. For the hypermultiplet, we will take a further twist with the maximal torus of the flavor symmetry:

$$U(1)'_{\mathcal{C}} \hookrightarrow U(1)_{\mathcal{C}} \times U(1)_R \times U(1)_{F,\check{F}}, \tag{7.3.12}$$

where $U(1)_{F,\check{F}}$ is the maximal torus of the $SU(2)$ flavor group or the $SU(2)$ auxiliary group. This is not really necessary but it will fix the spins of the resulting two-dimenisonal symplectic bosons to be integers. One can always undo this further twist. The tables 7.5 and 7.6 show the quantum numbers of the component fields in the hypermultiplet under the twist.

	q_{11}	q_{12}	q_{21}	q_{22}
$U(1)_{\mathcal{C}}$	0	0	0	0
$U(1)_{\mathcal{C}^\perp}$	0	0	0	0
$U(1)_R$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$U(1)_r$	0	0	0	0
$U(1)_F$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$U(1)'_{\mathcal{C}}$	0	-1	1	0
$U(1)'_{\mathcal{C}^\perp}$	0	0	0	0

Table 7.5: $\mathcal{N}=2$ hypermultiplet, scalars

	ψ_{+1}	ψ_{-1}	ψ_{+2}	ψ_{-2}	$\tilde{\psi}_{\dot{+}1}$	$\tilde{\psi}_{\dot{-}1}$	$\tilde{\psi}_{\dot{+}2}$	$\tilde{\psi}_{\dot{-}2}$
$U(1)_{\mathcal{C}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$U(1)_{\mathcal{C}^\perp}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$U(1)_R$	0	0	0	0	0	0	0	0
$U(1)_r$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$U(1)_F$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$U(1)'_{\mathcal{C}}$	1	0	0	-1	1	0	0	-1
$U(1)'_{\mathcal{C}^\perp}$	1	0	1	0	-1	0	-1	0

Table 7.6: $\mathcal{N}=2$ hypermultiplet, fermions

We define correspondingly

$$\begin{aligned}
&q_z \equiv q_{21}, \quad q_{\bar{z}} \equiv -q_{12}, \quad \tilde{q} \equiv q_{22}, \quad \tilde{q}^\dagger = q_{11}, \\
&\psi_{zw} \equiv \psi_{+1}, \quad \psi_{z\bar{z}} \equiv \psi_{-1}, \quad \tilde{\psi}_{z\bar{w}} \equiv \tilde{\psi}_{\dot{+}1}, \quad \tilde{\psi}_{z\bar{z}} \equiv \tilde{\psi}_{\dot{-}1} \\
&\psi_w \equiv \psi_{+2}, \quad \psi_{\bar{z}} \equiv \psi_{-2}, \quad \tilde{\psi}_{\bar{w}} \equiv \tilde{\psi}_{\dot{+}2}, \quad \tilde{\psi}_{\bar{z}} \equiv \tilde{\psi}_{\dot{-}2}, \\
&F_z \equiv F_{21}, \quad F_{\bar{z}} \equiv F_{12}, \quad \tilde{F} \equiv F_{22}, \quad \tilde{F}^\dagger = -F_{11}.
\end{aligned} \tag{7.3.13}$$

Let us take the hypermultiplet to be valued in a unitary representation R of the gauge group ($\bar{R}$ denotes the complex conjugate representation which is isomorphic to the dual representation). We take the convention such that the component fields q_z, $\tilde{q}^\dagger$, ψ_{zw}, $\psi_{z\bar{z}}$, $\tilde{\psi}_{z\bar{w}}$, $\tilde{\psi}_{z\bar{z}}$, F_z, $\tilde{F}^\dagger$ are valued in R while $q_{\bar{z}}$, $\tilde{q}$, ψ_w, $\psi_{\bar{z}}$, $\tilde{\psi}_{\bar{w}}$, $\tilde{\psi}_{\bar{z}}$, $F_{\bar{z}}$, $\tilde{F}$ are valued in $\bar{R}$. The $\mathcal{N}=2$ supersymmetry variations are given by

$$\begin{aligned}
&\delta q_{AI} = -i\zeta_A\psi_I + i\tilde{\zeta}_A\tilde{\psi}_I \\
&\delta\psi_I = -2\sigma^\mu\tilde{\zeta}^A D_\mu q_{AI} + 4i\zeta^A(\tilde{\phi}\cdot q_A)_I - \sigma^\mu D_\mu\tilde{\zeta}^A q_{AI} - 2\check{\zeta}^{\check{A}} F_{\check{A}I} \\
&\delta\tilde{\psi}_I = -2\tilde{\sigma}^\mu\zeta^A D_\mu q_{AI} + 4i\tilde{\zeta}^A(\phi\cdot q_A)_I - \tilde{\sigma}^\mu D_\mu\zeta^A q_{AI} - 2\check{\tilde{\zeta}}^{\check{A}} F_{\check{A}I} \\
&\delta F_{\check{A}I} = i\check{\zeta}_{\check{A}}\sigma^\mu D_\mu\tilde{\psi}_I^{\dot{\alpha}} - i\check{\tilde{\zeta}}_{\check{A}}\tilde{\sigma}^\mu D_\mu\psi_I - 2(\phi\cdot\check{\zeta}_{\check{A}}\psi)_I - 2(\check{\zeta}_{\check{A}}\lambda_B\cdot q^B)_I + 2(\tilde{\phi}\cdot\check{\tilde{\zeta}}_{\check{A}}\tilde{\psi})_I + 2(\check{\tilde{\zeta}}_{\check{A}}\tilde{\lambda}_B\cdot q^B)_I,
\end{aligned} \tag{7.3.14}$$

where the fermionic parameters $\check{\zeta}_A$, $\check{\tilde{\zeta}}_A$ should satisfy the constraints

$$\zeta_A\check{\zeta}_{\check{B}} - \tilde{\zeta}_A\check{\tilde{\zeta}}_{\check{B}} = 0, \quad \zeta^A\zeta_A + \check{\tilde{\zeta}}^{\check{A}}\check{\tilde{\zeta}}_{\check{A}} = 0, \quad \tilde{\zeta}^A\tilde{\zeta}_A + \check{\zeta}^{\check{A}}\check{\zeta}_{\check{A}} = 0, \quad \zeta^A\sigma^\mu\tilde{\zeta}_A + \check{\zeta}^{\check{A}}\sigma^\mu\check{\tilde{\zeta}}_{\check{A}} = 0, \tag{7.3.15}$$

to ensure the off-shell invariance of the supersymmetry. Since the holomorphic-topological supercharge is $\mathcal{Q} = \mathcal{Q}^1_- + \widetilde{\mathcal{Q}}^1_{\dot{-}}$, we have to find the solutions for $\check{\zeta}_A$ and $\check{\tilde{\zeta}}_A$ for $\zeta_1^- = 1$ and $\tilde{\zeta}_1^{\dot{-}} = 1$. It is not hard to find that $\check{\zeta}_2^+ = 1$ and $\check{\tilde{\zeta}}_{2\dot{-}} = -1$ satisfy the equations (7.3.15). Now it is straightforward to write out all the variations of component fields under the action of the holomorphic-topological supercharge:

$$\begin{aligned}
&\mathcal{Q}q_z = 0, \quad \mathcal{Q}\tilde{q} = 0, \quad \mathcal{Q}q_{\bar{z}} = i\psi_{\bar{z}} + i\tilde{\psi}_{\bar{z}}, \quad \mathcal{Q}\tilde{q}^\dagger = -\frac{i}{2}g^{z\bar{z}}(\psi_{z\bar{z}} + \tilde{\psi}_{z\bar{z}})\\
&\mathcal{Q}\psi_{zw} = 4i\mathcal{D}_w q_z, \quad \mathcal{Q}\psi_{z\bar{z}} = -4iD_{\bar{z}}q_z + 2\tilde{F}^\dagger\\
&\mathcal{Q}\psi_w = 4i\mathcal{D}_w\tilde{q}, \quad \mathcal{Q}\psi_{\bar{z}} = -4iD_{\bar{z}}\tilde{q} - 2F_{\bar{z}}\\
&\mathcal{Q}\tilde{\psi}_{z\bar{w}} = -4i\mathcal{D}_{\bar{w}}q_z, \quad \mathcal{Q}\tilde{\psi}_{z\bar{z}} = 4iD_{\bar{z}}q_z - 2\tilde{F}^\dagger\\
&\mathcal{Q}\tilde{\psi}_{\bar{w}} = -4i\mathcal{D}_{\bar{w}}\tilde{q}, \quad \mathcal{Q}\tilde{\psi}_{\bar{z}} = 4iD_{\bar{z}}\tilde{q} + 2F_{\bar{z}}\\
&\mathcal{Q}F_{\bar{z}} = 0, \quad \mathcal{Q}\tilde{F}^\dagger = 0\\
&\mathcal{Q}F_z = 2\mathcal{D}_{\bar{w}}\psi_{zw} + 2\mathcal{D}_w\tilde{\psi}_{z\bar{w}} - 2D_z(\psi_{z\bar{z}} + \tilde{\psi}_{z\bar{z}}) - 2(\tilde{\lambda}_z - \lambda_z)\cdot\tilde{q}^\dagger + 2(\lambda - \tilde{\lambda})\cdot q_z\\
&\mathcal{Q}\tilde{F} = 2\mathcal{D}_{\bar{w}}\psi_w + 2\mathcal{D}_w\tilde{\psi}_{\bar{w}} - 2D_z(\psi_{\bar{z}} + \tilde{\psi}_{\bar{z}}) + 2(\tilde{\lambda}_z - \lambda_z)\cdot q_{\bar{z}} + 2(\lambda - \tilde{\lambda})\cdot\tilde{q}.
\end{aligned} \tag{7.3.16}$$

Finally the action for the hypermultiplet is given by

$$\begin{aligned}
S_{\text{hyp}} =& \frac{1}{g^2}\int d^4x\,\Big[\frac{1}{2}D_\mu q^A D^\mu q_A - q^A\{\phi,\tilde{\phi}\}q_A + \frac{i}{2}q^A D_{AB}q^B - \frac{i}{2}\tilde{\psi}\tilde{\sigma}^\mu D_\mu\psi - \frac{1}{2}F^{\check{A}}F_{\check{A}}\\
&\qquad -\frac{1}{2}\psi\phi\psi + \frac{1}{2}\tilde{\psi}\tilde{\phi}\tilde{\psi} - q^A\lambda_A\psi + \tilde{\psi}\tilde{\lambda}_A q^A\Big].
\end{aligned} \tag{7.3.17}$$

To ensure the positive-definiteness of the action, we impose the following reality properties for the scalars,

$$\overline{(q_{AI})} = \Omega^{IJ}\epsilon^{AB}q_{BJ}, \quad \overline{(F_{\check{A}I})} = -\Omega^{IJ}\epsilon^{\check{A}\check{B}}F_{\check{B}J}, \tag{7.3.18}$$

while requiring the fermions to be Ω-symplectic Majorana

$$\overline{(\psi_{\alpha I})} = \epsilon^{\alpha\beta}\Omega^{IJ}\psi_{\beta J}, \quad \overline{(\tilde{\psi}_{\dot{\alpha}I})} = \epsilon^{\dot{\alpha}\dot{\beta}}\Omega^{IJ}\tilde{\psi}_{\dot{\beta}J}, \tag{7.3.19}$$

where Ω^{IJ} is the real antisymmetric $Sp(r)$-invariant tensor satisfying

$$(\Omega^{IJ})^* = -\Omega_{IJ}, \quad \Omega^{IJ}\Omega_{JK} = \delta^I_K. \tag{7.3.20}$$

Repeating the argument made for the vector multiplet, the holomorphic-topological supercharge is preserved for any product manifold after the twist. Hence we would like to write the supersymmetry variations in metric-independent fashion. This is achieved by making a bunch of re-definition of fields,

$$\begin{aligned}
&\sigma \equiv \frac{1}{4i}(\psi_w dw - \tilde{\psi}_{\bar{w}}d\bar{w}), \quad \xi_z \equiv \frac{1}{4i}(\psi_{zw}dw - \tilde{\psi}_{z\bar{w}}d\bar{w}), \quad \gamma \equiv -\frac{i}{2}g^{z\bar{z}}(\psi_{z\bar{z}} + \tilde{\psi}_{z\bar{z}}),\\
&\chi \equiv -\frac{ig^{z\bar{z}}(\psi_{z\bar{z}} - \tilde{\psi}_{z\bar{z}})}{4}dw\wedge d\bar{w}, \quad \eta_{\bar{z}} \equiv \frac{i(\psi_{\bar{z}} - \tilde{\psi}_{\bar{z}})}{2}dw\wedge d\bar{w}, \quad \zeta_{\bar{z}} \equiv i(\psi_{\bar{z}} + \tilde{\psi}_{\bar{z}}),\\
&h_z \equiv \frac{i}{8}(F_z + 2iD_z\tilde{q}^\dagger)dw\wedge d\bar{w}, \quad h \equiv \frac{i}{8}(\tilde{F} - ig^{z\bar{z}}D_z q_{\bar{z}})dw\wedge d\bar{w},\\
&h^\dagger \equiv -2i(\tilde{F}^\dagger - ig^{z\bar{z}}D_{\bar{z}}q_z)dw\wedge d\bar{w}, \quad h_{\bar{z}} \equiv -2i(F_{\bar{z}} + 2iD_{\bar{z}}\tilde{q})dw\wedge d\bar{w}.
\end{aligned} \tag{7.3.21}$$

In terms of these new fields, the holomorphic-topological supercharge is represented in a simple manner,

$$
\begin{aligned}
&\mathcal{Q}q_z = 0, \quad \mathcal{Q}\xi_z = \mathcal{D}_{\mathcal{C}^\perp} q_z, \quad \mathcal{Q}h_z = \mathcal{D}_{\mathcal{C}^\perp}\xi_z + i\nu \cdot q_z \\
&\mathcal{Q}\tilde{q} = 0, \quad \mathcal{Q}\sigma = \mathcal{D}_{\mathcal{C}^\perp}\tilde{q}, \quad \mathcal{Q}h = \mathcal{D}_{\mathcal{C}^\perp}\sigma + i\nu \cdot \tilde{q}, \\
&\mathcal{Q}\chi = h^\dagger, \quad \mathcal{Q}h^\dagger = 0, \quad \mathcal{Q}\eta_{\bar{z}} = h_{\bar{z}}, \quad \mathcal{Q}h_{\bar{z}} = 0, \\
&\mathcal{Q}\tilde{q}^\dagger = \gamma, \quad \mathcal{Q}\gamma = 0, \quad \mathcal{Q}q_{\bar{z}} = \zeta_{\bar{z}} \quad \mathcal{Q}\zeta_{\bar{z}} = 0.
\end{aligned}
\tag{7.3.22}
$$

Also in terms of the re-defined fields, the hypermultiplet action can be written as a linear combination of $\mathcal{Q}$-closed part and a $\mathcal{Q}$-exact part,

$$
S_{\text{hyp}} = S_{\text{hyp,cl}} + S_{\text{hyp,ext}}, \tag{7.3.23}
$$

where the $\mathcal{Q}$-exact part is given by

$$
\begin{aligned}
S_{\text{hyp,ext}} = \mathcal{Q}\Bigg\{ \frac{1}{g^2}\int_{\mathcal{C}} d^2z \int_{\mathcal{C}^\perp} \eta_{\bar{z}} \left(\star_{\mathcal{C}^\perp} h_z - \frac{i}{2} D_z \tilde{q}^\dagger\right) + \chi\left(\star_{\mathcal{C}^\perp} h + \frac{i}{2} D_z q_{\bar{z}}\right) - \frac{1}{2}\left(\tilde{q}^\dagger \alpha \cdot \tilde{q} + q_{\bar{z}} \alpha \cdot q_z\right) \\
- \bar{\mathcal{D}}_{\mathcal{C}^\perp}\tilde{q}^\dagger \wedge \star_{\mathcal{C}^\perp}\sigma - \bar{\mathcal{D}}_{\mathcal{C}^\perp} q_{\bar{z}} \wedge \star_{\mathcal{C}^\perp}\xi_z - \frac{1}{2} q_{\bar{z}}\mu_z \cdot \tilde{q}^\dagger \Bigg\},
\end{aligned}
\tag{7.3.24}
$$

whereas the $\mathcal{Q}$-closed part is given by

$$
S_{\text{hyp,cl}} = \frac{8i}{g^2}\int_{\mathcal{C}} d^2z \int_{\mathcal{C}^\perp} \xi_z \wedge D_{\bar{z}}\sigma + h_z D_{\bar{z}}\tilde{q} - h D_{\bar{z}} q_z - i q_z \mathsf{D}_{\bar{z}} \cdot \tilde{q} - i\tilde{q}\xi_z \wedge \rho_{\bar{z}} - i q_z \rho_{\bar{z}} \wedge \sigma. \tag{7.3.25}
$$

Combining the vector multiplet action (7.3.11) and the hypermultiplet action (7.3.24), (7.3.25), we obtain the full action of the holomorphic-topological theory

$$
S = S_{\text{vec}} + S_{\text{hyp,cl}} + S_{\text{hyp,ext}}. \tag{7.3.26}
$$

It should be reminded that, as firstly discovered in [189], the dependence on the metric on $\mathcal{C}^\perp$

and the Kähler form on $\mathcal{C}$ enters only through the $\mathcal{Q}$-exact terms, ensuring that the theory is topological along $\mathcal{C}^\perp$ and holomorphic along $\mathcal{C}$. Also note that we can absorb the gauge coupling in the $\mathcal{Q}$-closed part into the fields, so that the dependence on the gauge coupling also becomes absent. We also absorb the irrelevant numerical prefactors in some terms in S_{vec} by rescaling the metric on $\mathcal{C}$. Now we may take the theory on a general product metric background of $\mathcal{C} \times \mathcal{C}^\perp$ while its component fields take values of appropriate differential forms.

For later use, it is convenient to define the following specific combinations of component fields:

$$\begin{aligned} Q_z &\equiv q_z + \xi_z + h_z \\ \tilde{Q} &= \tilde{q} + \sigma + h \\ \mathcal{A}_{\bar{z}} &= A_{\bar{z}} + \rho_{\bar{z}} + \mathsf{D}_{\bar{z}}, \end{aligned} \tag{7.3.27}$$

on which our localizing supercharge will act as the equivariant differential on $\mathcal{C}^\perp$. Note that we can re-write the $\mathcal{Q}$-closed part of the action (7.3.25) using these combinations as

$$S_{\text{hyp,cl}} = 8i \int_{\mathcal{C}} d^2 z \int_{\mathcal{C}^\perp} Q_z \wedge (\partial_{\bar{z}} - i\mathcal{A}_{\bar{z}} \cdot)\tilde{Q}. \tag{7.3.28}$$

where $\cdot$ denotes the action according to the representation under the gauge group. This expression will turn out to be useful in finding the action of the localized theory on $\mathcal{C}$.

7.3.2 Ω-deformation

Suppose there is a vector field $V = \text{Vect}(\mathcal{C}^\perp)$ which generates an isometry on $\mathcal{C}^\perp$. The Ω-deformation can be defined at the level of supersymmetry variations of component fields, so that the deformed supercharge squares to this isometry plus possibly a gauge transformation, in a similar manner with [193, 194] for two-dimensional theories. For the case at hand, the holomorphic-topological theory on $\mathcal{C} \times \mathcal{C}^\perp$, we can deform the supersymmetry variations in

(7.3.10) and (7.3.22) as

$$\begin{aligned}
&\mathcal{Q}_\varepsilon \mathcal{A} = \varepsilon \iota_V \nu, \quad \mathcal{Q}_\varepsilon \bar{\mathcal{A}} = \lambda - \varepsilon \iota_V \nu, \\
&\mathcal{Q}_\varepsilon \lambda = 2\varepsilon \iota_V F - 2i\varepsilon D_{\mathcal{C}^\perp} \iota_V \phi, \quad \mathcal{Q}_\varepsilon \nu = \mathcal{F}, \\
&\mathcal{Q}_\varepsilon \alpha = \mathsf{D}, \quad \mathcal{Q}_\varepsilon \mathsf{D} = \varepsilon \iota_V \mathcal{D}_{\mathcal{C}^\perp} \alpha, \\
&\mathcal{Q}_\varepsilon A_{\bar{z}} = \varepsilon \iota_V \rho_{\bar{z}}, \quad \mathcal{Q}_\varepsilon A_z = \theta_z, \\
&\mathcal{Q}_\varepsilon \rho_{\bar{z}} = \mathcal{F}_{\bar{z}} + \varepsilon \iota_V \mathsf{D}_{\bar{z}}, \quad \mathcal{Q}_\varepsilon \theta_z = \varepsilon \iota_V \mathcal{F}_z, \\
&\mathcal{Q}_\varepsilon \mathsf{D}_{\bar{z}} = \mathcal{D}_{\mathcal{C}^\perp} \rho_{\bar{z}} + D_{\bar{z}} \nu, \quad \mathcal{Q}_\varepsilon \mu_z = \mathsf{D}_z, \quad \mathcal{Q}_\varepsilon \mathsf{D}_z = \varepsilon \mathcal{D}_{\mathcal{C}^\perp} \iota_V \mu_z,
\end{aligned} \tag{7.3.29}$$

for the vector multiplet and

$$\begin{aligned}
&\mathcal{Q}_\varepsilon q_z = \varepsilon \iota_V \xi_z, \quad \mathcal{Q}_\varepsilon \xi_z = \mathcal{D}_{\mathcal{C}^\perp} q_z + \varepsilon \iota_V h_z, \quad \mathcal{Q}_\varepsilon h_z = \mathcal{D}_{\mathcal{C}^\perp} \xi_z + i\nu \cdot q_z \\
&\mathcal{Q}_\varepsilon \tilde{q} = \varepsilon \iota_V \sigma, \quad \mathcal{Q}_\varepsilon \sigma = \mathcal{D}_{\mathcal{C}^\perp} \tilde{q} + \varepsilon \iota_V h, \quad \mathcal{Q}_\varepsilon h = \mathcal{D}_{\mathcal{C}^\perp} \sigma + i\nu \cdot \tilde{q}, \\
&\mathcal{Q}_\varepsilon \chi = h^\dagger, \quad \mathcal{Q}_\varepsilon h^\dagger = \varepsilon \mathcal{D}_{\mathcal{C}^\perp} \iota_V \chi, \quad \mathcal{Q}_\varepsilon \eta_{\bar{z}} = h_{\bar{z}}, \quad \mathcal{Q}_\varepsilon h_{\bar{z}} = \varepsilon \mathcal{D}_{\mathcal{C}^\perp} \iota_V \eta_{\bar{z}}, \\
&\mathcal{Q}_\varepsilon \tilde{q}^\dagger = \gamma, \quad \mathcal{Q}_\varepsilon \gamma = \varepsilon \iota_V \mathcal{D}_{\mathcal{C}^\perp} \tilde{q}^\dagger, \quad \mathcal{Q}_\varepsilon q_{\bar{z}} = \zeta_{\bar{z}} \quad \mathcal{Q}_\varepsilon \zeta_{\bar{z}} = \varepsilon \iota_V \mathcal{D}_{\mathcal{C}^\perp} q_{\bar{z}}.
\end{aligned} \tag{7.3.30}$$

for each hypermultiplet. Note that

$$\mathcal{Q}_\varepsilon^2 = \varepsilon(\mathcal{D}_{\mathcal{C}^\perp} \iota_V + \iota_V \mathcal{D}_{\mathcal{C}^\perp}) = \varepsilon \mathcal{L}_V + \mathrm{Gauge}[\varepsilon \iota_V \mathcal{A}], \tag{7.3.31}$$

where the first term is the Lie derivative with respect to the vector field V and the second term is the infinitesimal gauge transformation generated by $\varepsilon \iota_V \mathcal{A}$. Hence the deformed supercharge squares to an isometry generated by V plus a gauge transformation. Also it is immediate that $\mathcal{Q}_\varepsilon$ reduces to the original holomorphic-topological supercharge $\mathcal{Q}$ when $\varepsilon = 0$. Hence $\mathcal{Q}_\varepsilon$ indeed implements the Ω-deformation of the holomorphic-topological theory on $\mathcal{C} \times \mathcal{C}^\perp$ with respect to the isometry V.

We should correspondingly deform the action so that it is annihilated by the deformed supercharge. The action for the vector multiplet can be taken as the variation under the deformed supercharge of the same expression:

$$
\begin{aligned}
S_{\text{vec},\varepsilon} = \mathcal{Q}_\varepsilon \int_{\mathcal{C}} d^2 z \int_{\mathcal{C}^\perp} \operatorname{Tr} \Big[& -\bar{\mathcal{F}} \star_{\mathcal{C}^\perp} \nu - \alpha \left(\star_{\mathcal{C}^\perp} \mathsf{D} - 2i D_{\mathcal{C}^\perp} \star_{\mathcal{C}^\perp} \phi - \star_{\mathcal{C}^\perp} F_{z\bar{z}} \right) \\
& + \bar{\mathcal{F}}_z \wedge \star_{\mathcal{C}^\perp} \rho_{\bar{z}} + \mu_z \star_{\mathcal{C}^\perp} \mathsf{D}_{\bar{z}} \Big] .
\end{aligned} \tag{7.3.32}
$$

Similarly the $\mathcal{Q}$-exact part of the hypermultiplet action can be modified to:

$$
\begin{aligned}
S_{\text{hyp,ext},\varepsilon} = \mathcal{Q}_\varepsilon \int_{\mathcal{C}} d^2 z \int_{\mathcal{C}^\perp} & \eta_{\bar{z}} \left(\star_{\mathcal{C}^\perp} h_z - \frac{i}{2} D_z \tilde{q}^\dagger \right) + \chi \left(\star_{\mathcal{C}^\perp} h + \frac{i}{2} D_z q_{\bar{z}} \right) - \frac{1}{2} \left(\tilde{q}\alpha \cdot \tilde{q}^\dagger + q_{\bar{z}} \alpha \cdot q_z \right) \\
& - \bar{\mathcal{D}}_{\mathcal{C}^\perp} \tilde{q}^\dagger \wedge \star_{\mathcal{C}^\perp} \sigma - \bar{\mathcal{D}}_{\mathcal{C}^\perp} q_{\bar{z}} \wedge \star_{\mathcal{C}^\perp} \xi_z - \frac{1}{2} q_{\bar{z}} \mu_z \cdot \tilde{q}^\dagger .
\end{aligned} \tag{7.3.33}
$$

Since V generates an isometry on $\mathcal{C}^\perp$, $\mathcal{L}_V$ leaves the metric invariant and commutes with $\star_{\mathcal{C}^\perp}$. Hence (7.3.31) guarantees that these actions are $\mathcal{Q}_\varepsilon$-invariant.

When there is no boundary on $\mathcal{C}^\perp$, the $\mathcal{Q}_\varepsilon$-closed part of the action can be taken as before (7.3.28). It is straightforward to check the $\mathcal{Q}_\varepsilon$-invariance of this action.

7.3.3 Gauge-fixing

Gauge-fixing is needed to properly evaluate the path integral. We implement the gauge-fixing by the standard BRST procedure. We introduce a ghost c, an antighost $\bar{c}$, and an auxiliary field p which are in adjoint representation of the gauge group. The BRST transformations of these fields are

$$
\begin{aligned}
& \mathcal{Q}_B c = -\frac{i}{2}\{c, c\}, \quad \mathcal{Q}_B \bar{c} = p, \quad \mathcal{Q}_B p = 0, \\
& \mathcal{Q}_B X = \text{Gauge}[c] X,
\end{aligned} \tag{7.3.34}
$$

where X denotes all other component fields introduced in previous section. We also postulate the $\mathcal{Q}_\varepsilon$-variations for these fields as

$$\mathcal{Q}_\varepsilon c = -\varepsilon \iota_V \mathcal{A}, \quad \mathcal{Q}_\varepsilon \bar{c} = 0, \quad \mathcal{Q}_\varepsilon p = \varepsilon \iota_V d_{\mathcal{C}^\perp} \bar{c}. \tag{7.3.35}$$

Now we define a new supercharge $\hat{\mathcal{Q}}$ as the combination of the Ω-deformed supercharge and the BRST supercharge, $\hat{\mathcal{Q}} = \mathcal{Q}_\varepsilon + \mathcal{Q}_B$. Then we observe that

$$\hat{\mathcal{Q}}^2 = \varepsilon(d_{\mathcal{C}^\perp}\iota_V + \iota_V d_{\mathcal{C}^\perp}) = \varepsilon \mathcal{L}_V \tag{7.3.36}$$

for all fields. Note that the supercharge now squares to the isometry generated by V without any gauge transformation. We use this supercharge $\hat{\mathcal{Q}}$ to construct our cohomological field theory.

Since (7.3.32) and (7.3.33) are defined as $\mathcal{Q}_\varepsilon$-variations of gauge invariant expressions, they are also automatically $\hat{\mathcal{Q}}$-exact:

$$\begin{aligned}
S_{\text{vec},\varepsilon} = \hat{\mathcal{Q}} \int_{\mathcal{C}} d^2z \int_{\mathcal{C}^\perp} \operatorname{Tr} \Big[&-\bar{\mathcal{F}} \star_{\mathcal{C}^\perp} \nu - \alpha \left(\star_{\mathcal{C}^\perp} \mathsf{D} - 2i D_{\mathcal{C}^\perp} \star_{\mathcal{C}^\perp} \phi - \star_{\mathcal{C}^\perp} F_{z\bar{z}} \right) \\
&+ \bar{\mathcal{F}}_z \wedge \star_{\mathcal{C}^\perp} \rho_{\bar{z}} + \mu_z \star_{\mathcal{C}^\perp} \mathsf{D}_{\bar{z}} \Big] \\
S_{\text{hyp,ext},\varepsilon} = \hat{\mathcal{Q}} \int_{\mathcal{C}} d^2z \int_{\mathcal{C}^\perp} & \eta_{\bar{z}} \left(\star_{\mathcal{C}^\perp} h_z - \frac{i}{2} D_z \tilde{q}^\dagger \right) + \chi \left(\star_{\mathcal{C}^\perp} h + \frac{i}{2} D_z q_{\bar{z}} \right) - \frac{1}{2} \left(\tilde{q}\alpha \cdot \tilde{q}^\dagger + q_{\bar{z}} \alpha \cdot q_z \right) \\
&- \bar{\mathcal{D}}_{\mathcal{C}^\perp} \tilde{q}^\dagger \wedge \star_{\mathcal{C}^\perp} \sigma - \bar{\mathcal{D}}_{\mathcal{C}^\perp} q_{\bar{z}} \wedge \star_{\mathcal{C}^\perp} \xi_z - \frac{1}{2} q_{\bar{z}} \mu_z \cdot \tilde{q}^\dagger,
\end{aligned} \tag{7.3.37}$$

It is also clear that (7.3.28) is $\hat{\mathcal{Q}}$-closed since it is gauge-invariant. To gauge-fix we introduce another $\hat{\mathcal{Q}}$-exact term to the action

$$S_{\text{fix}} = \hat{\mathcal{Q}} \int \operatorname{Tr} \bar{c}\, G_{\text{fix}}, \tag{7.3.38}$$

where G_{fix} is a properly chosen gauge-fixing function. We will take the standard Lorentz gauge-fixing function

$$G_{\text{fix}} = \nabla_\mu A^\mu, \quad \mu = w, \bar{w}, \tag{7.3.39}$$

where ∇ the Levi-Civita connection on $\mathcal{C}^\perp$, while keeping the gauge redundancy on $\mathcal{C}$ intact. We will fix the residual gauge redundancy after localizing the theory onto $\mathcal{C}$.

7.3.4 Localization

For the purpose of recovering the chiral CFT on $\mathcal{C}$, we will take $\mathcal{C}^\perp = \mathbb{R}^2$ from now on. Let us analyze the localization locus of the path integral. The auxiliary fields D_z, $h_{\bar{z}}$, and $h^\dagger$ only enters in the action in linear terms. Hence we can integrate them out to find

$$\begin{aligned} \mathsf{D}_{\bar{z}} &= \frac{1}{2} \star_{\mathcal{C}^\perp} q_{\bar{z}} \tilde{q}^\dagger \\ h_z &= \frac{i}{2} \star_{\mathcal{C}^\perp} D_z \tilde{q}^\dagger \\ h &= -\frac{i}{2} \star_{\mathcal{C}^\perp} D_z q_{\bar{z}}. \end{aligned} \tag{7.3.40}$$

The localization locus is given by the fixed point set of the supersymmetry variations. Hence we set the right hand sides of (7.3.29) and (7.3.30) to zero. Thus we have, among other equatoins,

$$\begin{aligned} &\mathcal{F} = 0, \quad \iota_V F - i D_{\mathcal{C}^\perp} \iota_V \phi = 0, \quad \mathsf{D} = 0, \\ &\mathcal{F}_{\bar{z}} + \varepsilon \iota_V \mathsf{D}_{\bar{z}} = 0, \quad \mathcal{D}_{\mathcal{C}^\perp} q_z + \varepsilon \iota_V h_z = 0, \quad \mathcal{D}_{\mathcal{C}^\perp} \tilde{q} + \varepsilon \iota_V h = 0. \end{aligned} \tag{7.3.41}$$

From the equations in the first low and the gauge-fixing condition, we get $A = 0$. Applying this to the equations in the second row yields, among other equations, $D_{\bar{z}}\phi = [\phi, q_z] = [\phi, \tilde{q}] = 0$ and thus $\phi = 0$ for non-trivial solutions of $A_{\bar{z}}$, q_z, and $\tilde{q}$. Then we arrive at

$$\mathcal{A} = 0. \tag{7.3.42}$$

With (7.3.40) and (7.3.42) the rest of the equations in the second row of (7.3.41) yield

$$\begin{aligned} d_{\mathcal{C}^\perp} A_{\bar{z}} &= -\frac{1}{2}\varepsilon\iota_V \star_{\mathcal{C}^\perp} q_{\bar{z}}\tilde{q}^\dagger \\ d_{\mathcal{C}^\perp} q_z &= -\frac{i}{2}\varepsilon\iota_V \star_{\mathcal{C}^\perp} D_z\tilde{q}^\dagger \\ d_{\mathcal{C}^\perp}\tilde{q} &= \frac{i}{2}\varepsilon\iota_V \star_{\mathcal{C}^\perp} D_z q_{\bar{z}}. \end{aligned} \tag{7.3.43}$$

Let us introduce the polar coordinate on $\mathcal{C}^\perp = \mathbb{R}^2$, where the flat metric on $\mathcal{C}^\perp$ is simply written as $ds^2_{\mathcal{C}^\perp} = dr^2 + r^2 d\varphi^2$. Then our generator of the isometry is $V = \partial_\varphi$. The equations (7.3.43) can be written in the polar coordinates as

$$\begin{aligned} &\partial_\varphi A_{\bar{z}} = 0, \quad \partial_\varphi q_z = 0, \quad \partial_\varphi \tilde{q} = 0, \\ &\partial_r A_{\bar{z}} = -\frac{1}{2}\varepsilon r q_{\bar{z}}\tilde{q}^\dagger, \quad \partial_r q_z = -\frac{i}{2}\varepsilon r D_z\tilde{q}^\dagger, \quad \partial_r\tilde{q} = \frac{i}{2}\varepsilon r D_z q_{\bar{z}}. \end{aligned} \tag{7.3.44}$$

By re-defining the radial coordinate by $t = \varepsilon\bar{\varepsilon}\frac{r^2}{2}$, the equations in the second line become

$$\partial_t A_{\bar{z}} = -\frac{1}{2\bar{\varepsilon}} q_{\bar{z}}\tilde{q}^\dagger, \quad \partial_t q_z = -\frac{i}{2\bar{\varepsilon}} D_z\tilde{q}^\dagger, \quad \partial_t\tilde{q} = \frac{i}{2\bar{\varepsilon}} D_z q_{\bar{z}}. \tag{7.3.45}$$

Solutions to these equations are precisely the gradient trajectories on which two-dimensional B-model on $\mathbb{R}^2$ localizes, as discussed in [193] with its full detail, generated by the function $\mathrm{Re}\left(\frac{\mathcal{W}}{\varepsilon}\right)$ where $\mathcal{W}$ is the holomorphic superpotential. For the case at hand, one could view the four-dimensional holomorphic-topological theory on $\mathcal{C} \times \mathcal{C}^\perp$ as a two-dimensional B-twisted gauge theory on $\mathcal{C}^\perp$, as done in [194] for the six-dimensional holomorphic-topological theory to obtain the four-dimensional Chern-Simons theory. The superpotential has to be chosen as $\mathcal{W} = \int_{\mathcal{C}} d^2z\, q_z D_{\bar{z}}\tilde{q}$ to reproduce the four-dimensional holomorphic-topological theory in the $\varepsilon \to 0$ limit. As we approch to infinity $t \to \infty$, $A_{\bar{z}}$, q_z, and $\tilde{q}$ should end on the critical points $\{d\mathcal{W} = 0\}$ of the superpotential to guarantee that the action (7.3.28) does not diverge [193].[2]

[2] Generally, when the critical points are non-isolated we can choose a Lagrangian submanifold of the critical points to have a constant one-loop determinant [194], so we make such a choice here.

Now that we identified the localization locus, let us evaluate the effective action of the localized path integral. Recall that our theory is holomorphic along $\mathcal{C}$, so that the localized path integral should define a two-dimensional chiral CFT on $\mathcal{C}$. Also note that the localization locus does not contain any non-trivial topological sector of gauge field configurations, so that all we have to do is to evaluate the $\hat{\mathcal{Q}}$-closed part of the action on the localization locus properly. This can be accomplished by performing an equivariant integration on $\mathcal{C}^{\perp} = \mathbb{R}^2$ for the action integral (7.3.28) as follows.

To facilitate the equivariant integration, it is crucial to note that $\hat{\mathcal{Q}}$ acts on the combinations (7.3.27) as the equivariant differential $d_{\mathcal{C}^{\perp}} + \varepsilon \iota_V$ on $\mathcal{C}^{\perp} = \mathbb{R}^2$ plus a *gauge covariant* contribution:

$$\begin{aligned} \hat{\mathcal{Q}} Q_z &= (d_{\mathcal{C}^{\perp}} + \varepsilon \iota_V - iC\cdot) Q_z \\ \hat{\mathcal{Q}} \tilde{Q} &= (d_{\mathcal{C}^{\perp}} + \varepsilon \iota_V - iC\cdot) \tilde{Q} \\ \hat{\mathcal{Q}} \mathcal{A}_{\bar{z}} &= (d_{\mathcal{C}^{\perp}} + \varepsilon \iota_V - iC\cdot) \mathcal{A}_{\bar{z}} - \partial_{\bar{z}} C, \end{aligned} \tag{7.3.46}$$

where

$$C \equiv c + \mathcal{A} + \nu \tag{7.3.47}$$

acts as if it is a *gauge connection* for those complexes. Note that the last term in the third equality ensures that $\partial_{\bar{z}} - i\mathcal{A}_{\bar{z}}$ can be treated as a *covariant derivative* as far as $\hat{\mathcal{Q}}$-variation is concerned, namely it preserves the gauge charge:

$$\hat{\mathcal{Q}} \left((\partial_{\bar{z}} - i\mathcal{A}_{\bar{z}}\cdot) \tilde{Q} \right) = (d_{\mathcal{C}^{\perp}} + \varepsilon \iota_V - iC\cdot) \left((\partial_{\bar{z}} - i\mathcal{A}_{\bar{z}}\cdot) \tilde{Q} \right). \tag{7.3.48}$$

This would have failed without the last term of the third equality of (7.3.46). Therefore, $\hat{\mathcal{Q}}$

acts as the equivariant differential on the gauge-invariant combination,

$$\hat{\mathcal{Q}}\left(Q_z\wedge(\partial_{\bar z}-i\mathcal{A}_{\bar z}\cdot)\tilde{Q}\right)=(d_{\mathcal{C}^\perp}+\varepsilon\iota_V)\left(Q_z\wedge(\partial_{\bar z}-i\mathcal{A}_{\bar z}\cdot)\tilde{Q}\right). \tag{7.3.49}$$

In other words, $Q_z\wedge(\partial_{\bar z}-i\mathcal{A}_{\bar z}\cdot)\tilde{Q}$ is equivariantly closed when it is viewed as an element in the $\hat{\mathcal{Q}}$-cohomology. Hence we apply the Atiyah-Bott equivariant localization formula for the action integral (7.3.28) (while absorbing irrelevant numerical constant in front into q_z and $\tilde q$) to obtain

$$S_{\text{hyp,cl}}=\frac{1}{\varepsilon}\int_{\mathcal{C}}d^2z\,q_zD_{\bar z}\tilde q, \tag{7.3.50}$$

where $q=q_zdz$ is a $(1,0)$-form in the representation R and $\tilde q$ is a 0-form in the representation $\bar R$ of the gauge group, respectively. Here q_z, $\tilde q$, and $A_{\bar z}$ are understood as solutions to the gradient trajectory equations (7.3.45) evaluated at the origin of $\mathcal{C}^\perp$, $w=\bar w=0$. Note that the Ω-deformation parameter ε appears in the denominator of the action since $\mathcal{C}^\perp=\mathbb{R}^2$ has the unit weight under the isometry of $V=\partial_\varphi$. Consequently ε plays the role of the Planck constant of the localized theory on $\mathcal{C}$, which therefore appears in the numerator of the OPEs. Hence we confirm the identification of the non-commutative deformation parameter and the Ω-deformation parameter.

Now we choose to fix the residual gauge by the gauge-fixing function $A_{\bar z}=0$, yielding the gauge fixing term in the action

$$\frac{1}{\varepsilon}\int_{\mathcal{C}}d^2z\,\mathrm{Tr}\,(-p_zA_{\bar z}+b_zD_{\bar z}c). \tag{7.3.51}$$

Hence when the auxiliary field p_z is integrated out, we are left with

$$\frac{1}{\varepsilon}\int_{\mathcal{C}}\left(\mathrm{Tr}\,b\bar\partial c+\sum_i q^i\bar\partial\tilde q^i\right), \tag{7.3.52}$$

where i enumerates all the hypermultiplets that we coupled to the vector multiplet to make the original $\mathcal{N}=2$ theory superconformal. The algebra generated by the local operators of this theory are nothing but the chiral algebra of the standard bc-$\beta\gamma$ system with the BRST charge

$$Q_{\mathrm{BRST}} = \frac{1}{\varepsilon}\oint \frac{dz}{2\pi i}\left(\mathrm{Tr}\, bcc - \sum_i q^i c\tilde{q}^i\right). \tag{7.3.53}$$

Hence we arrive at the result expected from [181].

7.4 Superconformal indices and vacuum characters

As a consequence of the SCFT/VOA correspondence in [181], the Schur index of the $\mathcal{N}=2$ SCFT and the vacuum character of the chiral algebra are identified by directly comparing their state-counting formulas. Here we discuss how the Ω-deformation approach provides a path integral point of view on the identification.

7.4.1 Schur index of $\mathcal{N}=2$ SCFT

The Schur index is defined by the Schur limit of the $\mathcal{N}=2$ superconformal index [196]. It is given as

$$\mathcal{I}_S = \mathrm{Tr}_{\mathcal{H}_S}\,(-1)^F q^{E-R}, \tag{7.4.1}$$

where E is the scaling dimension and R is the Cartan of the $SU(2)_R$ R-symmetry as before. The trace is over the $\frac{1}{4}$-BPS states satisfying

$$\mathcal{H}_S: \quad E-(j_1+j_2)-2R=0, \quad j_1-j_2+r=0. \tag{7.4.2}$$

The operators corresponding to these states are called Schur operators. It is straightforward to compute the single-letter indices for the vector multiplet and the hypermultiplet by finding those operators in the component fields. The full index is simply given by the plethystic exponential of the sum of all the single-letter indices, integrated over the gauge group. We will not reproduce the exact forms of those expressions here.

7.4.2 Schur index and vacuum character

In [197], the Schur index was derived by supersymmetric localization of $\mathcal{N} = 2$ SCFT partition function on $S^3 \times S^1$, up to a multiplicative factor of the Casimir energy. The metric background used in the computation was the following S^3-fibration over S^1:

$$ds^2 = l^2 \cos^2\theta \,(d\phi - (\beta_1 + \beta_2)dt)^2 + l^2 \sin^2\theta \,(d\chi - (\beta_1 - \beta_2)dt)^2 + l^2 d\theta^2 - l^2(\tau + (\beta_1 + \beta_2))^2 dt^2, \tag{7.4.3}$$

where ϕ, χ, and t are periodic coordinates with period 2π and $\theta \in [0, \frac{\pi}{2}]$. l is the radius of the three-sphere which was written as a torus fibration over the θ-interval. It was shown in [197] that the variations of β_1 and β_2 do not affect the partition function. Then β_1 and β_2 were chosen to be real and $\mathrm{Re}\,\tau = -(\beta_1 + \beta_2)$ so that above metric restricts to the Kähler metric on the torus at $\theta = 0$. Here, we may make a different choice of these parameters at our convenience: $\beta_1 = -\beta_2 = \beta$. Then the metric becomes

$$ds^2 = l^2(-\tau^2 dt^2 + \cos^2\theta d\phi^2) + l^2(d\theta^2 + \sin^2\theta(d\chi - 2\beta dt)^2). \tag{7.4.4}$$

First note that $\sin\theta \sim \theta$ as $\theta \sim 0$ and $\sin\theta \sim \text{const}$ as $\theta \sim \frac{\pi}{2}$. Hence the above metric describes the product of a torus (t, ϕ) and a cigar (θ, χ) with a twist along the t direction with respect to the isometry on the cigar $V = \partial_\chi$. This is precisely the context where the Ω-deformation is realized as a metric background [7, 31].

Hence we can make a direct connection between the partition functions in four-dimension and two-dimension. First we localize the Ω-deformed holomorphic-topological theory on $\mathcal{C}$, as we have shown in the previous section, we would get the chiral CFT of the gauged symplectic bosons on the torus $\mathcal{C}$ with the metric

$$ds^2_{\mathcal{C}} = l^2(-\tau^2 dt^2 + d\phi^2) = l^2 dz d\tilde{z}, \tag{7.4.5}$$

where $z = \phi + \tau t$ and $\tilde{z} = \phi - \tau t$. Hence the we arrive at the identification of the $S^3 \times S^1$ partition function of the four-dimensinoal SCFT with the torus partition function of the two-dimensional chiral CFT.

In the localizing supercharge related to the holomorphic-topological twist, the $S^3 \times S^1$ partition function of four-dimensinoal SCFT computes the Schur index of the theory up to some multiplicative factor of Casimir energy [197]. Also the torus partition functions of two-dimensional chiral CFT compute the characters of the chiral algebra. Hence we re-discover one of the consequences of the SCFT/VOA correspondence found in [181], the identification of the Schur index and the vaccum character. To fully justify the appearance of the vacuum character, we really have to understand how the boundary condition (periodicity) for the local operators follows from the metric (7.4.4). It should be inherited from the β-deformation of the metric from the usual $S^3 \times S^1$ to (7.4.4), so that a more detailed analysis is needed on the meric (7.4.4) as the Ω-background. We leave this to future work.

7.5 Discussion

In the $(\mathcal{Q}+\mathcal{S})$-cohomology construction of the chiral algebra [181], the bc-system is obtained by the cohomology of the Schur operators in the vector multiplets: the gaugini. In our notation, the relevant gaugini fields are precisely μ_z and θ_z in (7.3.8). However, it is still not very clear how these fields are related to the bc-system in our construction which arises in the two-dimensional gauge fixing. The main problem is that neither μ_z nor θ_z is $\mathcal{Q}$-closed, so that it is not immediate to see how they come into play in the $\mathcal{Q}$-cohomological field theory.

It would be nice if we can understand this issue more clearly.

An interesting observation was made in [198] for the SCFT/VOA correspondence at the level of $\mathcal{N} = 2$ superconformal indices. It discovered a relation between the Macdonald index, a refinement of the Schur index, of $\mathcal{N} = 2$ SCFTs and the refined character of VOAs. A conjectural construction of a filtration of the vacuum module was suggested, from which the refined character was defined by its associated graded vector space. In [199], the construction of such filtrations was analyzed in great detail. It would be nice if we can understand this relation through the Ω-deformation formulation of the chiral algebra. A path integral representation of the Macdonald index or the suggested refined character would be helpful for this study.

Finally, the Ω-deformation approach to the chiral algebra discussed so far only applies to Lagrangian SCFTs. It was observed that in some cases there are $\mathcal{N} = 1$ preserving deformations of $\mathcal{N} = 2$ SQCDs such that the renormalization group flows from the deformed SQCDs to non-Lagrangian $\mathcal{N} = 2$ SCFTs such as Argyres-Douglas theories and Minahan-Nemeschansky theories [200, 201, 202, 203, 204]. It would be nice if we could find a way to apply the Ω-deformation procedure to the non-Lagrangian SCFTs, perhaps by using such deformations.

Part IV

Appendices

Appendix A

Examples of split surface defect partition functions

A.1 $N = 2$

For $N = 2$ case we can compare the results from the gauge theory with the well-known Mathieu functions with half-period and whole-period, in [205] for example. We observe the precise match between the two.

i) $a_{01} = \varepsilon_1$

$$
\begin{aligned}
&\widetilde{\mathbf{\Psi}}_{\mathrm{id}}(a_{01}=\varepsilon_1,\boldsymbol{\varepsilon},\Lambda,\mathbf{z}) \pm \widetilde{\mathbf{\Psi}}_{(01)}(a_{01}=\varepsilon_1,\boldsymbol{\varepsilon},\Lambda,\mathbf{z}) \\
&= e^{\frac{1}{\varepsilon_2}\left(\pm\frac{\Lambda^2}{\varepsilon_1}+\frac{\Lambda^4}{4\varepsilon_1^3}\mp\frac{\Lambda^6}{12\varepsilon_1^5}+\frac{\Lambda^8}{96\varepsilon_1^7}\pm\frac{11\Lambda^{10}}{720\varepsilon_1^9}+\mathcal{O}(\Lambda^{12})\right)} \\
&\quad\left[z^{1/2}\pm z^{-1/2}+\frac{\Lambda^2}{\varepsilon_1^2}\frac{z^{3/2}\pm z^{-3/2}}{2}+\frac{\Lambda^4}{\varepsilon_1^4}\left(\frac{z^{5/2}\pm z^{-5/2}}{12}-\frac{z^{1/2}\pm z^{-1/2}}{8}-\frac{z^{-3/2}\pm z^{3/2}}{12}\right)\right. \\
&+\frac{\Lambda^6}{\varepsilon_1^6}\left(\frac{z^{7/2}\pm z^{-7/2}}{144}+\frac{\mp z^{5/2}-z^{-5/2}}{18}-\frac{z^{3/2}\pm z^{-3/2}}{48}+\frac{\pm z^{1/2}+z^{-1/2}}{8}\right) \\
&+\frac{\Lambda^8}{\varepsilon_1^8}\left(\frac{z^{9/2}\pm z^{-9/2}}{2880}+\frac{\mp z^{7/2}-z^{-7/2}}{192}+\frac{49(\pm z^{3/2}+z^{-3/2})}{28}-\frac{37(z^{1/2}\pm z^{-1/2})}{1152}\right) \\
&+\frac{\Lambda^{10}}{\varepsilon_1^{10}}\left(\frac{z^{11/2}\pm z^{-11/2}}{86400}+\frac{\mp z^{9/2}-z^{-9/2}}{3600}+\frac{z^{7/2}\pm z^{-7/2}}{5760}\right. \\
&\qquad\left.\left.+\frac{41(\pm z^{5/2}+z^{-5/2})}{1152}-\frac{317(z^{3/2}\pm z^{-3/2})}{2304}-\frac{121(\pm z^{1/2}+z^{-1/2})}{1728}\right)+\mathcal{O}(\Lambda^{12})+\mathcal{O}(\varepsilon_2)\right]
\end{aligned}
\tag{A.1.1}
$$

Using the dictionary (3.5.5) we compute

$$
E^{\pm}_{2,m=1} = \frac{\varepsilon_1^2}{8} \mp \frac{\Lambda^2}{2} - \frac{\Lambda^4}{4\varepsilon_1^2} \pm \frac{\Lambda^6}{8\varepsilon_1^4} - \frac{\Lambda^8}{48\varepsilon_1^6} \mp \frac{11\Lambda^{10}}{288\varepsilon_1^8} + \mathcal{O}(\Lambda^{12}). \tag{A.1.2}
$$

These split eigenvalues and split eigenstate wavefunctions exactly match with the known results for the Mathieu function.

ii) $a_{01} = 2\varepsilon_1$

$$\widetilde{\mathbf{\Psi}}_{\mathrm{id}}(a_{01} = 2\varepsilon_1, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z}) \pm \widetilde{\mathbf{\Psi}}_{(01)}(a_{01} = 2\varepsilon_1, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z})$$
$$= e^{\frac{1}{\varepsilon_2}\left(\left(-\frac{1}{3}\mp\frac{1}{2}\right)\frac{\Lambda^4}{\varepsilon_1^3}+\left(\frac{379}{864}\pm\frac{4}{9}\right)\frac{\Lambda^8}{\varepsilon_1^7}+\mathcal{O}(\Lambda^{12})\right)}$$
$$\Bigg[z \pm z^{-1} + \frac{\Lambda^2}{\varepsilon_1^2}\left(\frac{z^2 \pm z^{-2}}{3} - (1 \pm 1)\right) + \frac{\Lambda^4}{\varepsilon_1^4}\left(\frac{z^3 \pm z^{-3}}{24} - \left(\frac{5}{9} \pm \frac{1}{2}\right)(z \pm z^{-1})\right)$$
$$+ \frac{\Lambda^6}{\varepsilon_1^6}\left(\frac{z^4 \pm z^{-4}}{360} + \left(-\frac{7}{72} \mp \frac{1}{18}\right)(z^2 \pm z^{-2}) + \frac{49}{18}(1 \pm 1)\right)$$
$$+ \frac{\Lambda^8}{\varepsilon_1^8}\left(\frac{z^5 \pm z^{-5}}{8640} - \left(\frac{1}{120} \pm \frac{1}{576}\right)(z^3 \pm z^{-3}) + \left(\frac{25655}{10368} \pm \frac{133}{54}\right)(z \pm z^{-1})\right)$$
$$+ \frac{\Lambda^{10}}{\varepsilon_1^{10}}\left(\frac{z^6 \pm z^{-6}}{302400} - \left(\frac{11}{25920} \mp \frac{1}{14400}\right)(z^4 \pm z^{-4}) + \left(\frac{91283}{155520} \pm \frac{37}{64}\right)(z^2 \pm z^{-2}) - \frac{134855}{5184}(1 \pm 1)\right)$$
$$+\mathcal{O}(\Lambda^{12}) + \mathcal{O}(\varepsilon_2)\Bigg] \tag{A.1.3}$$

$$E^{\pm}_{2,m=2} = \frac{\varepsilon_1^2}{2} + \left(\frac{1}{3} \pm \frac{1}{2}\right)\frac{\Lambda^4}{\varepsilon_1^2} - \left(\frac{379}{432} \pm \frac{8}{9}\right)\frac{\Lambda^8}{\varepsilon_1^6} + \mathcal{O}(\Lambda^{12}) \tag{A.1.4}$$

iii) $a_{01} = 3\varepsilon_1$

$$\widetilde{\mathbf{\Psi}}_{\mathrm{id}}(a_{01} = 3\varepsilon_1, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z}) \pm \widetilde{\mathbf{\Psi}}_{(01)}(a_{01} = 3\varepsilon_1, \boldsymbol{\varepsilon}, \Lambda, \mathbf{z})$$
$$= e^{\frac{1}{\varepsilon_2}\left(-\frac{\Lambda^4}{8\varepsilon_1^3}\pm\frac{\Lambda^6}{12\varepsilon_1^5}-\frac{13\Lambda^8}{1280\varepsilon_1^7}\mp\frac{\Lambda^{10}}{64\varepsilon_1^9}+\mathcal{O}(\Lambda^{12})\right)}$$
$$\Bigg[z^{3/2} \mp z^{-3/2} + \frac{\Lambda^2}{\varepsilon_1^2}\left(\frac{z^{5/2} \mp z^{-5/2}}{4} - \frac{z^{1/2} \mp z^{-1/2}}{2}\right) + \frac{\Lambda^4}{\varepsilon_1^4}\left(\frac{z^{7/2} \mp z^{-7/2}}{40} - \frac{5(z^{3/2} \mp z^{-3/2})}{32} - \frac{z^{1/2} \mp z^{-1/2}}{4}\right)$$
$$+ \frac{\Lambda^6}{\varepsilon_1^6}\left(\frac{z^{9/2} \mp z^{-9/2}}{720} - \frac{11(z^{5/2} \mp z^{-5/2})}{640} + \frac{\mp z^{3/2} + z^{-3/2}}{8} + \frac{z^{1/2} \mp z^{-1/2}}{64}\right)$$
$$+ \frac{\Lambda^8}{\varepsilon_1^8}\left(\frac{z^{11/2} \mp z^{-11/2}}{20160} - \frac{11(z^{7/2} \mp z^{-7/2})}{11520} + \frac{\mp z^{5/2} + z^{-5/2}}{64} - \frac{1621(z^{3/2} \mp z^{-3/2})}{51200} + \frac{21(\pm z^{1/2} - z^{-1/2})}{128}\right)$$
$$+ \frac{\Lambda^{10}}{\varepsilon_1^{10}}\Bigg(\frac{z^{13/2} \mp z^{-13/2}}{806400} - \frac{z^{9/2} \pm z^{-9/2}}{32256} + \frac{3(\mp z^{7/2} + z^{-7/2})}{3200} - \frac{12329(z^{5/2} \pm z^{-5/2})}{1843200}$$
$$+\frac{9(\pm z^{3/2} - z^{-3/2})}{128} + \frac{14061(z^{1/2} \mp z^{-1/2})}{102400}\Bigg) + \mathcal{O}(\Lambda^{12}) + \mathcal{O}(\varepsilon_2)\Bigg] \tag{A.1.5}$$

$$E^{\pm}_{2,m=3} = \frac{9\varepsilon_1^2}{8} + \frac{\Lambda^4}{8\varepsilon_1^2} \mp \frac{\Lambda^6}{8\varepsilon_1^4} + \frac{13\Lambda^8}{640\varepsilon_1^6} \pm \frac{5\Lambda^{10}}{128\varepsilon_1^8} + \mathcal{O}(\Lambda^{12}) \tag{A.1.6}$$

A.2 $N = 3$

In the case of $N = 3$ we do not have a known result to compare to. Although the degenerate perturbative expansion can be done in principle for the non-Hermitian Hamiltonians, it quickly becomes tedious for increasing orders. The following results from gauge theory provides an alternative way to compute the split eigenfunctions and the split eigenvalues.

i) $a_{01} = a_{02} = \varepsilon_1$

$$\begin{aligned}
&[\widetilde{\mathbf{\Psi}}_{(012)}(\boldsymbol{a},\boldsymbol{\varepsilon},\Lambda,\mathbf{z}) + \zeta\widetilde{\mathbf{\Psi}}_{(021)}(\boldsymbol{a},\boldsymbol{\varepsilon},\Lambda,\mathbf{z}) + \zeta^2\widetilde{\mathbf{\Psi}}_{\mathrm{id}}(\boldsymbol{a},\boldsymbol{\varepsilon},\Lambda,\mathbf{z})]|_{a_{01}=a_{02}=\varepsilon_1} \\
&= e^{\frac{1}{\varepsilon_2}\left(\zeta\frac{\Lambda^2}{\varepsilon_1} + \zeta^2\frac{\Lambda^4}{2\varepsilon_1^3} + \frac{\Lambda^6}{12\varepsilon_1^5} - \zeta\frac{3\Lambda^8}{9\varepsilon_1^7} + \mathcal{O}(\Lambda^{10})\right)} z_0^{-\frac{a_0}{\varepsilon_1}} z_1^{-\frac{a_0}{\varepsilon_1}+1} z_2^{-\frac{a_0}{\varepsilon_1}+1} \\
&\Bigg[1 + \zeta\frac{z_0}{z_1} + \zeta^2\frac{z_0}{z_2} + \frac{\Lambda^2}{\varepsilon_1^2}\left(\frac{z_1}{2z_0} + \frac{z_2}{z_1} + \zeta\left(\frac{z_0 z_2}{2z_1^2} + \frac{z_0^2}{z_1 z_2}\right) + \zeta^2\left(\frac{z_0^2}{2z_2^2} + \frac{z_1}{z_2}\right)\right) \\
&+ \frac{\Lambda^4}{\varepsilon_1^4}\left(-\frac{z_0}{2z_1} - \frac{z_1^2}{12z_0^2} - \frac{z_0^2}{4z_2^2} - \frac{z_1}{2z_2} + \zeta\left(\frac{3z_0^2}{4z_1^2} - \frac{z_1}{4z_0} + \frac{z_0^3}{4z_1 z_2^2} - \frac{z_0}{2z_2}\right) + \zeta^2\left(-\frac{1}{2} + \frac{z_0^3}{12z_2^3} + \frac{3z_0 z_1}{4z_2^2} - \frac{z_0^2}{z_1 z_2} + \frac{z_1^2}{4z_0 z_2}\right)\right) \\
&+ \frac{\Lambda^6}{\varepsilon_1^6}\Bigg(\frac{1}{8} + \frac{z_1^3}{144z_0^3} - \frac{z_0^3}{z_2^3} - \frac{3z_0 z_1}{4z_2^2} - \frac{5z_0^2}{4z_1 z_2} - \frac{z_1^2}{6z_0 z_2} - \frac{z_0 z_2}{4z_1^2} + \frac{z_1 z_2}{6z_0^2} + \frac{z_2^2}{4z_0 z_1} + \frac{z_2^3}{36z_1^3} \\
&\qquad +\zeta\left(\frac{z_0}{8z_1} - \frac{z_1^2}{18z_0^2} + \frac{z_0^4}{36z_1 z_2^3} - \frac{z_0^2}{4z_2^2} + \frac{z_0^3}{4z_1^2 z_2} - \frac{5z_1}{4z_2} - \frac{5z_1}{4z_2} - \frac{3z_2}{4z_0} + \frac{z_0^2 z_2}{6z_1^3} - \frac{z_2^2}{6z_1^2} + \frac{z_0 z_2^3}{144z_1^4}\right) \\
&\qquad +\zeta^2\left(-\frac{3z_0^2}{4z_1^2} - \frac{z_1}{4z_0} + \frac{z_0^4}{144z_2^4} + \frac{z_0^2 z_1}{6z_2^3} - \frac{z_0^3}{6z_1 z_2^2} + \frac{z_1^2}{4z_2^2} + \frac{z_0}{8z_2^2} + \frac{z_1^3}{36z_0^2 z_2} - \frac{5z_2}{4z_1} - \frac{z_0 z_2^2}{18z_1^3}\right)\Bigg) \\
&+ \frac{\Lambda^8}{\varepsilon_1^8}\Bigg(-\frac{3z_0^2}{4z_1^2} + \frac{z_1}{4z_0} + \frac{z_1^4}{2880z_0^4} - \frac{z_0^4}{192z_2^4} - \frac{7z_0^2 z_1}{36z_2^3} - \frac{25z_0^3}{72z_1 z_2^2} - \frac{7z_1^2}{24z_2^2} + \frac{z_1^3}{48z_0^2 z_1} + \frac{13z_2}{8z_1} + \frac{5z_1^2 z_2}{288z_0^3} + \frac{5z_2^2}{72z_0^2} - \frac{z_0 z_2^2}{24z_1^3} + \frac{5z_2^3}{144z_0 z_1^2} + \frac{z_2^4}{576z_1^4} \\
&\qquad + \zeta\left(\frac{25}{16} + \frac{5z_0^3}{72z_1^3} - \frac{z_1^3}{192z_0^3} + \frac{z_0^5}{576z_1 z_2^4} - \frac{z_0^3}{24z_2^3} + \frac{5z_0^4}{144z_1^2 z_2^2} - \frac{3z_0 z_1}{4z_2^2} + \frac{13z_0^2}{8z_1 z_2} - \frac{25z_1^2}{72z_0 z_2} + \frac{z_0 z_2}{4z_1^2} - \frac{7z_1 z_2}{36z_0^2} + \frac{5z_0^2 z_2^2}{288z_1^4} - \frac{7z_2^2}{24z_0 z_1} - \frac{z_2^3}{48z_1^3} + \frac{z_0 z_2^4}{2880z_1^5}\right) \\
&\qquad +\zeta^2\left(+\frac{25z_0}{16z_1} - \frac{z_1^2}{24z_0^2} + \frac{z_0^5}{2880z_2^5} + \frac{5z_0^3 z_1}{288z_2^4} - \frac{z_0^4}{48z_1 z_2^3} + \frac{5z_0 z_1^2}{72z_2^3} + \frac{z_0^2}{4z_2^2} + \frac{5z_1^3}{144z_0 z_2^2} - \frac{7z_0^3}{24z_1^2 z_2} + \frac{13z_1}{8z_2} + \frac{z_1^4}{576z_0^3 z_2} - \frac{3z_2}{4z_0} - \frac{7z_0^2 z_2}{36z_1^3} - \frac{25z_2^2}{72z_1^2} - \frac{z_0 z_2^3}{192z_1^4}\right)\Bigg) \\
&+\mathcal{O}(\Lambda^{10}) + \mathcal{O}(\varepsilon_2)\Bigg]
\end{aligned} \tag{A.2.1}$$

From the dictionary (3.5.7) we compute

$$E^{\zeta}_{2,m=1} = \frac{\varepsilon_1^2}{3} - \zeta\Lambda^2 - \zeta^2\frac{\Lambda^4}{\varepsilon_1^2} - \frac{\Lambda^6}{4\varepsilon_1^4} + \zeta\frac{3\Lambda^8}{2\varepsilon_1^6} + \mathcal{O}(\Lambda^{10}) \tag{A.2.2}$$

$$E^{\zeta}_{3,m=1} = \frac{2\varepsilon_1^3}{27} + \zeta\Lambda^2(-2a_0 + \varepsilon_1) - \zeta^2\frac{2(a_0 - \varepsilon_1)\Lambda^4}{\varepsilon_1^2} - \frac{(2a_0 + \varepsilon_1)\Lambda^6}{4\varepsilon_1^4} + \zeta\frac{3(4a_0 - 3\varepsilon_1)\Lambda^8}{4\varepsilon_1^6} + \mathcal{O}(\Lambda^{10}). \tag{A.2.3}$$

ii) $a_{01} = a_{02} = 2\varepsilon_1$

$$
\begin{aligned}
&[\widetilde{\boldsymbol{\Psi}}_{(012)}(\boldsymbol{a},\boldsymbol{\varepsilon},\Lambda,\mathbf{z}) + \zeta\widetilde{\boldsymbol{\Psi}}_{(021)}(\boldsymbol{a},\boldsymbol{\varepsilon},\Lambda,\mathbf{z}) + \zeta^2\widetilde{\boldsymbol{\Psi}}_{\mathrm{id}}(\boldsymbol{a},\boldsymbol{\varepsilon},\Lambda,\mathbf{z})]|_{a_{01}=a_{02}=2\varepsilon_1} \\
&= e^{\frac{1}{\varepsilon_2}\left(-\zeta\frac{\Lambda^4}{2\varepsilon_1^3} - \frac{2\Lambda^6}{27\varepsilon_1^5} + \zeta^2\frac{5\Lambda^8}{16\varepsilon_1^7} + \mathcal{O}(\Lambda^{10})\right)} z_0^{-\frac{a_0}{\varepsilon_1}} z_1^{-\frac{a_0}{\varepsilon_1}+2} z_2^{-\frac{a_0}{\varepsilon_1}+2} \\
&\Bigg[1 + \zeta\frac{z_0^2}{z_1^2} + \zeta^2\frac{z_0^2}{z_2^2} + \frac{\Lambda^2}{\varepsilon_1^2}\left(\frac{z_1}{3z_0} - \frac{z_0}{z_2} + \frac{z_2}{z_1} + \zeta\left(-\frac{z_0}{z_1} + \frac{z_0^3}{z_1^2 z_2} + \frac{z_0^2 z_2}{z_1^3}\right) + \zeta^2\left(\frac{z_0^3}{3z_2^3} + \frac{z_0 z_1}{z_2^2} - \frac{z_0^2}{z_1 z_2}\right)\right) \\
&+ \frac{\Lambda^4}{\varepsilon_1^4}\left(\frac{z_1^2}{24z_0^2} - \frac{z_0^2}{2z_2^2} - \frac{2z_1}{z_2} + \frac{4z_2}{9z_0} + \frac{z_2^2}{4z_1^2} + \zeta\left(-\frac{1}{2} + \frac{4z_0^3}{9z_1^3} + \frac{z_0^4}{4z_1^2 z_2^2} - \frac{2z_0 z_2}{3z_1^2} + \frac{z_0^2 z_2^2}{24z_1^4}\right) + \zeta^2\left(-\frac{z_0^2}{2z_1^2} + \frac{z_0^4}{24z_2^4} + \frac{4z_0^2 z_1}{9z_2^3} - \frac{2z_0^3}{3z_1 z_2^2} + \frac{z_1^2}{4z_2^2}\right)\right) \\
&+ \frac{\Lambda^6}{\varepsilon_1^6}\left(\frac{14}{27} + \frac{z_1^3}{360z_0^3} - \frac{z_0^3}{18z_2^3} - \frac{z_0 z_1}{6z_2^2} + \frac{3z_0^2}{2z_1 z_2} + \frac{3z_0^2}{2z_1 z_2} - \frac{z_1^2}{8z_0 z_2} + \frac{z_0 z_2}{4z_1^2} + \frac{5z_1 z_2}{72z_0^2} + \frac{5z_2^2}{36z_0 z_1} + \frac{z_2^3}{36z_1^3}\right. \\
&\quad + \zeta\left(\frac{14z_0^2}{27z_1^2} - \frac{z_1}{18z_0} + \frac{z_0^5}{36z_1^2 z_2^3} + \frac{z_0^3}{4z_1 z_2^2} + \frac{3z_0}{2z_2} + \frac{5z_0^4}{36z_1^3 z_2} + \frac{5z_0^3 z_2}{72z_1^4} - \frac{z_2}{6z_1} - \frac{z_0 z_2^2}{8z_1^3} + \frac{z_0^2 z_2^3}{360z_1^5}\right) \\
&\quad \left. + \zeta^2\left(\frac{3z_0}{2z_1} + \frac{z_0^5}{360z_2^5} + \frac{5z_0^3 z_1}{72z_2^4} - \frac{z_0^4}{8z_1 z_2^3} + \frac{5z_0 z_1^2}{36z_2^3} + \frac{14z_0^2}{27z_2^2} + \frac{z_1^3}{36z_0 z_2^2} - \frac{z_0^3}{6z_1^2 z_2} + \frac{z_1}{4z_2} - \frac{z_0^2 z_2}{18z_1^3} + \frac{z_0 z_2^2}{8z_1^3}\right)\right) \\
&+ \frac{\Lambda^8}{\varepsilon_1^8}\left(\frac{35z_0^2}{32z_1^2} + \frac{29z_1}{162z_0} + \frac{z_1^4}{8640z_0^4} - \frac{z_0^4}{576z_2^4} + \frac{2z_0^2 z_1}{27z_2^3} + \frac{7z_0^3}{9z_1 z_2^2} - \frac{z_1^2}{96z_2^2} - \frac{44z_0}{27z_2} - \frac{z_1^3}{90z_0^2 z_2} + \frac{61z_2}{54z_1} + \frac{z_1^2 z_2}{180z_0^3} + \frac{5z_2^2}{192z_0^2} + \frac{z_0 z_2^2}{18z_1^3} + \frac{z_2^3}{54z_0 z_1^2} + \frac{z_2^4}{576z_1^4}\right. \\
&\quad + \zeta\left(\frac{5z_0^4}{192z_1^4} - \frac{44z_0}{27z_1} - \frac{z_1^2}{576z_0^2} + \frac{z_0^6}{576z_1^2 z_2^4} + \frac{z_0^4}{18z_1 z_2^3} + \frac{35z_0^2}{32z_2^2} + \frac{z_0^5}{54z_1^3 z_2^2} + \frac{61z_0^3}{54z_1^2 z_2} + \frac{7z_1}{9z_2} + \frac{2z_2}{27z_0} + \frac{29z_0^2 z_2}{162z_1^3} + \frac{z_0^3 z_2^2}{180z_1^5} - \frac{z_2^2}{96z_1^2} - \frac{z_0 z_2^3}{90z_1^4} + \frac{z_0^2 z_2^4}{8640z_1^6}\right) \\
&\quad \left. + \zeta^2\left(\frac{35}{32} + \frac{2z_0^3}{27z_1^3} + \frac{z_0^6}{8640z_2^6} + \frac{z_0^4 z_1}{180z_2^5} - \frac{z_0^5}{90z_1 z_2^4} + \frac{z_0^2 z_1^2}{192z_2^4} + \frac{29z_0^3}{162z_2^3} + \frac{z_1^3}{54z_2^3} - \frac{z_0^4}{96z_1^2 z_2^2} + \frac{61z_0 z_1}{54z_2^2} + \frac{z_1^4}{576z_0^2 z_2^2} - \frac{44z_0^2}{27z_1 z_2} + \frac{z_1^2}{18z_0 z_2} + \frac{7z_0 z_2}{9z_1^2} - \frac{z_0^2 z_2^2}{576z_1^4}\right)\right) \\
&+ \mathcal{O}(\Lambda^{10}) + \mathcal{O}(\varepsilon_2)\Bigg]
\end{aligned}
\tag{A.2.4}
$$

$$
E^{\zeta}_{2,m=2} = \frac{4\varepsilon_1^2}{3} + \zeta\frac{\Lambda^4}{\varepsilon_1^2} + \frac{2\Lambda^6}{9\varepsilon_1^4} - \zeta^2\frac{5\Lambda^8}{4\varepsilon_1^6} + \mathcal{O}(\Lambda^{10}) \tag{A.2.5}
$$

$$
E^{\zeta}_{3,m=2} = \frac{16\varepsilon_1^3}{27} + \zeta\frac{2(a_0-\varepsilon_1)\Lambda^4}{\varepsilon_1^2} + \frac{4(a_0-4\varepsilon_1)\Lambda^6}{9\varepsilon_1^4} - \zeta^2\frac{(5a_0-6\varepsilon_1)\Lambda^8}{2\varepsilon_1^6} + \mathcal{O}(\Lambda^{10}) \tag{A.2.6}
$$

Appendix B

Generalized hypergeometric function

A generalized hypergeometric function ${}_pF_q\left(a_1, \cdots, a_p; b_1, \cdots, b_q; z\right)$ is defined as the power series

$$ {}_pF_q\left(a_1, \cdots, a_p; b_1, \cdots, b_q; z\right) = \sum_{n=0}^{\infty} \frac{a_1^{\bar{n}} \cdots a_p^{\bar{n}}}{b_1^{\bar{n}} \cdots b_q^{\bar{n}}} \frac{z^n}{n!}, \tag{B.0.1} $$

where the parameters $a_1, \cdots, a_p, b_1, \cdots, b_q$ are complex numbers. The notation $x^{\bar{n}}$ is the rising factorial or Pochhammer function, which is defined for real values of n using the Gamma function provided that x and $x+n$ are real numbers that are not negative integers,

$$ x^{\bar{n}} = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{B.0.2} $$

The generalized hypergeometric function ${}_pF_q\left(a_1, \cdots, a_p; b_1, \cdots, b_q; z\right)$ is a solution to the generalized hypergeometric function,

$$ \left[z \prod_{n=1}^{p} \left(z\frac{d}{dz} + a_n\right) - z\frac{d}{dz} \prod_{n=1}^{q} \left(z\frac{d}{dz} + b_n - 1\right)\right] {}_pF_q\left(a_1, \cdots, a_p; b_1, \cdots, b_q; z\right) = 0. \tag{B.0.3} $$

Clearly, the order of the parameters $\{a_1, \cdots, a_p\}$, or the order of the parameters $\{b_1, \cdots, b_q\}$ can be changed without changing the value of the function. The standard hypergeometric function ${}_2F_1\left(a, b; c; z\right)$ is simply a special case of the generalized hypergeometric function when $p = 2$ and $q = 1$.

If any a_j is a non-positive integer, then the series (B.0.1) only has a finite number of terms and becomes a polynomial of degree $-a_j$. If any b_k is a non-positive integer (excepting the previous case with $b_k < a_j$), then the series (B.0.1) is undefined. Excluding these special cases for which the numerator or the denominator of the coefficients can be 0, the radius of convergence can be determined using the ratio test. In this dissertation, we are interested in the case $p = q + 1$. The ratio of coefficients tends to one, implying that the series (B.0.1) converges for $|z| < 1$ and diverges for $|z| > 1$.

Appendix C

$U(1)$ factor

In this appendix, we derive the $U(1)$ factor using the non-perturbative Dyson-Schwinger equations.

When $N = 1$, we have $\mathcal{Y}_0(x) = x - \bar{a}_0$, and the non-perturbative Dyson-Schwinger equations lead to

$$\begin{aligned} 0 &= \langle [\mathcal{Y}_0(x)\mathcal{G}_r(x;t)]^{(-1)} \rangle \rangle \\ &= \langle \mathcal{G}_r^{(-2)}(t) \rangle - \bar{a}_0 \langle \mathcal{G}_r^{(-1)}(t) \rangle \\ &= \langle U_r[2] \rangle + \langle U_r[1,1] \rangle - \varepsilon \langle U_r[0,1] \rangle - \bar{a}_0 \langle U_r[1] \rangle \\ &= \sum_{i=0}^{r} u_i \left(\langle \zeta_{i,2} \rangle - \bar{a}_0 \langle \zeta_{i,1} \rangle \right) + \sum_{0 \le i_1 < i_2 \le r} u_{i_1} u_{i_2} \left(\langle \zeta_{i_1,1} \zeta_{i_2,1} \rangle - \varepsilon \langle \zeta_{i_2,1} \rangle \right), \end{aligned} \tag{C.0.1}$$

where

$$\zeta_{j,1} = \bar{a}_j - \bar{a}_{j+1} + \varepsilon, \tag{C.0.2}$$

$$\zeta_{j,2} = \bar{a}_j \left(\bar{a}_j - \bar{a}_{j+1} + \varepsilon \right) - \varepsilon_1 \varepsilon_2 \left(k_i - k_{i+1} \right). \tag{C.0.3}$$

Picking up the residue at $t = -z_j^{-1}$, we obtain

$$
\begin{aligned}
\varepsilon_1\varepsilon_2\langle k_j - k_{j+1}\rangle &= (\bar{a}_j - \bar{a}_0)(\bar{a}_j - \bar{a}_{j+1} + \varepsilon) \\
&\quad + \sum_{i=j+1}^{r} \frac{z_i}{z_i - z_j}(\bar{a}_j - \bar{a}_{j+1})(\bar{a}_i - \bar{a}_{i+1} + \varepsilon) \\
&\quad + \sum_{i=0}^{j-1} \frac{z_i}{z_i - z_j}(\bar{a}_i - \bar{a}_{i+1})(\bar{a}_j - \bar{a}_{j+1} + \varepsilon) \\
&= \sum_{i=j+1}^{r} \frac{z_i}{z_i - z_j}(\bar{a}_j - \bar{a}_{j+1})(\bar{a}_i - \bar{a}_{i+1} + \varepsilon) \\
&\quad + \sum_{i=0}^{j-1} \frac{z_j}{z_i - z_j}(\bar{a}_i - \bar{a}_{i+1})(\bar{a}_j - \bar{a}_{j+1} + \varepsilon).
\end{aligned} \tag{C.0.4}
$$

From the structure of the instanton partition function, we see that

$$
\langle k_j - k_{j+1}\rangle = z_j \frac{d}{dz_j} \log Z^{\text{instanton}}. \tag{C.0.5}
$$

Therefore, we have the $U(1)$ part of the instanton partition function

$$
Z^{\text{instanton}} = \prod_{0 \le i < j \le r} \left(1 - \frac{z_j}{z_i}\right)^{-\frac{(\bar{a}_i - \bar{a}_{i+1})\left(\bar{a}_j - \bar{a}_{j+1} + \varepsilon\right)}{\varepsilon_1\varepsilon_2}}. \tag{C.0.6}
$$

Appendix D

The accessory operator $\widehat{H}_2$

We present the full expression for the accessory operator $\widehat{H}_2(z,\mathfrak{q})$ in $\widehat{\widehat{\mathfrak{D}}}_3$ below.

$$\begin{aligned}
&\widehat{H}_2(z,\mathfrak{q})\\
&= -(1-\mathfrak{q})\left[\frac{1}{3}\left\langle \mathcal{O}_3\right\rangle_{A_2} + \frac{\varepsilon_1\varepsilon_2(12\bar{a}_0 - 4\mathcal{A}_1^{(1)} + 2\mathcal{A}_2^{(1)} + 15\varepsilon_1 + 8\varepsilon_2)}{6}\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} - \frac{1}{3}\sum_{\alpha=1}^{3} a_{2,\alpha}^3 + \frac{2\left(\mathcal{A}_1^{(1)}\right)^2\mathcal{A}_2^{(1)}}{9}\right.\\
&\qquad - \left(\mathcal{A}_2^{(1)}\right)^2\left(\frac{3\bar{a}_0 - \mathcal{A}_1^{(1)}}{3} + \frac{33\varepsilon_1 + 22\varepsilon_2}{36}\right) + \frac{\mathcal{A}_1^{(2)}\mathcal{A}_2^{(1)}}{3} + \mathcal{A}_2^{(2)}\left(-\bar{a}_0 + \frac{4\mathcal{A}_1^{(1)} + 2\mathcal{A}_2^{(1)} - 9\varepsilon_1 - 2\varepsilon_2}{12}\right) + \frac{\mathcal{A}_2^{(3)}}{3}\\
&\qquad - \frac{8\bar{a}_0 + 7\varepsilon_1 + 2\varepsilon_2}{6}\mathcal{A}_1^{(1)}\mathcal{A}_2^{(1)} - \frac{2\varepsilon_1\varepsilon_2}{3}(\Delta_0 - \Delta_a)\mathcal{A}_1^{(1)} - \frac{\left(\mathcal{A}_2^{(1)}\right)^3}{18} + \frac{\varepsilon_1\varepsilon_2(\Delta_0 - \Delta_a)}{6}(12\bar{a}_0 + 15\varepsilon_1 + 8\varepsilon_2)\\
&\qquad \left. + \left(\frac{\varepsilon_1\varepsilon_2(\Delta_0 + \Delta_\infty - \Delta_a)}{3} + \frac{7\bar{a}_0\varepsilon_1}{2} + \frac{\prod_{\alpha<\alpha'} a_{0,\alpha}a_{0,\alpha'} + \varepsilon_2(2a_{0,\beta} + 3\bar{a}_0)}{3} + \frac{51\varepsilon_1^2 + 48\varepsilon_1\varepsilon_2 + 2\varepsilon_2^2}{18}\right)\mathcal{A}_2^{(1)}\right]\\
&- \mathfrak{q}\left[\varepsilon_1\varepsilon_2\left(-\mathcal{A}_1^{(1)} + \mathcal{A}_2^{(1)} + 2\varepsilon\right)\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} + \frac{\mathcal{A}_1^{(3)} + \mathcal{A}_2^{(2)}}{3} + \frac{\mathcal{A}_1^{(2)}\left(-12\bar{a}_0 + 2\mathcal{A}_1^{(1)} + 8\mathcal{A}_2^{(1)} + 3\varepsilon_1 + 10\varepsilon_2\right)}{12}\right.\\
&\quad + \frac{\mathcal{A}_2^{(2)}\left(-12\bar{a}_0 + 8\mathcal{A}_1^{(1)} + 2\mathcal{A}_2^{(1)} - 21\varepsilon_1 - 14\varepsilon_2\right)}{12} + \frac{\left(\mathcal{A}_1^{(1)}\right)^3 - 5\left(\mathcal{A}_2^{(1)}\right)^3}{18} + \frac{(-12\bar{a}_0 + 4\mathcal{A}_2^{(1)} - 3\varepsilon_1 - 10\varepsilon_2)\left(\mathcal{A}_1^{(1)}\right)^2}{36}\\
&\quad + \frac{\left(-12\bar{a}_0 + 28\mathcal{A}_1^{(1)} - 39\varepsilon_1 - 46\varepsilon_2\right)\left(\mathcal{A}_2^{(1)}\right)^2}{36} + \mathcal{A}_1^{(1)}\mathcal{A}_2^{(1)}\left(-2\bar{a}_0 - \frac{\varepsilon_1}{6} + \frac{8\varepsilon_2}{9}\right)\\
&\quad + \mathcal{A}_1^{(1)}\left(\varepsilon_1\varepsilon_2(\Delta_a - \Delta_0) - \frac{\bar{a}_0\varepsilon_1}{2} - 2\bar{a}_0\varepsilon_2 + \frac{a_{0,\beta}\varepsilon_2}{3} + \frac{3\varepsilon_1^2 + 9\varepsilon_1\varepsilon_2 + 38\varepsilon_2^2}{18} + \frac{\varepsilon_1\varepsilon_2\Delta_\infty - \varepsilon^2 + \prod_{\alpha<\alpha'} a_{0,\alpha}a_{0,\alpha'}}{3}\right)\\
&\quad + \mathcal{A}_2^{(1)}\left(\varepsilon_1\varepsilon_2(\Delta_0 - \Delta_a) + \bar{a}_0\left(\frac{3\varepsilon_1}{2} - \varepsilon_2\right) + \frac{4a_{0,\beta}\varepsilon_2}{3} + \frac{12\varepsilon_1^2 - 15\varepsilon_1\varepsilon_2 - 16\varepsilon_2^2}{18} + \frac{\varepsilon_1\varepsilon_2\Delta_\infty - \varepsilon^2 + \prod_{\alpha<\alpha'} a_{0,\alpha}a_{0,\alpha'}}{3}\right)\\
&\quad \left. + 2\varepsilon_1\varepsilon_2\varepsilon(\Delta_0 - \Delta_a) + \frac{\varepsilon_2(a_{0,\beta} - \bar{a}_0)(2a_{0,\beta} - 2\bar{a}_0 - 3\varepsilon_1)}{2} + \frac{\varepsilon_2^2(4a_{0,\beta} - 4\bar{a}_0 + 3\varepsilon_1)}{6} - \frac{35\varepsilon_2^3}{9}\right]\\
&+ \frac{\mathfrak{q}^2}{3(1-\mathfrak{q})}\mathcal{A}_2^{(1)}\left(\mathcal{A}_1^{(1)} - 3\varepsilon\right)\left(\mathcal{A}_1^{(1)} - \mathcal{A}_2^{(1)} - 2\varepsilon\right),
\end{aligned} \tag{D.0.1}$$

where we have defined

$$\Delta_a \equiv \frac{1}{\varepsilon_1\varepsilon_2}\left(\varepsilon^2 - \frac{(a_{2,1}-a_{2,2})^2+(a_{2,1}-a_{2,3})^2-(a_{2,1}-a_{2,2})(a_{2,1}-a_{2,3})}{3}\right). \tag{D.0.2}$$

The accessory parameter H_2 can be obtained simply by taking the limit $\varepsilon_2 \to 0$.

Appendix E

Computing the non-regular parts of $\mathcal{X}_\omega$

We present the explicit expressions for the non-regular parts of the fundamental refined qq-character $\mathcal{X}_\omega$ for the $N=3$ case, i.e., the $(2,1)$-type $\mathbb{Z}_2$-orbifold surface defect. The computation for the $N=2$ case is easier and can be done in a similar way.

$$
\begin{aligned}
&[x^{-1}]\ \mathcal{X}_0(x) \\
&= \frac{\varepsilon_1^2}{2}(k_0-k_1)^2 - \frac{\varepsilon_1^2}{2}(k_0-k_1) + \varepsilon_1\varepsilon_2 k_1 + (\varepsilon - a_\beta)\varepsilon_1(k_0-k_1) + \varepsilon_1\left(\sum_{K_0} c_\square - \sum_{K_1} c_\square\right) \\
&+ \mathfrak{q}_0\left[\frac{1}{2}\left(\sum_{\bar\beta\neq\beta} a_{\bar\beta} - \sum_\alpha m_{+,\alpha} + \varepsilon_1(k_0-k_1)\right)^2 + \frac{1}{2}\sum_{\bar\beta\neq\beta} a_{\bar\beta}^2 - \frac{1}{2}\sum_\alpha m_{+,\alpha}^2 + \frac{\varepsilon_1^2}{2}(k_0-k_1) - \varepsilon_1\varepsilon_2 k_1\right. \\
&\qquad \left. + \varepsilon_1\left(\sum_{K_0} c_\square - \sum_{K_1} c_\square\right)\right]
\end{aligned}
\tag{E.0.1}
$$

$$
\begin{aligned}
&[x^{-2}]\, \mathcal{X}_0(x) \\
&= \frac{\varepsilon_1^3}{6}(k_0-k_1)^3 - \frac{\varepsilon_1^3}{2}(k_0-k_1)^2 + \varepsilon_1^2\varepsilon_2 k_1(k_0-k_1) + \frac{\varepsilon_1^3}{3}(k_0-k_1) - \varepsilon_1\varepsilon_2\varepsilon k_1 + 2\varepsilon_1\varepsilon_2 \sum_{K_1} c_\square \\
&+ \varepsilon_1^2(k_0-k_1)\left(\sum_{K_0} c_\square - \sum_{K_1} c_\square\right) - \varepsilon_1^2\left(\sum_{K_0} c_\square - \sum_{K_1} c_\square\right) + \varepsilon_1\left(\sum_{K_0} c_\square^2 - \sum_{K_1} c_\square^2\right) \\
&+ (\varepsilon - a_\beta)\left(\frac{\varepsilon_1^2}{2}(k_0-k_1)^2 - \frac{\varepsilon_1^2}{2}(k_0-k_1) + \varepsilon_1\varepsilon_2 k_1 + \varepsilon_1\left(\sum_{K_0} c_\square - \sum_{K_1} c_\square\right)\right) \\
&+ \mathfrak{q}_0\left[\frac{1}{6}\left(\sum_{\bar\beta\neq\beta} a_{\bar\beta} - \sum_\alpha m_{+,\alpha} + \varepsilon_1(k_0-k_1)\right)^2 + \frac{\varepsilon_1}{2}\left(\sum_{\bar\beta\neq\beta} a_{\bar\beta}^2 - \sum_\alpha m_{+,\alpha}^2\right)(k_0-k_1)\right. \\
&+ \frac{\varepsilon_1^3}{2}(k_0-k_1)^2 - \varepsilon_1^2\varepsilon_2 k_1(k_0-k_1) + \varepsilon_1^2(k_0-k_1)\left(\sum_{K_0} c_\square - \sum_{K_1} c_\square\right) + \frac{\varepsilon_1^3}{3}(k_0-k_1) - 2\varepsilon_1\varepsilon_2\sum_{K_1} c_\square \\
&+ \frac{1}{3}\left(\sum_{\bar\beta\neq\beta} a_{\bar\beta}^3 - \sum_\alpha m_{+,\alpha}^3\right) + \varepsilon_1^2\left(\sum_{K_0} c_\square - \sum_{K_1} c_\square\right) + \varepsilon_1\left(\sum_{K_0} c_\square^2 - \sum_{K_1} c_\square^2\right) - \varepsilon_1\varepsilon_2\varepsilon k_1 \\
&\left. + \left(\sum_{\bar\beta\neq\beta} a_{\bar\beta} - \sum_\alpha m_{+,\alpha}\right)\left(\frac{1}{2}\sum_{\bar\beta\neq\beta} a_{\bar\beta}^2 - \frac{1}{2}\sum_\alpha m_{+,\alpha}^2 + \frac{\varepsilon_1^2}{2}(k_0-k_1) - \varepsilon_1\varepsilon_2 k_1 + \varepsilon_1\left(\sum_{K_0} c_\square - \sum_{K_1} c_\square\right)\right)\right]
\end{aligned}
\tag{E.0.2}
$$

$$
\begin{aligned}
&[x^{-1}]\, \mathcal{X}_1(x) \\
&= -\frac{\varepsilon_1^3}{6}(k_0-k_1)^3 - \frac{\varepsilon_1^3}{2}(k_0-k_1)^2 - \frac{\varepsilon_1^3}{3}(k_0-k_1) - \varepsilon_1^2\varepsilon_2 k_0(k_0-k_1) + \varepsilon_1^2(k_0-k_1)\left(\sum_{K_0} c_\Box - \sum_{K_1} c_\Box\right) \\
&+ \varepsilon_1^2\left(\sum_{K_0} c_\Box - \sum_{K_1} c_\Box\right) - \varepsilon_1\left(\sum_{K_0} c_\Box^2 - \sum_{K_1} c_\Box^2\right) + 2\varepsilon_1\varepsilon_2\sum_{K_0} c_\Box - \varepsilon_1\varepsilon_2\varepsilon k_0 - \varepsilon_1(k_0-k_1)\prod_{\bar\beta\neq\beta}(\varepsilon - a_{\bar\beta}) \\
&+ \left(2\varepsilon - \sum_{\bar\beta\neq\beta} a_{\bar\beta}\right)\left(\frac{\varepsilon_1^2}{2}(k_0-k_1)^2 + \frac{\varepsilon_1^2}{2}(k_0-k_1) + \varepsilon_1\varepsilon_2 k_0 - \varepsilon_1\left(\sum_{K_0} c_\Box - \sum_{K_1} c_\Box\right)\right) \\
&+ \mathfrak{q}_1\left[\frac{1}{6}\left(a_\beta - \sum_\alpha m_{-,\alpha} - \varepsilon_1(k_0-k_1)\right)^3 + \frac{\varepsilon_1^3}{2}(k_0-k_1)^2 + \varepsilon_1^2\varepsilon_2 k_0(k_0-k_1) - \frac{\varepsilon_1^3}{3}(k_0-k_1)\right. \\
&- \frac{\varepsilon_1}{2}\left(a_\beta^2 - \sum_\alpha m_{-,\alpha}^2\right)(k_0-k_1) - \varepsilon_1\varepsilon_2\varepsilon k_0 - 2\varepsilon_1\varepsilon_2\sum_{K_0} c_\Box + \frac{1}{3}a_\beta^3 - \frac{1}{3}\sum_\alpha m_{-,\alpha}^3 \\
&+ \varepsilon_1^2(k_0-k_1)\left(\sum_{K_0} c_\Box - \sum_{K_1} c_\Box\right) - \varepsilon_1^2\left(\sum_{K_0} c_\Box - \sum_{K_1} c_\Box\right) - \varepsilon_1\left(\sum_{K_0} c_\Box^2 - \sum_{K_1} c_\Box^2\right) \\
&\left. + \left(a_\beta - \sum_\alpha m_{-,\alpha}\right)\left(-\frac{\varepsilon_1^2}{2}(k_0-k_1) - \varepsilon_1\varepsilon_2 k_0 - \varepsilon_1\left(\sum_{K_0} c_\Box - \sum_{K_1} c_\Box\right) + \frac{a_\beta^2}{2} - \frac{\sum_\alpha m_{-,\alpha}^2}{2}\right)\right]
\end{aligned}
\tag{E.0.3}
$$

Appendix F

Computing the Poisson brackets

Let us first recap some definitions. Let $N \approx \mathbb{C}^N$ be a vector space with a volume form. Let

$$g_i, M_i \in \mathrm{End}(N), , \quad i = -1, 0, 1, \ldots, r+1 \tag{F.0.1}$$

be $SL(N)$ matrices, such that

$$\begin{aligned}
&M_{-1} = g_{-1} \\
&g_i = \mathfrak{m}_i \left(1 + \left(\mathfrak{m}_i^{-N} - 1\right) \Pi_i\right) , \qquad i = 0, 1, \ldots, r \\
&M_i = g_{-1} g_0 g_1 \ldots g_i = \sum_{\alpha=1}^{N} \mathfrak{m}_i^{(\alpha)} \Pi_i^{(\alpha)} , \qquad i = 0, 1, \ldots, r \\
&M_{r+1} = \mathbb{1}_N,
\end{aligned} \tag{F.0.2}$$

where the projection operators Π are written in terms of

$$\begin{aligned}
&E_i, E_i^{(\alpha)} \in N, \quad \tilde{E}_i, \tilde{E}_i^{(\alpha)} \in N^*, \\
&\tilde{E}_i(E_i) = 1, \quad \tilde{E}_i^{(\alpha)}(E_i^{(\beta)}) = \delta_{\alpha,\beta}
\end{aligned} \tag{F.0.3}$$

as

$$
\begin{aligned}
&\Pi_i = E_i \otimes \tilde{E}_i, \quad i = 0, 1, \cdots, r-1, \\
&\Pi_i^{(\alpha)} = E_i^{(\alpha)} \otimes \tilde{E}_i^{(\alpha)}, \quad i = -1, 0, 1, \cdots, r, \ \ \alpha = 1, \cdots, N.
\end{aligned}
\tag{F.0.4}
$$

The following formulas are useful throughout the computation: for any $\alpha, \beta = 1, \ldots, N$,

$$
\begin{aligned}
\tilde{E}_{i+1}^{(\alpha)}(E_i^{(\beta)}) &= \frac{\mathfrak{m}_{i+1}^{(\alpha)}(\mathfrak{m}_{i+1}^{N} - 1)}{\mathfrak{m}_{i+1}\mathfrak{m}_i^{(\beta)} - \mathfrak{m}_{i+1}^{(\alpha)}}\, \tilde{E}_{i+1}^{(\alpha)}(E_{i+1})\tilde{E}_{i+1}(E_i^{(\beta)}) \\
\tilde{E}_i^{(\alpha)}(E_{i+1}^{(\beta)}) &= \frac{\mathfrak{m}_i^{(\alpha)}(\mathfrak{m}_{i+1}^{-N} - 1)}{\mathfrak{m}_{i+1}^{-1}\mathfrak{m}_{i+1}^{(\beta)} - \mathfrak{m}_i^{(\alpha)}} \tilde{E}_i^{(\alpha)}(E_{i+1})\tilde{E}_{i+1}(E_{i+1}^{(\beta)}).
\end{aligned}
\tag{F.0.5}
$$

We packaged the Darboux coordinates into

$$
\begin{aligned}
\mathbb{A}_i(x) &\equiv \mathrm{Tr}_N\ (x - M_i)^{-1} = \sum_{l=0}^{\infty} \frac{1}{x^{l+1}} \mathrm{Tr}_N\, M_i^l \\
\mathbb{B}_i(x) &\equiv \mathrm{Tr}_N\, \Pi_i\, (x - M_i)^{-1}\, \Pi_{i+1} = e^{\tilde{\beta}_i} \sum_{\alpha=1}^{N} e^{-\tilde{\beta}_i^{(\alpha)}}\, \frac{\mathrm{Tr}_N\, \Pi_i \Pi_i^{(\alpha)}}{x - m_i^{(\alpha)}},
\end{aligned}
\tag{F.0.6}
$$

where we express $\mathbb{B}_i(x)$ via

$$
\begin{aligned}
\mathbb{D}_i(x) &\equiv \mathrm{Tr}_N\, g_i\, (x - M_i)^{-1}\, g_{i+1} \\
&= \mathfrak{m}_i\mathfrak{m}_{i+1}(\mathfrak{m}_i^{-N} - 1)(\mathfrak{m}_{i+1}^{-N} - 1)\mathbb{B}_i(x) + \mathfrak{m}_i\mathfrak{m}_{i+1}x^{-1}\left(\frac{P_{i-1}(\mathfrak{m}_i^{-1}x)}{P_i(x)} - 1\right) \\
&\quad - \mathfrak{m}_i\mathfrak{m}_{i+1}^{1-N}x^{-1}\left(\frac{P_{i+1}(\mathfrak{m}_{i+1}x)}{P_i(x)} - 1\right) + \mathfrak{m}_i\mathfrak{m}_{i+1}\mathbb{A}_i(x).
\end{aligned}
\tag{F.0.7}
$$

The brackets remained to be computed are

$$
\Big\{\mathbb{D}_i(x), \mathbb{A}_i(y)\Big\}, \qquad \Big\{\mathbb{D}_i(x), \mathbb{D}_{i+1}(y)\Big\}, \quad \text{and} \quad \Big\{\mathbb{D}_i(x), \mathbb{D}_i(y)\Big\}. \tag{F.0.8}
$$

Using the geometric representation (5.5.24), the first Poisson bracket is computed as (see

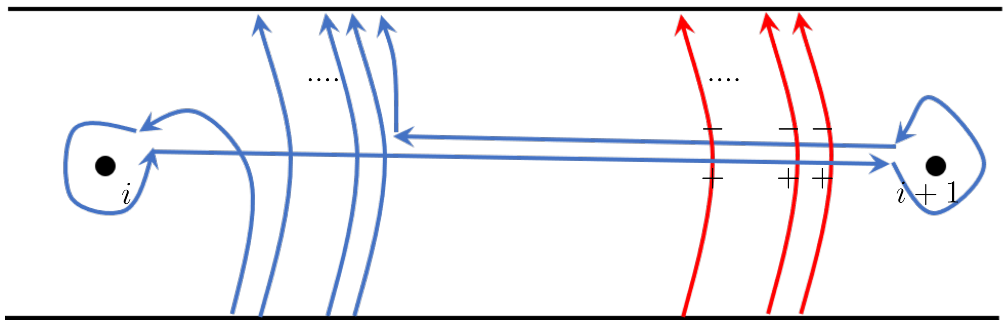

Figure F.1: The geometric picture for $\{\mathbb{D}_i(x), \mathbb{A}_i(y)\}$.

Figure F.1)

$$\begin{aligned}\left\{\mathbb{D}_i(x), \mathbb{A}_i(y)\right\} &= \mathrm{Tr}_N\left(\frac{M_i}{(y-M_i)^2}g_i\frac{1}{x-M_i}g_{i+1}\right) - \mathrm{Tr}_N\left(g_i\frac{M_i}{(y-M_i)^2}\frac{1}{x-M_i}g_{i+1}\right)\\ &= \mathfrak{m}_i\mathfrak{m}_{i+1}(\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1)\,\mathrm{Tr}_N\left[\frac{M_i}{(y-M_i)^2},\Pi_i\right]\frac{1}{x-M_i}\Pi_{i+1}.\end{aligned} \tag{F.0.9}$$

On the other hand, a direct computation gives (we omit the $2\pi i$ in front of the $\boldsymbol{\alpha}$ coordinates)

$$\begin{aligned}\left\{\mathbb{D}_i(x), \mathbb{A}_i(y)\right\} &= \mathfrak{m}_i\mathfrak{m}_{i+1}(\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1)\left\{e^{\tilde{\beta}_i}\sum_{\alpha=1}^{N}\frac{e^{-\tilde{\beta}_i^{(\alpha)}}\mathrm{Tr}\,\Pi_i\Pi_i^{(\alpha)}}{x-\mathfrak{m}_i^{(\alpha)}}, \sum_{\beta=1}^{N}\frac{1}{x-\mathfrak{m}_i^{(\beta)}}\right\}\\ &= \mathfrak{m}_i\mathfrak{m}_{i+1}(\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1)\sum_{\alpha,\beta}\frac{\mathfrak{m}_i^{(\alpha)}\,\mathrm{Tr}\,\Pi_i\Pi_i^{(\beta)}\Pi_{i+1}}{(x-\mathfrak{m}_i^{(\beta)})(y-\mathfrak{m}_i^{(\alpha)})^2}\{\boldsymbol{\alpha}_i^{(\alpha)}, \tilde{\boldsymbol{\beta}}_i^{(\beta)}\}\\ &\quad - \mathfrak{m}_i\mathfrak{m}_{i+1}(\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1)\sum_{\alpha,\beta,\gamma}\frac{\mathfrak{m}_i^{(\alpha)}\mathrm{Tr}\,\Pi_i\Pi_i^{(\beta)}\Pi_{i+1}\Pi_i^{(\gamma)}}{(x-\mathfrak{m}_i^{(\beta)})(y-\mathfrak{m}_i^{(\alpha)})^2}\{\boldsymbol{\alpha}_i^{(\alpha)}, \tilde{\boldsymbol{\beta}}_i^{(\gamma)}\}\end{aligned} \tag{F.0.10}$$

By comparing the two expressions, we derive:

$$\left\{\tilde{\boldsymbol{\beta}}_i^{(\alpha)}, \boldsymbol{\alpha}_i^{(\beta)}\right\} = \delta_{\alpha,\beta}, \quad i = 0, 1, \cdots, r-1,\ \alpha, \beta = 1, \cdots, N. \tag{F.0.11}$$

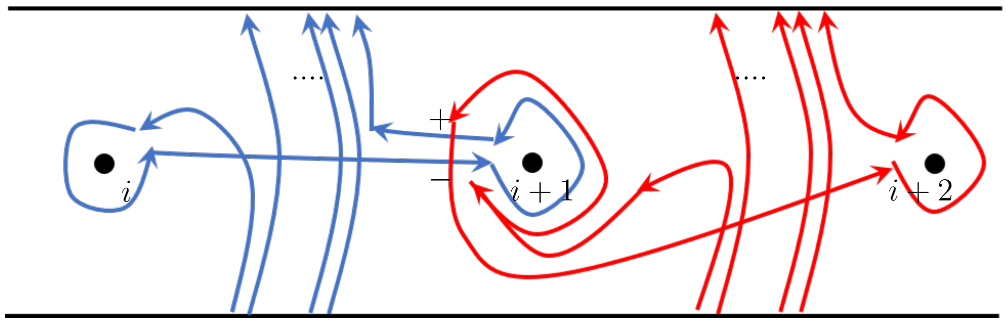

Figure F.2: The geometric picture for $\{\mathbb{D}_i(x), \mathbb{D}_{i+1}(y)\}$.

Next, we compute from the geometric representation (see Figure F.2)

$$\left\{\mathbb{D}_i(x), \mathbb{D}_{i+1}(y)\right\} = \mathrm{Tr}_N \left(\left[g_{i+1}, g_i(x-M_i)^{-1}\right] g_{i+1}(y-M_{i+1})^{-1} g_{i+2}\right). \tag{F.0.12}$$

On the other hand, a direct computation gives

$$\begin{aligned}\left\{\mathbb{D}_i(x), \mathbb{D}_{i+1}(y)\right\} &= \mathfrak{m}_i \mathfrak{m}_{i+1}^2 \mathfrak{m}_{i+2} (\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1)(\mathfrak{m}_{i+2}^{-N}-1)\{\mathbb{B}_i(x), \mathbb{B}_{i+1}(y)\} \\ &+ \mathfrak{m}_i \mathfrak{m}_{i+1}^2 \mathfrak{m}_{i+2} (\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1) y^{-1} \left\{\mathbb{B}_i(x), \frac{P_i(\mathfrak{m}_{i+1}^{-1} y)}{P_{i+1}(y)}\right\} \\ &- \mathfrak{m}_i \mathfrak{m}_{i+1}^{2-N} \mathfrak{m}_{i+2} (\mathfrak{m}_{i+1}^{-N}-1)(\mathfrak{m}_{i+2}^{-N}-1) x^{-1} \left\{\frac{P_{i+1}(\mathfrak{m}_{i+1} x)}{P_i(x)}, \mathbb{B}_{i+1}(y)\right\}.\end{aligned} \tag{F.0.13}$$

Each term can be explicitly computed. By comparing the results we derive

$$\left\{\tilde{\boldsymbol{\beta}}_i^{(\alpha)}, \tilde{\boldsymbol{\beta}}_{i+1}^{(\beta)}\right\} = 0, \quad i = 0, 1, \cdots, r-1,\ \alpha, \beta = 1, \cdots, N. \tag{F.0.14}$$

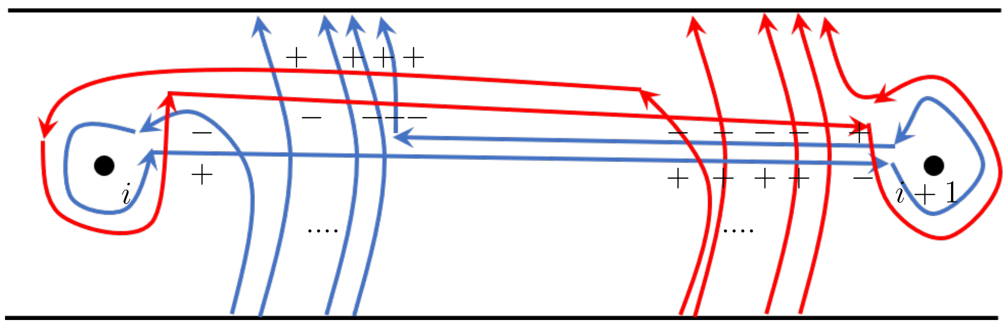

Figure F.3: The geometric picture for $\{\mathbb{D}_i(x), \mathbb{D}_i(y)\}$.

Finally, we compute from the geometric representation (see Figure F.3)

$$\begin{aligned}\Big\{\mathbb{D}_i(x), \mathbb{D}_i(y)\Big\} &= \mathrm{Tr}_N\left(\left[\frac{1}{y-M_i}g_{i+1}, g_{i+1}\right]\frac{1}{x-M_i}g_{i+1}g_i\frac{M_i}{x-M_i}\right)\\ &+ \mathrm{Tr}_N\left(\left[g_i\frac{1}{x-M_i}, g_{i+1}\right]\frac{M_i}{y-M_i}g_{i+1}g_i\frac{1}{y-M_i}\right)\\ &+ \mathrm{Tr}_N\left(g_i\frac{1}{y-M_i}g_{i+1}\left[g_i, \frac{1}{x-M_i}g_{i+1}\right]\right)\\ &+ \mathrm{Tr}_N\left(g_i\frac{1}{y-M_i}g_{i+1}\left[g_{i+1}, g_i\frac{1}{x-M_i}\right]\right).\end{aligned} \tag{F.0.15}$$

On the other hand, a direct computation gives

$$\begin{aligned}&\Big\{\mathbb{D}_i(x), \mathbb{D}_i(y)\Big\}\\ &= \mathfrak{m}_i^2\mathfrak{m}_{i+1}^2(\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1)\left(\{\mathbb{B}_i(x), \mathbb{A}_i(y)\} + \{\mathbb{A}_i(x), \mathbb{B}_i(y)\}\right)\\ &+ \mathfrak{m}_i^2\mathfrak{m}_{i+1}^2(\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1)\left(y^{-1}\left\{\mathbb{B}_i(x), \frac{P_{i-1}(\mathfrak{m}_i^{-1}y)}{P_i(y)}\right\} + x^{-1}\left\{\frac{P_{i-1}(\mathfrak{m}_i^{-1}x)}{P_i(x)}, \mathbb{B}_i(y)\right\}\right)\\ &- \mathfrak{m}_i^2\mathfrak{m}_{i+1}^{2-N}(\mathfrak{m}_i^{-N}-1)(\mathfrak{m}_{i+1}^{-N}-1)\left(y^{-1}\left\{\mathbb{B}_i(x), \frac{P_{i+1}(\mathfrak{m}_{i+1}y)}{P_i(y)}\right\} + x^{-1}\left\{\frac{P_{i+1}(\mathfrak{m}_{i+1}x)}{P_i(x)}, \mathbb{B}_i(y)\right\}\right)\\ &+ \mathfrak{m}_i^2\mathfrak{m}_{i+1}^2(\mathfrak{m}_i^{-N}-1)^2(\mathfrak{m}_{i+1}^{-N}-1)^2\{\mathbb{B}_i(x), \mathbb{B}_i(y)\},\end{aligned} \tag{F.0.16}$$

in which all the brackets are explicitly computable. By comparing the results we derive

$$\left\{ \widetilde{\boldsymbol{\beta}}_i^{(\alpha)}, \widetilde{\boldsymbol{\beta}}_i^{(\beta)} \right\} = 0, \quad i = 0, 1, \cdots, r = 1,\ \alpha, \beta = 1, \cdots, N. \tag{F.0.17}$$

Therefore, we confirm that the Poisson brackets for the coordinates $\boldsymbol{\alpha}_i^{(\alpha)}, \tilde{\boldsymbol{\beta}}_i^{(\alpha)}$ are canonical.

Appendix G

Quantum toroidal algebra of $\mathfrak{gl}(p)$

In this appendix, we remind the definition of the quantum toroidal algebra of $\mathfrak{gl}(p)$, give its vertical and horizontal representations, and comment on the reduction $\nu_1 = -\nu_2 = 1$ of the algebraic relations 6.3.2.

G.1 Definition

Quantum toroidal algebras were introduced by V. Ginzburg and M. Kapranov and E. Vasserot in [171]. In general, they can be built over an affine Kac-Moody algebra $\hat{\mathfrak{g}}$, but we will focus in this appendix on the case of an algebra of type $A_{p-1}^{(1)}$, also called quantum toroidal $\mathfrak{g} = \mathfrak{gl}(p)$ algebra. This algebra is formulated in terms of the Drinfeld currents

$$x_\omega^\pm(z) = \sum_{k\in\mathbb{Z}} z^{-k} x_{\omega,k}^\pm, \quad \psi_\omega^\pm(z) = \sum_{k\geq 0} z^{\mp k} \psi_{\omega,\pm k}^\pm. \tag{G.1.1}$$

Like the Chevalley generators, the operators $x_\omega^\pm(z)$ are associated to the simple roots α_ω of $\hat{\mathfrak{g}}$. On the other hand, the operators $\psi_\omega^\pm(z)$ describe the Cartan sector of the algebra, they are naturally associated to the coroots $\alpha_\omega^\vee$. We denote the Cartan matrix $\beta_{\omega\omega'} = \langle\alpha_\omega^\vee, \alpha_{\omega'}\rangle$, in the case of $\mathfrak{gl}(p)$, we have $\beta_{\omega\omega'} = 2\delta_{\omega,\omega'} - \delta_{\omega,\omega'+1} - \delta_{\omega,\omega'-1}$ (here $\delta_{\omega,\omega'}$ denotes the Kronecker delta with indices taken modulo p). In this case, the original relations can be deformed by an extra central parameter κ, using the antisymmetric matrix $m_{\omega\omega'} = \delta_{\omega,\omega'-1} - \delta_{\omega,\omega'+1}$ [160]:[1]

$$[\psi_\omega^\pm(z), \psi_{\omega'}^\pm(w)] = 0, \quad \psi_\omega^+(z)\psi_{\omega'}^-(w) = \frac{g_{\omega\omega'}(q^c z/w)}{g_{\omega\omega'}(q^{-c}z/w)}\psi_{\omega'}^-(w)\psi_\omega^+(z), \quad x_\omega^\pm(z)x_{\omega'}^\pm(w) = g_{\omega\omega'}(z/w)^{\pm 1}x_{\omega'}^\pm(w)x_\omega^\pm(z$$

$$\psi_\omega^+(z)x_{\omega'}^\pm(w) = g_{\omega\omega'}(q^{\pm c/2}z/w)^{\pm 1}x_{\omega'}^\pm(w)\psi_\omega^+(z), \quad \psi_\omega^-(z)x_{\omega'}^\pm(w) = g_{\omega\omega'}(q^{\mp c/2}z/w)^{\pm 1}x_{\omega'}^\pm(w)\psi_\omega^-(z)$$

$$[x_\omega^+(z), x_{\omega'}^-(w)] = \frac{\delta_{\omega,\omega'}}{q-q^{-1}}\left[\delta(q^{-c}z/w)\psi_\omega^+(q^{-c/2}z) - \delta(q^c z/w)\psi_\omega^-(q^{c/2}z)\right]$$

$$\sum_{\sigma\in S_2}\left[x_\omega^\pm(z_{\sigma(1)})x_\omega^\pm(z_{\sigma(2)})x_{\omega\pm1}^\pm(w) - (q+q^{-1})x_\omega^\pm(z_{\sigma(1)})x_{\omega\pm1}^\pm(w)x_\omega^\pm(z_{\sigma(2)}) + x_{\omega\pm1}^\pm(w)x_\omega^\pm(z_{\sigma(1)})x_\omega^\pm(z_{\sigma(2)})\right] = 0, \tag{G.1.3}$$

and $\psi_{\omega,0}^+\psi_{\omega,0}^- = \psi_{\omega,0}^-\psi_{\omega,0}^+ = 1$. In these relations, $q \in \mathbb{C}^\times$, c is a central element, and the matrix $g_{\omega\omega'}(z)$ writes[2]

$$g_{\omega\omega'}(z) = q^{-\beta_{\omega\omega'}}\frac{1-q^{\beta_{\omega\omega'}}\kappa^{m_{\omega\omega'}}z}{1-q^{-\beta_{\omega\omega'}}\kappa^{m_{\omega\omega'}}z}, \quad g_{\omega\omega'}(z^{-1}) = g_{\omega'\omega}(z)^{-1} = g_{\omega\omega'}(\kappa^{-2m_{\omega\omega'}}z)^{-1}. \tag{G.1.5}$$

In order to compare with the gauge theory quantities, we should set $q = q_3^{1/2}$, $\kappa = (q_1/q_2)^{1/2}$, then

$$g_{\omega\omega'}(z) = \left(q_3^{-1}\frac{1-q_3 z}{1-q_3^{-1}z}\right)^{\delta_{\omega,\omega'}}\left(q_3^{1/2}\frac{1-q_1 z}{1-q_2^{-1}z}\right)^{\delta_{\omega,\omega'-1}}\left(q_3^{1/2}\frac{1-q_2 z}{1-q_1^{-1}z}\right)^{\delta_{\omega,\omega'+1}}. \tag{G.1.6}$$

[1]The generators of this algebra are sometimes denoted $x_\omega^+(z) \to E_i(z)$, $x_\omega^-(z) \to F_i(z)$, $\psi_\omega^\pm(z) \to K_i^\pm(z)$. More rigorously the $x^\pm - x^\pm$ exchange relation should be written

$$(\kappa^{m_{\omega\omega'}}z - q^{\pm\beta_{\omega\omega'}}w)x_\omega^\pm(z)x_{\omega'}^\pm(w) = (\kappa^{m_{\omega\omega'}}q^{\pm\beta_{\omega\omega'}}z - w)x_{\omega'}^\pm(w)x_\omega^\pm(z). \tag{G.1.2}$$

This subtlety only affects the colliding points $z = q^{\pm\beta_{\omega\omega'}}\kappa^{-m_{\omega\omega'}}w$. The parameter κ here bears not connection with the Chern-Simons levels κ_ω of the gauge theory.

[2]This matrix is sometimes also written

$$g_{\omega\omega'}(z) = \theta_{\beta_{\omega\omega'}}(\kappa^{m_{\omega\omega'}}z), \quad \theta_m(z) = q^{-m}\frac{1-q^m z}{1-q^{-m}z}, \quad \theta_m(z^{-1}) = \theta_m(z)^{-1} = \theta_{-m}(z). \tag{G.1.4}$$

Modes decomposition The algebraic relations G.1.3 can also be directly written for the modes of the Drinfeld currents. In particular, introducing

$$\psi_\omega^\pm(z) = \psi_{\omega,0}^\pm \exp\left(\pm\sum_{k\geq 1} z^{\mp k} a_{\omega,\pm k}\right), \tag{G.1.7}$$

we find,

$$\psi_{\omega,0}^+ x_{\omega'}^\pm(z) = q^{\pm\beta_{\omega\omega'}} x_{\omega'}^\pm(z)\psi_{\omega,0}^+, \quad [a_{\omega,k}, a_{\omega',l}] = (q^{kc} - q^{-kc})c_{\omega\omega'}^{(k)}\delta_{k+l}, \quad [a_{\omega,k}, x_{\omega',l}^\pm] = \pm q^{\mp|k|c/2} c_{\omega\omega'}^{(k)} x_{\omega',k+l}^\pm, \tag{G.1.8}$$

where the coefficients $c_{\omega\omega'}^{(k)}$ appear in the expansion of $\log g_{\omega\omega'}(z)$:[3]

$$[g_{\omega\omega'}(z)]_\pm = q^{\pm\beta_{\omega\omega'}} \exp\left(\pm\sum_{k>0} z^{\mp k} c_{\omega\omega'}^{(\pm k)}\right), \quad c_{\omega\omega'}^{(k)} = c_{\omega'\omega}^{(-k)} = \frac{1}{k}\kappa^{-km_{\omega\omega'}}(q^{k\beta_{\omega\omega'}} - q^{-k\beta_{\omega\omega'}}), \tag{G.1.10}$$

where $[\cdots]_\pm$ denotes the expansion in powers of $z^{\mp 1}$. In addition to the central charge c, it possible to define a second central charge using the zero modes of the Cartan currents:

$$\prod_{\omega=0}^{p-1} \psi_{\omega,0}^\pm = q^{\mp\bar{c}}. \tag{G.1.11}$$

Finally, the algebra can be supplemented with the following grading operators,

$$\begin{aligned} q^d x_\omega^\pm(z) q^{-d} &= x_\omega^\pm(q^{-1}z), \quad q^d \psi_\omega^\pm(z) q^{-d} = \psi_\omega^\pm(q^{-1}z), \\ q^{\bar{d}_\omega} x_{\omega'}^\pm(z) q^{-\bar{d}_\omega} &= q^{\pm\delta_{\omega,\omega'}} x_{\omega'}^\pm(z), \quad q^{\bar{d}_\omega} \psi_{\omega'}^\pm(z) q^{-\bar{d}_\omega} = \psi_{\omega'}^\pm(z). \end{aligned} \tag{G.1.12}$$

[3] Alternatively,

$$c_{\omega\omega'}^{(k)} = \frac{1}{k}\left[(q_3^k - q_3^{-k})\delta_{\omega,\omega'} + (q_2^k - q_1^{-k})\delta_{\omega,\omega'-1} + (q_1^k - q_2^{-k})\delta_{\omega,\omega'+1}\right] = k(\beta_{\omega\omega'}^{[k]} - \beta_{\omega'\omega}^{[-k]}), \tag{G.1.9}$$

where $\beta_{\omega\omega'}^{[k]} = (1 + q_3^k)\delta_{\omega\omega'} - q_1^{-k}\delta_{\omega,\omega'-1} - q_2^{-k}\delta_{\omega,\omega'+1}$ is the mass-deformed Cartan matrix of Kimura and Pestun [173] with the mass $\mu_e = q_1$ associated to each link $e : \omega \to \omega + 1$ of the necklace quiver.

G.2 Horizontal representation

Representations of this type have central charge $c = 1$, they have been constructed by Saito in [160] under the name *vertex representations*. We review here this construction.

For $c \neq 0$, the Cartan modes $a_{\omega,k}$ define p coupled Heisenberg subalgebras. For later convenience we introduce the rescaled modes

$$\alpha_{\omega,k} = \frac{k}{q^k - q^{-k}}\rho^{(H)}(a_{\omega,k}) \quad \Rightarrow \quad [\alpha_{\omega,k}, \alpha_{\omega',l}] = k\delta_{k+l}\frac{q^{k\beta_{\omega\omega'}} - q^{-k\beta_{\omega\omega'}}}{q^k - q^{-k}}\kappa^{-km_{\omega\omega'}}. \tag{G.2.1}$$

The representation of the currents $x_\omega^\pm(z)$ and $\psi_\omega^\pm(z)$ can be factorized into two commuting parts: a *zero mode* part $(X_\omega^\pm(z),\ Y_\omega^\pm(z))$, and a vertex operator part $(\eta_\omega^\pm(z),\ \varphi_\omega^\pm(z))$ built over the Cartan modes $\alpha_{\omega,k}$:

$$\rho^{(H)}(x_\omega^\pm(z)) = X_\omega^\pm(z)\eta_\omega^\pm(z), \quad \rho^{(H)}(\psi_\omega^\pm(z)) = Y_\omega^\pm\varphi_\omega^\pm(z). \tag{G.2.2}$$

We focus first on the vertex operators part, it writes

$$\eta_\omega^\pm(z) =: \exp\left(\mp\sum_{k\in\mathbb{Z}}\frac{z^{-k}}{k}q^{\mp|k|/2}\alpha_{\omega,k}\right) :, \quad \varphi_\omega^\pm(z) = \exp\left(\pm\sum_{k>0}\frac{z^{\mp k}}{k}(q^k - q^{-k})\alpha_{\omega,\pm k}\right), \tag{G.2.3}$$

note that $\varphi_\omega^\pm(z) =: \eta_\omega^+(q^{\pm1/2}z)\eta_\omega^-(q^{\mp1/2}z) :$. The Fock vacuum $|\varnothing\rangle$ is annihilated by positive modes $\alpha_{\omega,k>0}$, and we define accordingly the normal ordering $: \cdots :$ by writing positive modes on the right. It is a matter of simple algebra to derive the following normal-ordering relations:

$$
\begin{aligned}
&\eta_\omega^+(z)\eta_{\omega'}^+(w) = S_{\omega'\omega}(w/z)^{-1} : \eta_\omega^+(z)\eta_{\omega'}^+(w) :, \quad \eta_\omega^-(z)\eta_{\omega'}^-(w) = S_{\omega'\omega}(q^2 w/z)^{-1} : \eta_\omega^-(z)\eta_{\omega'}^-(w) :,\\
&\eta_\omega^\pm(z)\eta_{\omega'}^\mp(w) = S_{\omega'\omega}(qw/z) : \eta_\omega^\pm(z)\eta_{\omega'}^\mp(w) :,\\
&\varphi_\omega^+(q^{\mp 1/2}z)\eta_{\omega'}^\pm(w) = \left(\frac{S_{\omega'\omega}(q^2 w/z)}{S_{\omega'\omega}(w/z)}\right)^{\pm 1} : \varphi_\omega^+(q^{\mp 1/2}z)\eta_{\omega'}^\pm(w) :,\\
&\eta_\omega^\pm(z)\varphi_{\omega'}^-(q^{\pm 1/2}w) = \left(\frac{S_{\omega'\omega}(q^2 w/z)}{S_{\omega'\omega}(w/z)}\right)^{\pm 1} : \eta_\omega^\pm(z)\varphi_{\omega'}^-(q^{\pm 1/2}w) :,\\
&\varphi_\omega^+(z)\varphi_{\omega'}^-(w) = \left(\frac{S_{\omega'\omega}(qw/z)^2}{S_{\omega'\omega}(q^{-1}w/z)S_{\omega'\omega}(q^3 w/z)}\right) : \varphi_\omega^+(z)\varphi_{\omega'}^-(w) :,
\end{aligned}
\tag{G.2.4}
$$

with the function

$$
S_{\omega\omega'}(z) = \exp\left(\sum_{k>0} \frac{1}{k} z^k q^{-k} \kappa^{k m_{\omega\omega'}} \frac{q^{k\beta_{\omega\omega'}} - q^{-k\beta_{\omega\omega'}}}{q^k - q^{-k}}\right). \tag{G.2.5}
$$

In fact, it is possible to resum the infinite series and write the matrix elements $S_{\omega\omega'}(z)$ as simple rational functions:

$$
S_{\omega\omega'}(z) = \prod_{r=0}^{|\beta_{\omega\omega'}|-1} (1 - \kappa^{m_{\omega\omega'}} q^{2r - |\beta_{\omega\omega'}|} z)^{-\text{sign }(\beta_{\omega\omega'})} = \frac{(1 - q_1 z)^{\delta_{\omega,\omega'-1}} (1 - q_2 z)^{\delta_{\omega,\omega'+1}}}{(1-z)^{\delta_{\omega,\omega'}} (1 - q_1 q_2 z)^{\delta_{\omega,\omega'}}}. \tag{G.2.6}
$$

We then observe the crossing symmetry,

$$
S_{\omega\omega'}(q^2/z) = f_{\omega\omega'}(z) S_{\omega'\omega}(z), \quad f_{\omega\omega'}(z) = F_{\omega\omega'} z^{\beta_{\omega\omega'}}, \quad F_{\omega\omega'} = (-q)^{-\beta_{\omega\omega'}} \kappa^{-m_{\omega\omega'}\beta_{\omega\omega'}}, \tag{G.2.7}
$$

and $F_{\omega\omega'} F_{\omega'\omega} = q^{-2\beta_{\omega\omega'}}$, $f_{\omega\omega'}(q^2/z) f_{\omega'\omega}(z) = 1$. The structure function $g_{\omega\omega'}(z)$ can be written as a ratio of functions $S_{\omega\omega'}(z)$ with shifted arguments,

$$
g_{\omega\omega'}(z) = q^{-\beta_{\omega\omega'}} \frac{S_{\omega\omega'}(z)}{S_{\omega\omega'}(q^2 z)} = f_{\omega'\omega}(qz) \frac{S_{\omega\omega'}(z)}{S_{\omega'\omega}(z^{-1})}. \tag{G.2.8}
$$

We now turn to the analysis of the zero-modes. In [160], Saito introduces the symbols e^{α_ω}

associated to the roots α_ω, and obeying the commutation relations $e^{\alpha_\omega}e^{\alpha_{\omega'}} = (-1)^{\beta_{\omega\omega'}}e^{\alpha_{\omega'}}e^{\alpha_\omega}$ (in particular symbols attached to the same root commute). These symbols, together with the operators $a_{\omega,0}$ and ∂_{α_ω} act on states parameterized by a root $\alpha = \sum_{\omega\in\mathbb{Z}_p} r_\omega\alpha_\omega$ $(r_\omega \in \mathbb{Z})$ and a fundamental weight Λ_{ω_0},

$$
\begin{aligned}
&e^{\alpha_\omega}\left|\alpha,\Lambda_{\omega_0}\right\rangle = \prod_{\omega'<\omega}(-1)^{r_{\omega'}\beta_{\omega\omega'}}\left|\alpha+\alpha_\omega,\Lambda_{\omega_0}\right\rangle, \quad \partial_{\alpha_\omega}\left|\alpha,\Lambda_{\omega_0}\right\rangle = \left\langle\alpha_\omega^\vee,\beta+\Lambda_{\omega_0}\right\rangle\ \left|\alpha,\Lambda_{\omega_0}\right\rangle,\\
&z^{a_{\omega,0}}\left|\alpha,\Lambda_{\omega_0}\right\rangle = z^{\left\langle\alpha_\omega^\vee,\alpha+\Lambda_{\omega_0}\right\rangle}\prod_{\omega'}\kappa^{r_{\omega'}\beta_{\omega\omega'}m_{\omega\omega'}/2}\left|\alpha,\Lambda_{\omega_0}\right\rangle.
\end{aligned}
\tag{G.2.9}
$$

In this representation,

$$
q^{\partial_{\alpha_\omega}}e^{\alpha_{\omega'}} = q^{\beta_{\omega\omega'}}e^{\alpha_{\omega'}}q^{\partial_{\alpha_\omega}}, \quad z^{a_{\omega,0}}e^{\alpha_{\omega'}} = z^{\beta_{\omega\omega'}}\kappa^{m_{\omega\omega'}\beta_{\omega\omega'}/2}e^{\alpha_{\omega'}}z^{a_{\omega,0}}. \tag{G.2.10}
$$

Thus, introducing $X_\omega^\pm(z) = e^{\pm\alpha_\omega}z^{1\pm a_{\omega,0}}$ and $Y_\omega^\pm = q^{\pm\partial_{\alpha_\omega}}$, we find the algebraic relations

$$
\begin{aligned}
&X_\omega^\pm(z)X_{\omega'}^\pm(w) = f_{\omega'\omega}(qz/w)X_{\omega'}^\pm(w)X_\omega^\pm(z), \quad X_\omega^\pm(z)X_{\omega'}^\mp(w) = f_{\omega'\omega}(qz/w)^{-1}X_{\omega'}^\mp(w)X_\omega^\pm(z),\\
&Y_\omega^+X_{\omega'}^\pm(w) = q^{\pm\beta_{\omega\omega'}}X_{\omega'}^\pm(w)Y_\omega^+, \quad Y_\omega^-X_{\omega'}^\pm(w) = q^{\mp\beta_{\omega\omega'}}X_{\omega'}^\pm(w)Y_\omega^-, \quad [Y_\omega^+,Y_{\omega'}^-] = 0.
\end{aligned}
\tag{G.2.11}
$$

It is easy to verify that these are indeed the factors needed to reproduce the algebraic relations G.1.3. The only difficulty appears in the verification of the commutation relation $[x_\omega^+, x_{\omega'}^-]$ for which we need to use the property $z^{a_{\omega,0}}w^{-a_{\omega,0}} = z^{\partial_{\alpha_\omega}}w^{-\partial_{\alpha_\omega}}$ to treat the zero mode dependence. The value of the central charge $\bar{c}$ can be recovered by noticing that

$$
\sum_{\omega\in\mathbb{Z}_p}\left\langle\alpha_\omega^\vee,\beta+\Lambda_{\omega_0}\right\rangle = 1 \quad\Rightarrow\quad \prod_{\omega\in\mathbb{Z}_p}q^{\partial_{\alpha_\omega}} = q. \tag{G.2.12}
$$

This representation has been extended to higher level $\bar{c}$ in [154]. Note however that in the definition of the (ν_1,ν_2)-deformed horizontal representation, a set of $4p$ Heisenberg algebras will be employed to define to the zero-modes $X_\omega^\pm$ and $Y_\omega^\pm$ instead of the symbols introduced in G.2.9.

G.3 Vertical representations

The vertical representations have central charge $c = 0$ and thus the Cartan currents $\psi_\omega^\pm(z)$ commute. They are diagonal in the basis of states $|\boldsymbol{\lambda}\rangle\!\rangle$ labeled by m-tuple Young diagram $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(m)})$. The representation depends on an m-vector of weights $\boldsymbol{v} = (v_1, \cdots, v_m)$ and a choice of coloring c_α for each component v_α. We denote $m_\omega = |C_\omega(m)|$ the number of weights v_α of color $c_\alpha = \omega$ (obviously, $m = \sum_{\omega\in\mathbb{Z}_p} m_\omega$). The action of the Drinfeld currents on the states $|\boldsymbol{\lambda}\rangle\!\rangle$ reads

$$\begin{aligned}
\rho^{(V)}(x_\omega^+(z))\,|\boldsymbol{\lambda}\rangle\!\rangle &= (qz)^{-\beta_\omega^{[\boldsymbol{\lambda}]}} \prod_{\square\in\boldsymbol{\lambda}} (-\kappa)^{-m_{\omega c(\square)}/2} \sum_{\square\in A_\omega(\boldsymbol{\lambda})} \delta(z/\chi_\square) \operatorname*{Res}_{z=\chi_\square} \frac{1}{z\tilde{\mathcal{Y}}_\omega^{[\boldsymbol{\lambda}]}(z)}\,|\boldsymbol{\lambda}+\square\rangle\!\rangle,\\
\rho^{(V)}(x_\omega^-(z))\,|\boldsymbol{\lambda}\rangle\!\rangle &= qz^{\beta_\omega^{[\boldsymbol{\lambda}]}+2} \prod_{\square\in\boldsymbol{\lambda}} (-\kappa)^{m_{\omega c(\square)}/2} \sum_{\square\in R_\omega(\boldsymbol{\lambda})} \delta(z/\chi_\square) \operatorname*{Res}_{z=\chi_\square} z^{-1}\tilde{\mathcal{Y}}_\omega^{[\boldsymbol{\lambda}]}(q_3^{-1}z)\,|\boldsymbol{\lambda}-\square\rangle\!\rangle,\\
\rho^{(V)}(\psi_\omega^\pm(z))\,|\boldsymbol{\lambda}\rangle\!\rangle &= \left[\tilde{\Psi}_\omega^{[\boldsymbol{\lambda}]}(z)\right]_\pm |\boldsymbol{\lambda}\rangle\!\rangle.
\end{aligned} \tag{G.3.1}$$

In the first two lines, summations are performed over the set of boxes of color ω that can be added ($A_\omega(\boldsymbol{\lambda})$) to or removed from ($R_\omega(\boldsymbol{\lambda})$) the m-tuple Young diagram $\boldsymbol{\lambda}$. The summands are expressed in terms of residues involving the functions $\tilde{\mathcal{Y}}_\omega^{[\boldsymbol{\lambda}]}(z)$, and the action of the Cartan is given as an expansion of the functions $\tilde{\Psi}_\omega^{[\boldsymbol{\lambda}]}(z)$ in powers of $z^{\mp1}$. These two sets of functions are defined as follows:

$$\begin{aligned}
&\tilde{\Psi}_\omega^{[\boldsymbol{\lambda}]}(z) = q^{-m_\omega} \prod_{\alpha\in C_\omega(m)} \frac{1-q^2v_\alpha/z}{1-v_\alpha/z} \prod_{\square\in\boldsymbol{\lambda}} g_{\omega c(\square)}(z/\chi_\square) = q^{-\beta_\omega^{[\boldsymbol{\lambda}]}} \frac{\tilde{\mathcal{Y}}_\omega^{[\boldsymbol{\lambda}]}(q^{-2}z)}{\tilde{\mathcal{Y}}_\omega^{[\boldsymbol{\lambda}]}(z)},\\
&\tilde{\mathcal{Y}}_\omega^{[\boldsymbol{\lambda}]}(z) = \prod_{\alpha\in C_\omega(m)} (1-v_\alpha/z) \prod_{\square\in\boldsymbol{\lambda}} S_{c(\square)\omega}(\chi_\square/z), \quad \beta_\omega^{[\boldsymbol{\lambda}]} = m_\omega - \sum_{\square\in\boldsymbol{\lambda}} \beta_{\omega c(\square)}.
\end{aligned} \tag{G.3.2}$$

The zero modes of the Cartan act as

$$\rho^{(V)}(\psi^{\pm}_{\omega,0})\,|\boldsymbol{\lambda}\rangle\!\rangle = q^{\mp\beta^{[\boldsymbol{\lambda}]}_{\omega}}\,|\boldsymbol{\lambda}\rangle\!\rangle. \tag{G.3.3}$$

and, taking the product over the index ω, we deduce the level $\rho^{(V)}(\bar{c}) = m$.

G.4 Relation with the (ν_1, ν_2)-deformed algebra

The physical quantity we need to reproduce is the scattering function $S_{\omega\omega'}(z)$ defined in 6.2.16. In this scope, it is easier to compare the horizontal representations, and reproduce the commutation of the Heisenberg subalgebras 6.3.22 using the $\mathfrak{gl}(p)$ formula G.2.1. This leads to identify the Cartan matrix with the matrix $\beta_{\omega\omega'}$ defined in 6.2.18. Furthermore, the factor $\kappa^{m_{\omega\omega'}}$ has to be replaced with a more general matrix $\kappa_{\omega\omega'}$ that reads[4]

$$\kappa_{\omega\omega'} = q^{-1} q_3^{\delta_{\omega,\bar{\omega}'}} q_1^{-\delta_{\omega,\omega'+\nu_1}} q_2^{-\delta_{\omega,\omega'+\nu_2}}. \tag{G.4.1}$$

Using this identification, the crossing symmetry relation 6.2.17 reduces to G.2.7 for $\nu_3 = 0$, and the formula G.2.7 for $F_{\omega\omega'}$ reproduces the definition 6.2.18. The function $S_{\omega\omega'}(z)$ defined in 6.2.16 is recovered by replacing the expression G.2.5 with

$$S_{\omega\omega'}(z) = \exp\left(\sum_{k>0} \frac{z^k}{k} q^{-k} \kappa_{\omega'\omega}^{-k} \frac{q^{k\beta_{\omega'\omega}} - q^{-k\beta_{\omega'\omega}}}{q^k - q^{-k}}\right). \tag{G.4.2}$$

The definition of the structure function $g_{\omega\omega'}(z)$ is a little more difficult because of the freedom in defining the *zero-mode* factor. Comparing with the formula G.2.8 for the case of $\mathfrak{gl}(p)$, the most natural choice would be

$$g^{(1)}_{\omega\omega'}(z) = f_{\omega\omega'}(qz) \frac{S_{\omega\omega'}(z)}{S_{\omega'\omega}(z^{-1})} = q^{\beta_{\omega\omega'}} F^2_{\omega\omega'} \prod_{i=1,2,3} \frac{(1 - q_i z)^{\delta_{\omega,\omega'-\nu_i}}}{(1 - q_i^{-1} z)^{\delta_{\omega,\omega'+\nu_i}}}. \tag{G.4.3}$$

[4]We could also express the coefficients $\sigma^{[k]}_{\omega\omega'} = k\beta_{\omega\omega'}\kappa^{-k}_{\omega\omega'} = kq_3^{k/2}\beta^{[k]}_{\omega\omega'}$ in terms of the mass-deformed Cartan matrix $\beta^{[k]}_{\omega\omega'} = \delta_{\omega,\omega'} + q_3^k \delta_{\omega,\bar{\omega}'} - q_1^{-k}\delta_{\omega,\omega'+\nu_1} - q_2^{-k}\delta_{\omega,\omega'+\nu_2}$.

Unfortunately, with this definition, the identity $g^{(1)}_{\omega\omega'}(z)g^{(1)}_{\omega'\omega}(z^{-1}) = 1$ is NOT satisfied, yet it is necessary for the consistency of the algebraic relations. This prompts us to propose instead the definition given in 6.3.4 where the factor $f_{\omega\omega'}(qz)$ is missing. Unfortunately, this redefinition of the structure function $g_{\omega\omega'}(z)$ breaks the natural symmetry between positive and negative currents, and makes the definition of the central charge $\bar{c}$ more difficult. Note also that another possibility could have been to define

$$g^{(2)}_{\omega\omega'}(z) = \prod_{i=1,2,3} \frac{(1-q_i z)^{\delta_{\omega,\omega'-\nu_i}}}{(1-q_i z^{-1})^{\delta_{\omega,\omega'+\nu_i}}}, \tag{G.4.4}$$

but this would require us to modify the definition of the function $S_{\omega\omega'}(z)$:

$$g^{(2)}_{\omega\omega'}(z) = -\frac{S^{(2)}_{\omega\omega'}(z)}{S^{(2)}_{\omega'\omega}(z^{-1})}, \quad S^{(2)}_{\omega\omega'}(z) = \frac{(1-q_1 z)^{\delta_{\omega,\omega'-\nu_1}}(1-q_2 z)^{\delta_{\omega,\omega'-\nu_2}}}{(1-z^{-1})^{\delta_{\omega,\omega'}}(1-q_3 z^{-1})^{\delta_{\omega,\omega'-\nu_1-\nu_2}}}. \tag{G.4.5}$$

Appendix H

Shell formula

We provide here a short derivation of the shell formula 6.2.23 for the functions $\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)$. Since $K_\omega(\boldsymbol{\lambda})$ is a direct sum of $K_\omega(\lambda^{(\alpha)})$, the function $\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)$ factorizes into contributions of the individual Young diagrams $\mathcal{Y}_\omega^{[\lambda^{(\alpha)}]}(z)$. Thus, it is possible to focus on the case of a single Young diagram $\lambda^{(\alpha)}$ corresponding to a weight v_α of color c_α. The proof will be done by recursion on the number of boxes. We start with an empty Young diagram, for which $R_\omega(\varnothing) = \varnothing$. The box $\square = (1,1)$ is of color $c(\square) = c_\alpha$, thus $A_\omega(\varnothing) = \{(1,1)\}$ if $\omega = c_\alpha$ and $A_\omega(\varnothing) = \varnothing$ otherwise. Accordingly, we recover $\mathcal{Y}_\omega^{[\varnothing]}(z) = (1 - v_\alpha/z)^{\delta_{\omega,c_\alpha}}$.

Now, let's add a box $\square$ to $\lambda^{(\alpha)}$. From the definition 6.3.13, we have

$$\frac{\mathcal{Y}_\omega^{[\lambda^{(\alpha)}+\square]}(z)}{\mathcal{Y}_\omega^{[\lambda^{(\alpha)}]}(z)} = \frac{(1-q_1\chi_\square/z)^{\delta_{c(\square),\omega-\nu_1}}(1-q_2\chi_\square/z)^{\delta_{c(\square),\omega-\nu_2}}}{(1-\chi_\square/z)^{\delta_{c(\square),\omega}}(1-q_1q_2\chi_\square/z)^{\delta_{c(\square),\omega-\nu_1-\nu_2}}}. \tag{H.0.1}$$

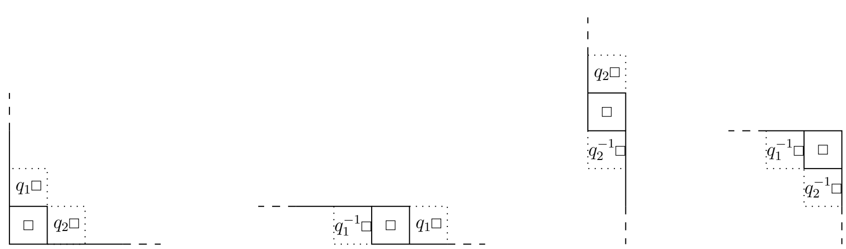

Figure H.1: Generic and degenerate configurations of a box added to the corner of a Young diagram

There are four possible configurations for adding box in $\lambda^{(\alpha)}$, all represented in figure H.1. We start with the generic case, for which

$$\begin{aligned}
&A_\omega(\lambda^{(\alpha)}+\square)=A_\omega(\lambda^{(\alpha)})\setminus\{\square\mid c(\square)=\omega\}\cup\{q_1\square\mid c(\square)=\omega-\nu_1\}\cup\{q_2\square\mid c(\square)=\omega-\nu_2\},\\
&R_{\omega-\nu_1-\nu_2}(\lambda^{(\alpha)}+\square)=R_{\omega-\nu_1-\nu_2}(\lambda^{(\alpha)})\cup\{\square\mid c(\square)=\omega-\nu_1-\nu_2\}.
\end{aligned}\tag{H.0.2}$$

We employed here the shortcut notation $q_1^{\pm1}\square$ ($q_2^{\pm1}\square$) to designate the box of coordinate $(i\pm1,j)$ (resp. $(i,j\pm1)$) next to $\square=(i,j)$. In this generic case, the factors induced by the variation of the content of the sets A_ω and $R_{\omega-\nu_1-\nu_2}$ reproduce the extra factor $S_{c(\square)\omega}(\chi_\square/z)$ in the RHS of H.0.1.

We now turn to the first case of the degenerate configurations represented on figure H.1. In this case, only one more box can be added to $A_\omega(\lambda^{(\alpha)}+\square)$. On the other hand, the addition of the box $\square$ prevents the removal of the box $q_1^{-1}\square$. As a result,

$$\begin{aligned}
&A_\omega(\lambda^{(\alpha)}+\square)=A_\omega(\lambda^{(\alpha)})\setminus\{\square\mid c(\square)=\omega\}\cup\{q_1\square\mid c(\square)=\omega-\nu_1\},\\
&R_{\omega-\nu_1-\nu_2}(\lambda^{(\alpha)}+\square)=R_{\omega-\nu_1-\nu_2}(\lambda^{(\alpha)})\cup\{\square\mid c(\square)=\omega-\nu_1-\nu_2\}\setminus\{q_1^{-1}\square\mid c(\square)=\omega-\nu_2\}.
\end{aligned}\tag{H.0.3}$$

Once again, we observe the agreement between the variation of the RHS 6.2.23 and the recursion relation H.0.1. The other two cases are treated in the same way.

Appendix I

Representations of the extended algebra

I.1 Vertical representation

The vertical representation is of the highest weight type. The highest state $|\varnothing\rangle\rangle$, also called *vacuum state*, is annihilated by the currents $x_\omega^-(z)$, while $x_\omega^+(z)$ create excitations. The excited states $|\boldsymbol{\lambda}\rangle\rangle$ are parameterized by an m-tuple Young diagram $\boldsymbol{\lambda}$. The weights $\boldsymbol{v} = (v_1, \cdots, v_m)$ parameterize the action of the Cartan $\psi_\omega^\pm(z)$ on the vacuum state. The two Cartan currents commute, they are diagonal in the basis $|\boldsymbol{\lambda}\rangle\rangle$, with the eigenvalue $[\Psi_\omega^{[\boldsymbol{\lambda}]}(z)]_\pm$ where $\pm$ denotes an expansion in powers of $z^{\mp 1}$. The action of $x_\omega^\pm(z)$ add/remove a box of color ω. In order to produce the Dirac δ-function in the commutator $[x^+, x^-]$, it is natural to assume that modes $x_{\omega,k}^\pm$ depends on the index k only through a factor of $\chi_\square^k$ where $\square$ is the box that is added/removed. Taking all these assumptions in consideration, we arrive at the following ansatz:

$$
\begin{aligned}
x^+_\omega(z)\,|\boldsymbol{\lambda}\rangle\!\rangle &= \sum_{\Box\in A_\omega(\boldsymbol{\lambda})} \delta(z/\chi_\Box) A^{+[\boldsymbol{\lambda}]}_\omega(x)\,|\boldsymbol{\lambda}+\Box\rangle\!\rangle,\\
x^-_\omega(z)\,|\boldsymbol{\lambda}\rangle\!\rangle &= \sum_{\Box\in R_\omega(\boldsymbol{\lambda})} \delta(z/\chi_\Box) A^{-[\boldsymbol{\lambda}]}_\omega(x)\,|\boldsymbol{\lambda}-\Box\rangle\!\rangle,\\
\psi^\pm_\omega(z)\,|\boldsymbol{\lambda}\rangle\!\rangle &= \left[\Psi^{[\boldsymbol{\lambda}]}_\omega(z)\right]_\pm |\boldsymbol{\lambda}\rangle\!\rangle,
\end{aligned}
\tag{I.1.1}
$$

where $A^{\pm[\boldsymbol{\lambda}]}_\omega(x)$ are the coefficients to be determined.

When the central charge c is vanishing, the algebra 6.3.2 simplifies drastically,

$$
\begin{aligned}
&x^\pm_\omega(z)x^\pm_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm1}x^\pm_{\omega'}(w)x^\pm_\omega(z), \quad [\psi^+_\omega(z),\psi^-_{\omega'}(w)] = [\psi^\pm_\omega(z),\psi^\pm_{\omega'}(w)] = 0,\\
&\psi^+_\omega(z)x^\pm_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm1}x^\pm_{\omega'}(w)\psi^+_\omega(z), \quad \psi^-_\omega(z)x^\pm_{\omega'}(w) = g_{\omega\omega'}(z/w)^{\pm1}x^\pm_{\omega'}(w)\psi^-_\omega(z),\\
&[x^+_\omega(z),x^-_{\omega'}(w)] = \Omega\delta_{\omega,\omega'}\delta(z/w)\left[\psi^+_\omega(z)-\psi^-_\omega(z)\right].
\end{aligned}
\tag{I.1.2}
$$

Plugging in the ansatz I.1.1, and the expression 6.3.13 for $\Psi^{[\boldsymbol{\lambda}]}_\omega(z)$, we find that these relations are satisfied provided that

$$
\begin{aligned}
&A^{-[\boldsymbol{\lambda}]}_\omega(x)A^{+[\boldsymbol{\lambda}-\Box]}_\omega(x) = \Omega \operatorname*{Res}_{z=\chi_\Box} z^{-1}\Psi^{[\boldsymbol{\lambda}]}_\omega(z), \quad \Box\in R_\omega(\lambda),\\
&A^{+[\boldsymbol{\lambda}]}_\omega(x)A^{-[\boldsymbol{\lambda}+\Box]}_\omega(x) = -\Omega \operatorname*{Res}_{z=\chi_\Box} z^{-1}\Psi^{[\boldsymbol{\lambda}]}_\omega(z), \quad \Box\in A_\omega(\lambda),\\
&\frac{A^{\pm[\boldsymbol{\lambda}\pm x]}_\omega(y)}{A^{\pm[\boldsymbol{\lambda}]}_\omega(y)} = g_{\omega\omega'}(\chi_y/\chi_\Box)^{\pm1}\frac{A^{\pm[\boldsymbol{\lambda}\pm y]}_{\omega'}(x)}{A^{\pm[\boldsymbol{\lambda}]}_{\omega'}(x)}, \quad c(\Box)=\omega', \quad c(y)=\omega.
\end{aligned}
\tag{I.1.3}
$$

The first two relations come from the projection of the commutator $[x^+,x^-]$ on the basis $|\boldsymbol{\lambda}\rangle\!\rangle$, decomposing the RHS as

$$
\left[\Psi^{[\boldsymbol{\lambda}]}_\omega(z)\right]_+ - \left[\Psi^{[\boldsymbol{\lambda}]}_\omega(z)\right]_- = \sum_{\Box\in A_\omega(\boldsymbol{\lambda})}\delta(z/\chi_\Box)\operatorname*{Res}_{z=\chi_\Box} z^{-1}\Psi^{[\boldsymbol{\lambda}]}_\omega(z) + \sum_{\Box\in R_\omega(\boldsymbol{\lambda})}\delta(z/\chi_\Box)\operatorname*{Res}_{z=\chi_\Box} z^{-1}\Psi^{[\boldsymbol{\lambda}]}_\omega(z).
\tag{I.1.4}
$$

The last equation in I.1.3 arises from the exchange relations $x^\pm x^\pm$. Then, it is simply a matter of calculation to check that the following coefficients do indeed satisfy the relations I.1.3,

$$\begin{aligned}
A_\omega^{+[\boldsymbol{\lambda}]}(\square) &= F^{1/2} \operatorname*{Res}_{z=\chi_\square} z^{-1}\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)^{-1} = \Omega\mathcal{Y}_\omega^{[\boldsymbol{\lambda}+\square]}(\chi_\square)^{-1},\\
A_\omega^{-[\boldsymbol{\lambda}]}(\square) &= \mathring{f}_{\bar\omega}^{[\boldsymbol{\lambda}]}(q_3^{-1}z) \operatorname*{Res}_{z=\chi_\square} z^{-1}\mathcal{Y}_{\bar\omega}^{[\boldsymbol{\lambda}]}(q_3^{-1}z) = -\Omega F^{1/2}\mathring{f}_{\bar\omega}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)\mathcal{Y}_{\bar\omega}^{[\boldsymbol{\lambda}-\square]}(q_3^{-1}\chi_\square).
\end{aligned} \tag{I.1.5}$$

I.2 Horizontal representation

Here the strategy is to start by computing the algebraic relations satisfied by the vertex operators $\eta_\omega^\pm$ and $\varphi_\omega^\pm$, compare them with 6.3.2, and introduce the zero-modes factors to compensate unwanted factors. Using the definition 6.3.24, we can compute the normal-ordering relations

$$\begin{aligned}
&\eta_\omega^+(z)\eta_{\omega'}^+(w) = S_{\omega'\omega}(w/z)^{-1} :\eta_\omega^+(z)\eta_{\omega'}^+(w):, \quad \eta_\omega^+(z)\eta_{\omega'}^-(w) = S_{\omega'\omega}(w/z) :\eta_\omega^+(z)\eta_{\omega'}^-(w):,\\
&\eta_\omega^-(z)\eta_{\omega'}^-(w) = S_{\omega'\bar\omega}(q_3w/z)^{-1} :\eta_\omega^-(z)\eta_{\omega'}^-(w):= f_{\omega'\omega}(z/w)^{-1}S_{\omega\omega'}(z/w)^{-1} :\eta_\omega^-(z)\eta_{\omega'}^-(w):,\\
&\eta_\omega^-(z)\eta_{\omega'}^+(w) = S_{\omega'\ \bar\omega}(q_3w/z) :\eta_\omega^-(z)\eta_{\omega'}^+(w):= f_{\omega'\omega}(z/w)S_{\omega\omega'}(z/w) :\eta_\omega^-(z)\eta_{\omega'}^+(w):,
\end{aligned} \tag{I.2.1}$$

and, since $\varphi_\omega^+(z) =: \eta_\omega^+(z)\eta_\omega^-(z):$ and $\varphi_\omega^-(z) =: \eta_{\bar\omega}^+(q_3^{-1}z)\eta_\omega^-(z):$,

$$\begin{aligned}
\varphi_\omega^+(z)\eta_{\omega'}^\pm(w) &= f_{\omega'\omega}(z/w)^{\pm1}g_{\omega\omega'}(z/w)^{\pm1} :\varphi_\omega^+(z)\eta_{\omega'}^\pm(w):,\\
\eta_\omega^+(z)\varphi_{\omega'}^-(w) &= f_{\bar\omega'\omega}(q_3z/w)g_{\omega\bar\omega'}(q_3z/w) :\eta_\omega^+(z)\varphi_{\omega'}^-(w):\\
\eta_\omega^-(z)\varphi_{\omega'}^-(w) &= f_{\omega'\omega}(z/w)^{-1}g_{\omega\omega'}(z/w)^{-1} :\eta_\omega^-(z)\varphi_{\omega'}^-(w):,\\
\varphi_\omega^+(z)\varphi_{\omega'}^-(w) &= \frac{f_{\bar\omega'\omega}(q_3z/w)}{f_{\omega'\omega}(z/w)}\frac{g_{\omega\bar\omega'}(q_3z/w)}{g_{\omega\omega'}(z/w)} :\varphi_\omega^+(z)\varphi_{\omega'}^-(w): .
\end{aligned} \tag{I.2.2}$$

We deduce the algebraic relations between vertex operators. Comparing them with the currents algebra 6.3.2 at $c=1$, we observe that the latter are satisfied provided that we set

$$\rho_u^{(H)}(x_\omega^\pm(z)) = X_\omega^\pm(z)\eta_\omega^\pm(z), \quad \rho_u^{(H)}(\psi_\omega^\pm(z)) = Y_\omega^\pm(z)\varphi_\omega^\pm(z), \tag{I.2.3}$$

with[1]

$$
\begin{aligned}
&X^+_\omega(z)X^+_{\omega'}(w) = X^+_{\omega'}(w)X^+_\omega(z), \quad X^-_\omega(z)X^-_{\omega'}(w) = \frac{f_{\omega'\omega}(z/w)}{f_{\omega\omega'}(w/z)}X^-_{\omega'}(w)X^-_\omega(z),\\
&X^+_\omega(z)X^-_{\omega'}(w) =: X^+_\omega(z)X^-_{\omega'}(w) := f_{\omega\omega'}(w/z)X^-_{\omega'}(w)X^+_\omega(z),\\
&Y^+_\omega(z)X^\pm_{\omega'}(w) = f_{\omega'\omega}(z/w)^{\mp 1}X^\pm_\omega(z)Y^+_{\omega'}(w),\\
&X^+_\omega(z)Y^-_{\omega'}(w) = f_{\bar\omega'\omega}(q_3 z/w)^{-1}Y^-_{\omega'}(w)X^+_\omega(z), \quad X^-_\omega(z)Y^-_{\omega'}(w) = f_{\omega'\omega}(z/w)Y^-_{\omega'}(w)X^-_\omega(z),\\
&Y^+_\omega(z) = F^{-1/2} : X^+_\omega(z)X^-_\omega(z) :, \quad Y^-_\omega(z) = F^{1/2} : X^+_{\bar\omega}(q_3^{-1}z)X^-_\omega(z) : .
\end{aligned}
\tag{I.2.5}
$$

The last two relations come from the commutator $[x^+, x^-]$, they have been obtained using the pole decomposition of the function $S_{\omega'\omega}(w/z)$ which brings

$$
[S_{\omega'\omega}(w/z)]_+ - [S_{\omega'\omega}(w/z)]_- = \Omega\left[\delta_{\omega,\omega'}\delta(z/w)F^{-1/2} - \delta_{\omega,\bar\omega'}\delta(q_3 z/w)F^{1/2}\right]. \tag{I.2.6}
$$

The relations I.2.5 are satisfied if we set

$$
X^+_\omega(z) = Q_\omega(z), \quad X^-_\omega(z) = Q_\omega(z)^{-1}P_{\bar\omega}(q_3^{-1}z), \quad Y^+_\omega(z) = F^{-1/2}P_{\bar\omega}(q_3^{-1}z), \quad Y^-_\omega(z) = F^{1/2}\frac{Q_{\bar\omega}(q_3^{-1}z)}{Q_\omega(z)}P_{\bar\omega}(q_3^{-1}
\tag{I.2.7}
$$

where $Q_\omega(z)$ and $P_{\omega'}(w)$ obey 6.3.21. These operators can be constructed in terms of $2p$ Heisenberg algebras $[p_\omega, q_{\omega'}] = \delta_{\omega,\omega'}$ and $[\tilde p_\omega, \tilde q_{\omega'}] = \delta_{\omega,\omega'}$ by setting

$$
Q_\omega(z) = e^{q_\omega + \tilde q_\omega \log z}, \quad P_\omega(z) = z^{-\sum_{\omega'}\beta_{\omega\omega'}p_{\omega'}}e^{\sum_{\omega'}\beta_{\omega\omega'}\tilde p_{\omega'}}(-1)^{p_\omega}(-q_3)^{-p_{\omega+\nu_3}}(-q_1)^{-p_{\omega-\nu_1}}(-q_2)^{-p_{\omega-\nu_2}}.
\tag{I.2.8}
$$

Combining the operators $X^\pm_\omega$, $Y^\pm_\omega$ and the vertex operators $\eta^\pm_\omega$, $\psi^\pm_\omega$, we find the representation 6.3.25. The dependence in the weights u_ω and levels n_ω is recovered using the freedom to

[1] Note that these relations imply

$$
Y^+_\omega(z)Y^-_{\omega'}(w) = \frac{f_{\omega'\omega}(z/w)}{f_{\bar\omega'\omega}(q_3 z/w)}Y^-_{\omega'}(w)Y^+_\omega(z). \tag{I.2.4}
$$

shift the operators q_ω, $\tilde{q}_\omega$ as $q_\omega \to q_\omega + \log(u_\omega)$, $\tilde{q}_\omega \to \tilde{q}_\omega - n_\omega$.

It remains to compute the central charge $\bar{c}$. The zero modes of the Cartan currents write

$$\rho^{(H)}(\psi^+_{\omega,0}) = F^{-1/2} P_{\bar{\omega}}(q_3^{-1}) \quad \rho^{(H)}(\psi^-_{\omega,0}) = F^{1/2} \frac{u_{\bar{\omega}}}{u_\omega} q_3^{n_{\bar{\omega}}} \frac{Q_{\bar{\omega}}(q_3^{-1})}{Q_\omega(1)} P_{\bar{\omega}}(q_3^{-1}). \tag{I.2.9}$$

We deduce that

$$\rho^{(H)}\left(\prod_{\omega\in\mathbb{Z}_p} \psi^+_{\omega,0}(\psi^-_{\omega,0})^{-1}\right) = q_3^{-n-\tilde{q}}, \quad \tilde{q} = \sum_{\omega\in\mathbb{Z}_p} \tilde{q}_\omega. \tag{I.2.10}$$

Since $[\tilde{q}_\omega, P_{\omega'}(w)] = \beta_{\omega'\omega} P_{\omega'}(w)$, the operator $\tilde{q}$ commute with $P_\omega(z)$, thus it is central in this representation. Moreover, since $Q_\omega(z)$ acts trivially on the dual state $\langle\varnothing|$, we have $\tilde{q} = 0$. Finally, we also have to take into account the non-commutation of the zero modes which brings the extra factor

$$\prod_{\substack{\omega,\omega'=0\\ \omega\leq\omega'}}^{p-1} \frac{F_{\omega'\omega}}{F_{\bar{\omega}'\omega}} = \prod_{\substack{\omega,\omega'=0\\ \omega\leq\omega'}}^{p-1} q_3^{\beta_{\omega\omega'}} F_{\omega\omega'} F_{\omega'\omega} = \prod_{\substack{\omega,\omega'=0\\ \omega\leq\omega'}}^{p-1} q_3^{\beta_{\omega\omega'}} \times F^{-p} \prod_{\omega,\omega'=0}^{p-1} F_{\omega\omega'} = \prod_{\substack{\omega,\omega'=0\\ \omega\leq\omega'}}^{p-1} q_3^{\beta_{\omega\omega'}}. \tag{I.2.11}$$

Since $\beta_{\omega\omega'}$ is circulant, it is easy to compute

$$\sum_{\substack{\omega,\omega'=0\\ \omega\leq\omega'}}^{p-1} \beta_{\omega\omega'} = \begin{cases} p & (\nu_1+\nu_2 < p), \\ 0 & \text{(else)}, \end{cases} \tag{I.2.12}$$

assuming $0 \leq \nu_1, \nu_2 \leq p-1$. This gives us the value of the central charge $\bar{c}$.

Appendix J

Automorphisms, gradings and modes expansion

J.1 Automorphisms and gradings

The algebraic relations 6.3.2 can be supplemented with the grading operators d and $\bar{d}_\omega$ ($\omega \in \mathbb{Z}_p$) acting on the currents as

$$e^{\alpha d} x_\omega^\pm(z) e^{-\alpha d} = x^\pm(e^\alpha z), \quad e^{\alpha d} \psi_\omega^\pm(z) e^{-\alpha d} = \psi_\omega^\pm(e^\alpha z),$$
$$e^{\alpha \bar{d}_\omega} x_{\omega'}^\pm(z) e^{-\alpha \bar{d}_\omega} = e^{\pm\alpha\delta_{\omega,\omega'}} x_{\omega'}^\pm(z), \quad e^{\alpha \bar{d}_\omega} \psi_{\omega'}^+(z) e^{-\alpha \bar{d}_\omega} = \psi_{\omega'}^+(z), \quad e^{\alpha \bar{d}_\omega} \psi_{\omega'}^-(z) e^{-\alpha \bar{d}_\omega} = e^{\alpha(\delta_{\omega,\omega'-\nu_3 c} - \delta_{\omega\omega'})} \psi_{\omega'}^-(z), \tag{J.1.1}$$

for any parameter $\alpha \in \mathbb{C}$. The grading operator d reflects the invariance of the algebra under rescaling of the variable $z \to e^\alpha z$, it defines the automorphisms τ_α acting on an element x of the algebra as $\tau_\alpha(x) = e^{\alpha d} x e^{-\alpha d}$. Similarly, the grading operators $\bar{d}_\omega$ defines the automorphisms $\bar{\tau}_{\omega,\alpha}(x) = e^{\alpha \bar{d}_\omega} x e^{-\alpha \bar{d}_\omega}$ associated to the invariance under the following rescaling of the currents for a fixed ω:

$$x_\omega^\pm(z) \to e^{\pm\alpha} x_\omega^\pm(z), \quad \psi_\omega^-(z) \to e^{-\alpha} \psi_\omega^-(z), \quad \psi_{\omega+\nu_3 c}^-(z) \to e^\alpha \psi_{\omega+\nu_3 c}^-(z), \tag{J.1.2}$$

while the currents $x^{\pm}_{\omega'\neq\omega}(z)$, $\psi^{+}_{\omega'}(z)$ and $\psi^{-}_{\omega'\neq\omega,\omega+\nu_3 c}(z)$ remain invariant.

In addition to the automorphisms τ_α and $\bar{\tau}_{\omega,\alpha}$, the algebraic relations are invariant under a third class of automorphisms $\tilde{\tau}_{\omega,\alpha}(x) = e^{\alpha\tilde{d}_\omega} x e^{-\alpha\tilde{d}_\omega}$ defined as

$$\begin{aligned}
&e^{\alpha\tilde{d}_\omega} x^{\pm}_{\omega'}(z) e^{-\alpha\tilde{d}_\omega} = z^{\pm\alpha\delta_{\omega,\omega'}} x^{\pm}_{\omega'}(z), \quad e^{\alpha\tilde{d}_\omega} \psi^{+}_{\omega'}(z) e^{-\alpha\tilde{d}_\omega} = \psi^{+}_{\omega'}(z), \\
&e^{\alpha\tilde{d}_\omega} \psi^{-}_{\omega'}(z) e^{-\alpha\tilde{d}_\omega} = (q_3^{-c} z)^{\alpha\delta_{\omega,\omega'-\nu_3 c}} z^{-\alpha\delta_{\omega\omega'}} \psi^{-}_{\omega'}(z).
\end{aligned} \tag{J.1.3}$$

This transformation is the generalization of the element $\mathcal{T}$ of the $\mathrm{SL}(2,\mathbb{Z})$ group of automorphisms for the quantum toroidal algebra of $\mathfrak{gl}_1$ (or Ding-Iohara-Miki algebra) [51]. With a slight abuse of terminology, we will also call $\tilde{d}_\omega$ a grading operator.

J.2 Modes expansion

In order to define properly the modes expansion of the currents $x^{\pm}_{\omega}(z)$ and $\psi^{\pm}_{\omega}(z)$, we need to remove some part of the zero modes factors. For this purpose, we use a twist by a combination of automorphisms to define the new currents $\tilde{x}^{\pm}_{\omega}(z)$ and $\tilde{\psi}^{\pm}_{\omega}(z)$ with proper modes expansion. First, we introduce the following combinations of grading operators,

$$F_\omega = (-1)^{\bar{d}_\omega} (-q_3)^{-\bar{d}_{\omega+\nu_3}} (-q_1)^{-\bar{d}_{\omega-\nu_1}} (-q_2)^{-\bar{d}_{\omega-\nu_2}}, \quad \beta_\omega = -\sum_{\omega'} \beta_{\omega\omega'} \bar{d}_{\omega'}, \quad D_\omega = e^{\sum_{\omega'} \beta_{\omega\omega'} \tilde{d}_{\omega'}}, \tag{J.2.1}$$

such that

$$\begin{aligned}
&F_\omega x^{\pm}_{\omega'}(w) F_\omega^{-1} = F_{\omega\omega'}^{\pm 1}\ x^{\pm}_{\omega'}(w), \quad z^{\beta_\omega} x^{\pm}_{\omega'}(w) z^{-\beta_\omega} = z^{\mp\beta_{\omega\omega'}} x^{\pm}_{\omega'}(w), \quad D_\omega x^{\pm}_{\omega'}(w) D_\omega^{-1} = w^{\pm\beta_{\omega\omega'}}\ x^{\pm}_{\omega'}(w), \\
&F_\omega \psi^{-}_{\omega'}(w) F_\omega^{-1} = \frac{F_{\omega\omega'-\nu_3 c}}{F_{\omega\omega'}}\ \psi^{-}_{\omega'}(w), \quad z^{\beta_\omega} \psi^{-}_{\omega'}(w) z^{-\beta_\omega} = z^{\beta_{\omega\omega'}-\beta_{\omega\omega'-\nu_3 c}} \psi^{-}_{\omega'}(w), \\
&D_\omega \psi^{-}_{\omega'}(w) D_\omega^{-1} = w^{-\beta_{\omega\omega'}+\beta_{\omega\omega'-\nu_3 c}}\ \psi^{-}_{\omega'}(w),
\end{aligned} \tag{J.2.2}$$

and $\psi^+_{\omega'}(w)$ remains invariant. Defining $\xi_\omega(z)=z^{\beta_\omega}D_\omega F_\omega$, we find

$$\begin{aligned}
&\xi_\omega(z)x^\pm_{\omega'}(w)=f_{\omega\omega'}(w/z)^{\pm1}x^\pm_{\omega'}(w)\xi_\omega(z),\quad [\xi_\omega(z),\xi_{\omega'}(w)]=0,\\
&\xi_\omega(z)\psi^+_{\omega'}(w)=\psi^+_{\omega'}(w)\xi_\omega(z),\quad \xi_\omega(z)\psi^-_{\omega'}(w)=\frac{f_{\omega\ \omega'-\nu_3c}(q_3^{-c}w/z)}{f_{\omega\omega'}(w/z)}\psi^-_{\omega'}(w)\xi_\omega(z).
\end{aligned}\tag{J.2.3}$$

The operator $\xi_\omega(z)$ is used to define the twisted currents

$$x^+_\omega(z)=\tilde{x}^+_\omega(z),\quad x^-_\omega(z)=\tilde{x}^-_\omega(z)\xi_{\bar\omega}(q_3^{-1}z),\quad \psi^+_\omega(z)=\tilde\psi^+_\omega(z)\xi_{\bar\omega}(q_3^{-1}z),\quad \psi^-_\omega(z)=\tilde\psi^-_\omega(z)\xi_{\omega-\nu_3c}(q_3^{-c}z),\tag{J.2.4}$$

that satisfy the following algebraic relations,

$$\begin{aligned}
&\tilde{x}^+_\omega(z)\tilde{x}^+_{\omega'}(w)=g_{\omega\omega'}(z/w)\tilde{x}^+_{\omega'}(w)\tilde{x}^+_\omega(z),\quad \tilde{x}^-_\omega(z)\tilde{x}^-_{\omega'}(w)=\frac{f_{\omega\omega'}(w/z)}{f_{\omega'\omega}(z/w)}g_{\omega\omega'}(z/w)^{-1}\tilde{x}^-_{\omega'}(w)\tilde{x}^-_\omega(z),\\
&\tilde\psi^+_\omega(z)\tilde{x}^\pm_{\omega'}(w)=f_{\omega'\omega}(z/w)^{\pm1}g_{\omega\omega'}(z/w)^{\pm1}\tilde{x}^\pm_{\omega'}(w)\tilde\psi^+_\omega(z),\\
&\tilde\psi^-_\omega(z)\tilde{x}^+_{\omega'}(w)=f_{\omega-\nu_3c\ \omega'}(q_3^cw/z)^{-1}g_{\omega-\nu_3c\ \omega'}(q_3^{-c}z/w)\tilde{x}^+_{\omega'}(w)\tilde\psi^-_\omega(z),\\
&\tilde\psi^-_\omega(z)\tilde{x}^-_{\omega'}(w)=f_{\omega\omega'}(w/z)g_{\omega\omega'}(z/w)^{-1}\tilde{x}^-_{\omega'}(w)\tilde\psi^-_\omega(z),\\
&\tilde\psi^+_\omega(z)\tilde\psi^-_{\omega'}(w)=\frac{f_{\omega'-\nu_3c\ \omega}(q_3^cz/w)g_{\omega\omega'-\nu_3c}(q_3^cz/w)}{f_{\omega'\omega}(z/w)g_{\omega\omega'}(z/w)}\tilde\psi^-_{\omega'}(w)\tilde\psi^+_\omega(z),\\
&\tilde{x}^+_\omega(z)\tilde{x}^-_{\omega'}(w)-f_{\omega\omega'}(w/z)^{-1}\tilde{x}^-_{\omega'}(w)\tilde{x}^+_\omega(z)=\Omega\delta_{\omega,\omega'}\delta\left(\frac{z}{w}\right)\tilde\psi^+_\omega(z)-\Omega\delta_{\omega,\omega'-\nu_3c}\delta\left(\frac{q_3^cz}{w}\right)\tilde\psi^-_{\omega'}(q_3^cz)\frac{\xi_\omega(z)}{\xi_{\bar\omega'}(q_3^{c-1}z)}.
\end{aligned}\tag{J.2.5}$$

The operators $\xi_\omega(z)$ do not fully decouple from the twisted algebra as it appears in the commutation relation $[\tilde{x}^+,\tilde{x}^-]$. The exchange relations $\tilde\psi^\pm-\tilde{x}$ now have the correct behavior as $z^{\pm1}\to\infty$ to define the expansions

$$\tilde{x}^\pm_\omega(z)=\sum_{k\in\mathbb{Z}}z^{-k}\tilde{x}^\pm_{\omega,k},\quad \tilde\psi^+_\omega(z)=\tilde\psi^+_{\omega,0}\exp\left(\sum_{k>0}z^{-k}a_{\omega,k}\right),\quad \tilde\psi^-_\omega(z)=\tilde\psi^-_{\omega,0}z^{\tilde{a}_{\omega,0}}\exp\left(\sum_{k>0}z^ka_{\omega,-k}\right).\tag{J.2.6}$$

The currents $\tilde\psi^-_\omega(z)$ still conserve a zero mode dependence $\tilde{a}_{\omega,0}$ that is required to reproduce the exchange relation J.2.3 with the grading operator $\xi_\omega(z)$. From the asymptotic behavior of the algebraic relations, we deduce that this zero mode operator $\tilde{a}_{\omega,0}$ commutes with all the twisted currents $\tilde{x}^{\pm}_{\omega}$, $\tilde{\psi}^{\pm}_{\omega}$ but not with the gradings $\xi_\omega(z)$:

$$\xi_\omega(z)w^{\tilde{a}_{\omega',0}} = w^{\beta_{\omega\omega'-\nu_3 c}-\beta_{\omega\omega'}}w^{\tilde{a}_{\omega',0}}\xi_\omega(z). \tag{J.2.7}$$

Note that this operator becomes central if $\nu_3 = 0$ or $c = 0$. Expanding in powers of the spectral parameters, the exchange relations $\tilde{\psi} - \tilde{x}$ and $\tilde{\psi} - \tilde{\psi}$ given in J.2.5 provide the commutation relations between the modes,

$$\begin{aligned}
&[a_{\omega,k>0}, a_{\omega',l}] = \delta_{k+l}(q_3^{-kc}c^{(k)}_{\omega\omega'-\nu_3 c} - c^{(k)}_{\omega\omega'}), \quad [a_{\omega,k>0}, \tilde{x}^{\pm}_{\omega',l}] = \pm c^{(k)}_{\omega\omega'}\tilde{x}^{\pm}_{\omega',l+k},\\
&[a_{\omega,-k<0}, \tilde{x}^{+}_{\omega',l}] = q_3^{-kc}c^{(-k)}_{\omega\omega'+\nu_3 c}\tilde{x}^{+}_{\omega',l-k}, \quad [a_{\omega,-k<0}, \tilde{x}^{-}_{\omega',l}] = -c^{(-k)}_{\omega\omega'}\tilde{x}^{+}_{\omega',l-k},
\end{aligned} \tag{J.2.8}$$

where[1]

$$c^{(k)}_{\omega\omega'} = c^{(-k)}_{\omega'\omega} = \frac{1}{k}\sum_{i=1,2,3}(q_i^k\delta_{\omega,\omega'+\nu_i} - q_i^{-k}\delta_{\omega,\omega'-\nu_i}). \tag{J.2.10}$$

In particular, when $c \neq 0$, the modes $a_{\omega,k}$ of the Cartan currents define p Heisenberg subalgebras. This property is used to build the horizontal representation in appendix I.2. The exchange relations $\tilde{x} - \tilde{x}$ can also be written in terms of modes by projecting the following relations:

$$\begin{aligned}
&z^{\beta_{\omega\omega'}}\prod_{i=1,2,3}(w - q_i^{-1}z)^{\delta_{\omega,\omega'+\nu_i}}\tilde{x}^{+}_{\omega}(z)\tilde{x}^{+}_{\omega'}(w) = F_{\omega\omega'}w^{\beta_{\omega'\omega}}\prod_{i=1,2,3}(w - q_i z)^{\delta_{\omega\omega'-\nu_i}}\tilde{x}^{+}_{\omega'}(w)\tilde{x}^{+}_{\omega}(z),\\
&z^{\beta_{\omega'\omega}}\prod_{i=1,2,3}(w - q_i z)^{\delta_{\omega,\omega'-\nu_i}}\tilde{x}^{-}_{\omega}(z)\tilde{x}^{-}_{\omega'}(w) = F^{-1}_{\omega'\omega}w^{\beta_{\omega\omega'}}\prod_{i=1,2,3}(w - q_i^{-1}z)^{\delta_{\omega\omega'+\nu_i}}\tilde{x}^{-}_{\omega'}(w)\tilde{x}^{-}_{\omega}(z).
\end{aligned} \tag{J.2.11}$$

A priori, the commutator $[\tilde{x}^+, \tilde{x}^-]$ could also be written in terms of modes, but the expression is rather cumbersome. On the other hand, the grading operators have simple actions on the

[1] These coefficients appear in the expansions

$$[g_{\omega\omega'}(z)]_+ = f_{\omega'\omega}(z)^{-1}\exp\left(\sum_{k>0}z^{-k}c^{(k)}_{\omega\omega'}\right), \quad [g_{\omega\omega'}(z)]_- = f_{\omega\omega'}(z^{-1})\exp\left(-\sum_{k>0}z^{k}c^{(-k)}_{\omega\omega'}\right). \tag{J.2.9}$$

modes:

$$
\begin{aligned}
&[d, \tilde{x}^{\pm}_{\omega',k}] = -k\tilde{x}^{\pm}_{\omega',k}, \quad [d, a_{\omega,k}] = -ka_{\omega,k}, \quad [d, \tilde{\psi}^{\pm}_{\omega',0}] = 0, \quad [d, \tilde{a}_{\omega',0}] = 0,\\
&[\bar{d}_\omega, \tilde{x}^{\pm}_{\omega',k}] = \pm\delta_{\omega,\omega'}\tilde{x}^{\pm}_{\omega',k}, \quad [\bar{d}_\omega, a_{\omega',k}] = 0, \quad [\bar{d}_\omega, \psi^{+}_{\omega',0}] = [\bar{d}_\omega, \tilde{a}_{\omega',0}] = 0, \quad [\bar{d}_\omega, \tilde{\psi}^{-}_{\omega',0}] = (\delta_{\omega,\omega'-\nu_3 c} - \delta_{\omega,\omega'})\tilde{\psi}^{-}_{\omega',0},\\
&e^{\tilde{d}_\omega}\tilde{x}^{\pm}_{\omega',k}e^{-\tilde{d}_\omega} = \tilde{x}^{\pm}_{\omega',k\pm\delta_{\omega,\omega'}}, \quad e^{\tilde{d}_\omega}\tilde{\psi}^{+}_{\omega',0}e^{-\tilde{d}_\omega} = \tilde{\psi}^{+}_{\omega',0}, \quad e^{\tilde{d}_\omega}\tilde{\psi}^{-}_{\omega',0}e^{-\tilde{d}_\omega} = q_3^{-c\delta_{\omega,\omega'-\nu_3 c}}\tilde{\psi}^{-}_{\omega',0},\\
&[\tilde{d}_\omega, a_{\omega',k}] = 0, \quad e^{\tilde{d}_\omega} z^{\tilde{a}_{\omega',0}} e^{-\tilde{d}_\omega} = z^{\delta_{\omega,\omega'-\nu_3 c} - \delta_{\omega,\omega'}} z^{\tilde{a}_{\omega',0}}.
\end{aligned}
\tag{J.2.12}
$$

J.3 Coproduct

The Hopf algebra structure can be extended to include the grading operators, provided we define the coproduct, counit and antipode as

$$
\begin{aligned}
&\Delta(d) = d\otimes 1 + 1\otimes d, \quad \Delta(\bar{d}_\omega) = \bar{d}_\omega \otimes 1 + 1\otimes \bar{d}_{\omega-\nu_3 c_{(1)}},\\
&\Delta(\tilde{d}_\omega) = \tilde{d}_\omega \otimes 1 + 1\otimes \tilde{d}_{\omega-\nu_3 c_{(1)}} + (\log q_3)\ c\otimes \bar{d}_{\omega-\nu_3 c_{(1)}},\\
&\varepsilon(d) = \varepsilon(\bar{d}_\omega) = \varepsilon(\tilde{d}_\omega) = 0, \quad S(d) = -d, \quad S(\bar{d}_\omega) = -d_{\omega+\nu_3 c},\\
&S(\tilde{d}_\omega) = -\tilde{d}_{\omega+\nu_3 c} + (\log q_3)\ c\bar{d}_{\omega+\nu_3 c}.
\end{aligned}
\tag{J.3.1}
$$

We deduce, for the composite operators,

$$
\begin{aligned}
&\Delta(\beta_\omega) = \beta_\omega\otimes 1 + 1\otimes \beta_{\omega-\nu_3 c_{(1)}}, \quad \varepsilon(\beta_\omega) = 0, \quad S(\beta_\omega) = -\beta_{\omega+\nu_3 c},\\
&\Delta(F_\omega) = F_\omega \otimes F_{\omega-\nu_3 c_{(1)}}, \quad \varepsilon(F_\omega) = 1, \quad S(F_\omega) = F^{-1}_{\omega+\nu_3 c},\\
&\Delta(D_\omega) = D_\omega \otimes q_3^{-c_{(1)}\beta_{\omega-\nu_3 c_{(1)}}} D_{\omega-\nu_3 c_{(1)}}, \quad \varepsilon(D_\omega) = 1, \quad S(F_\omega) = q_3^{c\beta_{\omega+\nu_3 c}} D^{-1}_{\omega+\nu_3 c},
\end{aligned}
\tag{J.3.2}
$$

and, finally,

$$
\Delta(\xi_\omega(z)) = \xi_\omega(z)\otimes \xi_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}} z), \quad \varepsilon(\xi_\omega(z)) = 1, \quad S(\xi_\omega(z)) = \xi_{\omega+\nu_3 c}(q_3^c z)^{-1}. \tag{J.3.3}
$$

We can also compute the coproduct for the twisted currents,

$$\begin{aligned}
&\Delta(\tilde{x}^+_\omega(z)) = \tilde{x}^+_\omega(z)\otimes 1 + \tilde{\psi}_{\omega+\nu_3 c_{(1)}}(q_3^{c_{(1)}}z)\xi_\omega(z)\otimes\tilde{x}^+_\omega(z)\\
&\Delta(\tilde{x}^-_\omega(z)) = \xi_{\bar{\omega}}(q_3^{-1}z)^{-1}\otimes\tilde{x}_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z) + \tilde{x}^-_\omega(z)\otimes\tilde{\psi}_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z),\\
&\Delta(\tilde{\psi}^+_\omega(z)) = \tilde{\psi}^+_\omega(z)\otimes\tilde{\psi}^+_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z),\\
&\Delta(\tilde{\psi}^-_\omega(z)) = \tilde{\psi}^-_{\omega-\nu_3 c_{(2)}}(q_3^{-c_{(2)}}z)\otimes\tilde{\psi}^-_{\omega-\nu_3 c_{(1)}}(q_3^{-c_{(1)}}z)\xi_{\omega-\nu_3 c_{(1)}-\nu_3 c_{(2)}}(q_3^{-c_{(1)}-c_{(2)}}z)\xi_{\omega-2\nu_3 c_{(1)}-\nu_3 c_{(2)}}(q_3^{-2c_{(1)}-c_{(2)}}z)^{-1},
\end{aligned} \tag{J.3.4}$$

and deduce

$$\begin{aligned}
&\Delta(a_{\omega,k>0}) = a_{\omega,k}\otimes 1 + q_3^{ck}\otimes a_{\omega-\nu_3 c_{(1)},k},\quad \Delta(a_{\omega,-k<0}) = a_{\omega-\nu_3 c_{(2)},-k}\otimes q_3^{-kc} + q_3^{-kc}\otimes a_{\omega-\nu_3 c_{(1)},-k},\\
&\Delta(a_{\omega,0}) = a_{\omega-\nu_3 c_{(2)},0}\otimes 1 + 1\otimes a_{\omega-\nu_3 c_{(1)},0} + 1\otimes\left(\beta_{\omega-\nu_3 c_{(1)}-\nu_3 c_{(2)}} - \beta_{\omega-2\nu_3 c_{(1)}-\nu_3 c_{(2)}}\right),
\end{aligned} \tag{J.3.5}$$

together with the coproduct of the zero modes $\psi^\pm_{\omega,0}$.

J.4 Vertical representation

In the vertical representation, the grading operators $\bar{d}_\omega$ and $\tilde{d}_\omega$ commute with the currents $\psi^\pm_\omega(z)$, therefore they are diagonal in the basis $|\boldsymbol{\lambda}\rangle\rangle$. Their eigenvalues can be determined recursively using the relations with the currents $x^\pm(z)$,

$$\rho_{\boldsymbol{v}}^{(0,m)}(\bar{d}_\omega)\,|\boldsymbol{\lambda}\rangle\rangle = |K_\omega(\boldsymbol{\lambda})|\,|\boldsymbol{\lambda}\rangle\rangle,\quad \rho_{\boldsymbol{v}}^{(0,m)}(\tilde{d}_\omega)\,|\boldsymbol{\lambda}\rangle\rangle = \left(\sum_{\square\in\boldsymbol{\lambda}}\delta_{\omega,c(\square)}\log\chi_\square\right)|\boldsymbol{\lambda}\rangle\rangle, \tag{J.4.1}$$

where the eigenvalues on the vacuum have been chosen to be zero. Then, the representation of $\xi_\omega(z)$ takes the simple form

$$\rho^{(V)}(\xi_\omega(z))\,|\boldsymbol{\lambda}\rangle\rangle = \mathring{f}^{[\boldsymbol{\lambda}]}_\omega(z)\,|\boldsymbol{\lambda}\rangle\rangle, \tag{J.4.2}$$

with the function $\mathring{f}^{[\boldsymbol{\lambda}]}_{\omega}(z)$ defined in 6.3.13. We find the representation of the twisted currents to be

$$\begin{aligned}
&\rho^{(V)}(\tilde{x}^+_\omega(z))\,|\boldsymbol{\lambda}\rangle\!\rangle = F^{1/2}\sum_{\square\in A_\omega(\boldsymbol{\lambda})}\delta(z/\chi_\square)\operatorname*{Res}_{z=\chi_\square} z^{-1}\mathcal{Y}^{[\boldsymbol{\lambda}]}_\omega(z)^{-1}\,|\boldsymbol{\lambda}+\square\rangle\!\rangle,\\
&\rho^{(V)}(\tilde{x}^-_\omega(z))\,|\boldsymbol{\lambda}\rangle\!\rangle = \sum_{\square\in R_\omega(\boldsymbol{\lambda})}\delta(z/\chi_\square)\operatorname*{Res}_{z=\chi_\square} z^{-1}\mathcal{Y}^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)\,|\boldsymbol{\lambda}-\square\rangle\!\rangle,\\
&\rho^{(V)}(\tilde{\psi}^+_\omega(z))\,|\boldsymbol{\lambda}\rangle\!\rangle = \mathring{f}^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1}\left[\Psi^{[\boldsymbol{\lambda}]}_\omega(z)\right]_+|\boldsymbol{\lambda}\rangle\!\rangle,\quad \rho^{(V)}(\tilde{\psi}^+_{\omega,0})=1,\\
&\rho^{(V)}(\tilde{\psi}^-_\omega(z))\,|\boldsymbol{\lambda}\rangle\!\rangle = \mathring{f}^{[\boldsymbol{\lambda}]}_{\omega}(z)^{-1}\left[\Psi^{[\boldsymbol{\lambda}]}_\omega(z)\right]_-|\boldsymbol{\lambda}\rangle\!\rangle,\quad \rho^{(V)}(\tilde{\psi}^+_{\omega,0})=\frac{\prod_{\alpha\in C_{\bar\omega}(m)}(-q_3v_\alpha)}{\prod_{\alpha\in C_\omega(m)}(-v_\alpha)},\\
&\rho^{(V)}(a_{\omega,k})\,|\boldsymbol{\lambda}\rangle\!\rangle = \left(\sum_{\square\in\boldsymbol{\lambda}}c^{(k)}_{\omega c(\square)}\chi_\square^k\right)|\boldsymbol{\lambda}\rangle\!\rangle,\quad \rho^{(V)}(\tilde{a}_{\omega,0})=m_\omega-m_{\bar\omega}.
\end{aligned}\tag{J.4.3}$$

J.5 Horizontal representation

Computing the exchange relations of the operators $e^{\alpha p_\omega}$ and $e^{\alpha\tilde{p}_\omega}$ with the Drinfeld currents leads to the identification of the representation for grading operators

$$\rho^{(H)}(\bar{d}_\omega)=p_\omega,\quad \rho^{(H)}(\tilde{d}_\omega)=\tilde{p}_\omega\quad\Rightarrow\quad \rho^{(H)}(\xi_\omega(z))=P_\omega(z).\tag{J.5.1}$$

Thus, we find the representation for the twisted currents,

$$\begin{aligned}
&\rho^{(H)}(\tilde{x}^+_\omega(z))=u_\omega z^{-n_\omega}Q_\omega(z)\eta^+_\omega(z),\quad \rho^{(1,n)}_u(\tilde{x}^-_\omega(z))=u_\omega^{-1}z^{n_\omega}Q_\omega(z)^{-1}\eta^-_\omega(z),\\
&\rho^{(H)}(\tilde{\psi}^+_\omega(z))=F^{-1/2}\varphi^+_\omega(z),\\
&\rho^{(H)}(\tilde{\psi}^-_\omega(z))=F^{1/2}\frac{u_{\bar\omega}}{u_\omega}q_3^{n_{\bar\omega}}z^{n_\omega-n_{\bar\omega}}\frac{Q_{\bar\omega}(q_3^{-1}z)}{Q_\omega(z)}\varphi^-_\omega(z),
\end{aligned}\tag{J.5.2}$$

and the modes

$$\begin{aligned}
&\rho^{(H)}(a_{\omega,k>0})=-\frac{1}{k}(q_3^{-k/2}\alpha_{\omega,k}-q_3^{k/2}\alpha_{\bar\omega,k}),\quad \rho^{(H)}(a_{\omega,-k<0})=-\frac{1}{k}(q_3^{-k}\alpha_{\bar\omega,-k}-\alpha_{\omega,-k}),\\
&\rho^{(H)}(\tilde{\psi}^+_{\omega,0})=F^{-1/2},\quad \rho^{(H)}(\tilde{\psi}^-_{\omega,0})=F^{1/2}\frac{u_{\bar\omega}}{u_\omega}q_3^{n_{\bar\omega}-\tilde{q}_{\bar\omega}}e^{q_{\bar\omega}-q_\omega},\quad \rho^{(H)}(\tilde{a}_{\omega,0})=n_\omega-n_{\bar\omega}+\tilde{q}_{\bar\omega}-\tilde{q}_\omega.
\end{aligned}\tag{J.5.3}$$

We can verify that $\tilde{a}_{\omega,0}$ commutes with the twisted currents, and satisfies the relation J.2.7 with the grading operator $\xi_\omega(z)$.

Appendix K

Derivation of the vertex operators

K.1 Definition of the vacuum components

Before sketching the derivation of the solution for the intertwining relations, we would like to provide a bold argument for the definition of the vacuum components $\Phi_\varnothing$ and $\Phi^*_\varnothing$ entering in the definition 6.4.4 of the intertwiners. In fact, the full partition function of the gauge theory, including classical, one-loop and instantons contributions, has a nice description in terms of the melting crystal picture [7, 206]. Indeed, the one-loop contribution can be written as a double product over the boxes of completely filled (infinite) Young diagrams $\boldsymbol{\lambda}^\infty = \{(\alpha, i, j) \diagup \alpha = 1\cdots m, i = 1\cdots\infty, j = 1\cdots\infty\}$, assuming a ζ_2-regularization for the infinite product. Then, the instanton correction of order $O(\mathfrak{q}^k)$ is obtained by removing k boxes to $\boldsymbol{\lambda}^\infty$, taking the double product over $\boldsymbol{\lambda}^{\boldsymbol{c}} = \boldsymbol{\lambda}^\infty \setminus \boldsymbol{\lambda} = \{(\alpha, i, j) \diagup \alpha = 1\cdots m, i = \lambda_j^{(\alpha)} + 1\cdots\infty, j = 1\cdots\infty\}$ and summing over the configurations $\boldsymbol{\lambda}$ of $k = |\boldsymbol{\lambda}|$ boxes. The vacuum component $\Phi_\varnothing$ of the intertwiner Φ is associated to this infinite product over boxes in $\boldsymbol{\lambda}^\infty$, so that formally

$$\Phi_\varnothing \simeq : \prod_{\square\in\boldsymbol{\lambda}^\infty} \eta^+_{c(\square)}(\chi_\square)^{-1} :, \quad \Phi_{\boldsymbol{\lambda}} \simeq t_{\boldsymbol{\lambda}} : \prod_{\square\in\boldsymbol{\lambda}^{\boldsymbol{c}}} \eta^+_{c(\square)}(\chi_\square)^{-1} :, \tag{K.1.1}$$

and similarly for $\Phi^*_{\boldsymbol{\lambda}}$, replacing $\eta^+_{c(\square)}(\chi_\square)$ with $\eta^-_{c(\square)+\nu_3}(q_3\chi_\square)$.

In order to develop this idea, we may introduce a very crude cut-off N such that $\boldsymbol{\lambda}^\infty$ is obtained as the limit $N\to\infty$ of m Young diagrams consisting of squares of size $(pN)\times(pN)$, i.e. $\boldsymbol{\lambda}_N=\{(\alpha,i,j)/\alpha=1\cdots m, i=1\cdots pN, j=1\cdots pN\}$. Then, we may consider the product over boxes $(\alpha,i,j)\in\boldsymbol{\lambda}_N$ and decompose the indices (i,j) as $i=\bar{i}+1+k_ip$, $j=\bar{j}+1+k_jp$ with $\bar{i},\bar{j}=0\cdots p-1$ and $k_i,k_j=0\cdots N-1$. We end up with

$$\begin{aligned}:\prod_{\square\in\boldsymbol{\lambda}_N}\eta^+_{c(\square)}(\chi_\square)^{-1}:=:\prod_{\alpha=1}^{m}\prod_{\bar{i},\bar{j}=0}^{p-1}&\exp\left(-\sum_{k>0}\frac{(v_\alpha q_1^{\bar{i}}q_2^{\bar{j}})^k}{k}\sum_{k_i,k_j=0}^{N-1}q_1^{pk_ik}q_2^{pk_jk}\alpha_{c_\alpha+\bar{i}\nu_1+\bar{j}\nu_2,-k}\right)\\ &\times\exp\left(\sum_{k>0}\frac{(v_\alpha q_1^{\bar{i}}q_2^{\bar{j}}\gamma)^{-k}}{k}\sum_{k_i,k_j=0}^{N-1}q_1^{-pk_ik}q_2^{-pk_jk}\alpha_{c_\alpha+\bar{i}\nu_1+\bar{j}\nu_2,k}\right):,\end{aligned}\tag{K.1.2}$$

Performing the sum over k_i and k_j, we find

$$\begin{aligned}:\prod_{\square\in\boldsymbol{\lambda}_N}\eta^+_{c(\square)}(\chi_\square)^{-1}:=:\prod_{\alpha=1}^{m}\prod_{\bar{i},\bar{j}=0}^{p-1}&\exp\left(-\sum_{k>0}\frac{(v_\alpha q_1^{\bar{i}}q_2^{\bar{j}})^k}{k}\frac{1-q_1^{pkN}}{1-q_1^{pk}}\frac{1-q_2^{pkN}}{1-q_2^{pk}}\alpha_{c_\alpha+\bar{i}\nu_1+\bar{j}\nu_2,-k}\right)\\ &\times\exp\left(\sum_{k>0}\frac{(v_\alpha q_1^{\bar{i}}q_2^{\bar{j}}\gamma)^{-k}}{k}\frac{1-q_1^{-pkN}}{1-q_1^{-pk}}\frac{1-q_2^{-pkN}}{1-q_2^{-pk}}\alpha_{c_\alpha+\bar{i}\nu_1+\bar{j}\nu_2,k}\right):,\end{aligned}\tag{K.1.3}$$

At this stage, the limit $N\to\infty$ is ill-defined because the first exponential converges when $|q_1|,|q_2|<1$ while the second exponential for $|q_1|,|q_2|>1$. However, we notice that each color can be treated independently, and their contribution written in terms of the vacuum component for the intertwiner describing instantons on a Ω-background with no orbifold [141, 51], with the replacement $\varepsilon_1,\varepsilon_2\to p\varepsilon_1,p\varepsilon_2$. Thus, we can borrow the corresponding operator and simply define

$$\Phi_\varnothing=:\prod_{\alpha=1}^{m}\prod_{\bar{i},\bar{j}=1}^{p-1}\exp\left(-\sum_{k>0}\frac{(v_\alpha q_1^{\bar{i}}q_2^{\bar{j}})^k}{k(1-q_1^{pk})(1-q_2^{pk})}\alpha_{c_\alpha+\bar{i}\nu_1+\bar{j}\nu_2,-k}\right)\exp\left(\sum_{k>0}\frac{(v_\alpha q_1^{\bar{i}}q_2^{\bar{j}}\gamma)^{-k}}{k(1-q_1^{-pk})(1-q_2^{-pk})}\alpha_{c_\alpha+\bar{i}\nu_1+\bar{j}\nu_2,k}\right):\tag{K.1.4}$$

The appearance of quantities defined on the background $\mathbb{C}_{p\varepsilon_1}\times\mathbb{C}_{p\varepsilon_2}\times S^1_R$ is reminiscent of the surface defect interpretation of the orbifold developed in [21, 1]. It may also be related to the *abelianization* procedure described in the case of $\mathfrak{gl}(p)$ (unrefined, i.e. $q_3=1$) in [154].

Using the definition K.1.4, we obtain the following normal-ordering relations[1]

$$\begin{aligned}
\eta^+_\omega(z)\Phi_\varnothing &= \prod_{\alpha\in C_\omega(m)}(1-v_\alpha/z)^{-1} :\eta^+_\omega(z)\Phi_\varnothing:, \quad \Phi_\varnothing\eta^+_\omega(z)=\prod_{\alpha\in C_{\bar\omega}(m)}(1-z/(q_3v_\alpha))^{-1} :\eta^+_\omega(z)\Phi_\varnothing:,\\
\eta^-_\omega(z)\Phi_\varnothing &= \prod_{\alpha\in C_{\bar\omega}(m)}(1-q_3v_\alpha/z) :\eta^-_\omega(z)\Phi_\varnothing:, \quad \Phi_\varnothing\eta^-_\omega(z)=\prod_{\alpha\in C_{\bar\omega}(m)}(1-z/(q_3v_\alpha)) :\eta^-_\omega(z)\Phi_\varnothing: .
\end{aligned}\tag{K.1.6}$$

Since $\varphi^\pm_\omega(z)$ can be expressed in terms of $\eta^\pm_\omega(z)$, we easily deduce the normal-ordering relations for these vertex operators as well. This argument can also be applied to $\Phi^*_\varnothing$, it leads to define

$$\begin{aligned}
\Phi^*_\varnothing =: \prod_{\alpha=1}^{m}\prod_{\bar i,\bar j=0}^{p-1} &\exp\left(\sum_{k>0}\frac{(v_\alpha q_1^{\bar i}q_2^{\bar j}q_3)^k}{k(1-q_1^{pk})(1-q_2^{pk})}\alpha_{c_\alpha+(\bar i-1)\nu_1+(\bar j-1)\nu_2,-k}\right)\\
&\times\exp\left(-\sum_{k>0}\frac{(v_\alpha q_1^{\bar i}q_2^{\bar j}\gamma)^{-k}}{k(1-q_1^{-pk})(1-q_2^{-pk})}\alpha_{c_\alpha+\bar i\nu_1+\bar j\nu_2,k}\right):,
\end{aligned}\tag{K.1.7}$$

and we obtain

$$\begin{aligned}
\eta^+_\omega(z)\Phi^*_\varnothing &= \prod_{\alpha\in C_{\bar\omega}(m)}(1-q_3v_\alpha/z) :\eta^+_\omega(z)\Phi^*_\varnothing:, \quad \Phi^*_\varnothing\eta^+_\omega(z)=\prod_{\alpha\in C_{\bar\omega}(m)}(1-z/(q_3v_\alpha)) :\eta^+_\omega(z)\Phi^*_\varnothing:,\\
\eta^-_\omega(z)\Phi^*_\varnothing &= \prod_{\alpha\in C_{\omega+2\nu_1+2\nu_2}(m)}(1-q_3^2v_\alpha/z)^{-1} :\eta^-_\omega(z)\Phi^*_\varnothing:, \quad \Phi^*_\varnothing\eta^-_\omega(z)=\prod_{\alpha\in C_{\bar\omega}(m)}(1-z/(q_3v_\alpha))^{-1} :\eta^-_\omega(z)\Phi^*_\varnothing: .
\end{aligned}\tag{K.1.8}$$

[1] We have used the following property to perform the sum over indices $\bar i,\bar j$:

$$\frac{1}{k}\sum_{\bar i,\bar j=0}^{p-1}(q_1^{\bar i}q_2^{\bar j})^k q_3^{-k/2}\sigma^{(k)}_{\omega,c_\alpha+\bar i\nu_1+\bar j\nu_2}=(1-q_1^{pk})(1-q_2^{pk})\delta_{\omega,c_\alpha}.\tag{K.1.5}$$

We can also compute

$$\begin{aligned}&\Phi_\varnothing\Phi'_\varnothing=\mathcal{G}(\boldsymbol{v}'|\boldsymbol{v})^{-1}:\Phi_\varnothing\Phi'_\varnothing:,\quad \Phi^*_\varnothing\Phi^{*\prime}_\varnothing=\mathcal{G}(\boldsymbol{v}'|q_3^{-1}\boldsymbol{v})^{-1}:\Phi^*_\varnothing\Phi^{*\prime}_\varnothing:,\\&\Phi_\varnothing\Phi^{*\prime}_\varnothing=\mathcal{G}(\boldsymbol{v}'|q_3^{-1}\boldsymbol{v}):\Phi_\varnothing\Phi^{*\prime}_\varnothing:,\quad \Phi^*_\varnothing\Phi'_\varnothing=\mathcal{G}(\boldsymbol{v}'|\boldsymbol{v}):\Phi^*_\varnothing\Phi'_\varnothing:,\end{aligned}\tag{K.1.9}$$

where we $\mathcal{G}(\boldsymbol{v}|\boldsymbol{v}')$ denotes the bifundamental contribution at one-loop expressed in terms of the function $\mathcal{G}_{q_1,q_2}(z)$,[2]

$$\mathcal{G}(\boldsymbol{v}|\boldsymbol{v}')=\prod_{\alpha=1}^m\prod_{\alpha'=1}^{m'}\prod_{\bar{i},\bar{j}=0}^{p-1}\mathcal{G}_{q_1^p,q_2^p}(v_\alpha q_1^{\bar{i}+1}q_2^{\bar{j}+1}/v'_{\alpha'})^{-\delta_{c'_{\alpha'},c_\alpha+(\bar{i}+1)\nu_1+(\bar{j}+1)\nu_2}},\quad \mathcal{G}_{q_1,q_2}(z)=\exp\left(-\sum_{k=1}^\infty\frac{1}{k}\frac{z^k}{(1-q_1^k)(1-q_2^k)}\right)\tag{K.1.11}$$

K.2 Solution of intertwining relations

Once projected on the vertical states using the decomposition 6.4.3, the intertwining relations 6.4.1 write

$$\begin{aligned}&x^+_\omega(z)\Phi_{\boldsymbol{\lambda}}=\Psi^{[\boldsymbol{\lambda}]}_\omega(z)\Phi_{\boldsymbol{\lambda}}x^+_\omega(z)+\rho^{(V)}(x^+_\omega(z))\cdot\Phi_{\boldsymbol{\lambda}},\\&x^-_\omega(z)\Phi_{\boldsymbol{\lambda}}=\Phi_{\boldsymbol{\lambda}}x^-_\omega(z)+\left[\rho^{(V)}(x^-_\omega(z))\cdot\Phi_{\boldsymbol{\lambda}}\right]\psi^+_\omega(z),\\&\psi^+_\omega(z)\Phi_{\boldsymbol{\lambda}}=\Psi^{[\boldsymbol{\lambda}]}_\omega(z)\Phi_{\boldsymbol{\lambda}}\psi^+_\omega(z),\quad \psi^-_\omega(z)\Phi_{\boldsymbol{\lambda}}(z)=\Psi^{[\boldsymbol{\lambda}]}_{\bar{\omega}}(q_3^{-1}z)\Phi_{\boldsymbol{\lambda}}\psi^-_\omega(z),\end{aligned}\tag{K.2.1}$$

and

$$\begin{aligned}&x^+_\omega(z)\Phi^*_{\boldsymbol{\lambda}}=\Phi^*_{\boldsymbol{\lambda}}x^+_\omega(z)-\psi^-_{\omega-\nu_1-\nu_2}(q_3z)\left[\rho^{(V)*}(x^+_\omega(z))\cdot\Phi^*_{\boldsymbol{\lambda}}\right],\\&\Psi^{[\boldsymbol{\lambda}]}_{\bar{\omega}}(q_3^{-1}z)x^-_\omega(z)\Phi^*_{\boldsymbol{\lambda}}=\Phi^*_{\boldsymbol{\lambda}}x^-_\omega(z)-\rho^{(V)*}(x^-_{\bar{\omega}}(q_3^{-1}z))\cdot\Phi^*_{\boldsymbol{\lambda}},\\&\psi^+_\omega(z)\Phi^*_{\boldsymbol{\lambda}}=\Psi^{[\boldsymbol{\lambda}]}_{\bar{\omega}}(q_3^{-1}z)^{-1}\Phi^*_{\boldsymbol{\lambda}}\psi^+_\omega(z),\quad \psi^-_\omega(z)\Phi^*_{\boldsymbol{\lambda}}(z)=\Psi^{[\boldsymbol{\lambda}]}_{\bar{\omega}}(q_3^{-1}z)^{-1}\Phi^*_{\boldsymbol{\lambda}}\psi^-_\omega(z).\end{aligned}\tag{K.2.2}$$

[2] Note that when the weights are shifted as $\boldsymbol{v}\to q_3\boldsymbol{v}$, we have to shift the colors $c_\alpha\to c_\alpha+\nu_3$ accordingly. For instance,

$$\mathcal{G}(\boldsymbol{v}|q_3^{-1}\boldsymbol{v}')=\prod_{\alpha=1}^m\prod_{\alpha'=1}^{m'}\prod_{\bar{i},\bar{j}=0}^{p-1}\mathcal{G}_{q_1^p,q_2^p}(v'_{\alpha'}q_1^{\bar{i}+1}q_2^{\bar{j}+1}/v_\alpha)^{\delta_{c_\alpha,c'_{\alpha'}+(\bar{i}+1)\nu_1+(\bar{j}+1)\nu_2}}.\tag{K.1.10}$$

To lighten the notations, we have omitted the horizontal representations $\rho^{(H)}$ and $\rho^{(H')}$ and indicated the vertical action with a central dot. In order to show that the operators $\Phi_{\boldsymbol{\lambda}}$ and $\Phi^*_{\boldsymbol{\lambda}}$ defined in 6.4.4 satisfy these relations, we need to compute the factors coming from the normal ordering of products with the Drinfeld currents in the horizontal representation. It is easier to treat separately the vertex operators part,

$$\begin{aligned}
&\eta^+_\omega(z)\Phi_{\boldsymbol{\lambda}} = \mathcal{Y}^{[\boldsymbol{\lambda}]}_\omega(z)^{-1} : \eta^+_\omega(z)\Phi_{\boldsymbol{\lambda}} :, \quad \Phi_{\boldsymbol{\lambda}}\eta^+_\omega(z) = f^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1}\mathcal{Y}^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1} : \eta^+_\omega(z)\Phi_{\boldsymbol{\lambda}} :,\\
&\eta^-_\omega(z)\Phi_{\boldsymbol{\lambda}} = \mathcal{Y}_{\bar\omega}(q_3^{-1}z) : \eta^-_\omega(z)\Phi_{\boldsymbol{\lambda}} :, \quad \Phi_{\boldsymbol{\lambda}}\eta^-_\omega(z) = f^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)\mathcal{Y}^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z) : \eta^-_\omega(z)\Phi_{\boldsymbol{\lambda}} :,\\
&\varphi^+_\omega(z)\Phi_{\boldsymbol{\lambda}} = \mathring{f}^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1}\Psi^{[\boldsymbol{\lambda}]}_\omega(z) : \varphi^+_\omega(z)\Phi_{\boldsymbol{\lambda}} :, \quad \Phi_{\boldsymbol{\lambda}}\varphi^+_\omega(z) =: \varphi^+_\omega(z)\Phi_{\boldsymbol{\lambda}} :,\\
&\varphi^-_\omega(z)\Phi_{\boldsymbol{\lambda}} =: \varphi^-_\omega(z)\Phi_{\boldsymbol{\lambda}} :, \quad \Phi_{\boldsymbol{\lambda}}\varphi^-_\omega(z) = f^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)\frac{\mathring{f}_{\omega+2\nu_1+2\nu_2}(q_3^{-2}z)}{f_{\omega+2\nu_1+2\nu_2}(q_3^{-2}z)}\Psi^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1} : \varphi^-_\omega(z)\Phi_{\boldsymbol{\lambda}} :,
\end{aligned}\tag{K.2.3}$$

$$\begin{aligned}
&\eta^+_\omega(z)\Phi^*_{\boldsymbol{\lambda}} = \mathcal{Y}^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z) : \eta^+_\omega(z)\Phi^*_{\boldsymbol{\lambda}} :, \quad \Phi^*_{\boldsymbol{\lambda}}\eta^+_\omega(z) = f^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)\mathcal{Y}_{\bar\omega}(q_3^{-1}z) : \eta^+_\omega(z)\Phi^*_{\boldsymbol{\lambda}} :,\\
&\eta^-_\omega(z)\Phi^*_{\boldsymbol{\lambda}} = \mathcal{Y}^{[\boldsymbol{\lambda}]}_{\omega+2\nu_1+2\nu_2}(q_3^{-2}z)^{-1} : \eta^-_\omega(z)\Phi^*_{\boldsymbol{\lambda}} :, \quad \Phi^*_{\boldsymbol{\lambda}}\eta^-_\omega(z) = f^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1}\mathcal{Y}_{\bar\omega}(q_3^{-1}z)^{-1} : \eta^-_\omega(z)\Phi^*_{\boldsymbol{\lambda}} :,\\
&\varphi^+_\omega(z)\Phi^*_{\boldsymbol{\lambda}} = \mathring{f}^{[\boldsymbol{\lambda}]}_{\omega+2\nu_1+2\nu_2}(q_3^{-2}z)\Psi^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1} : \varphi^+_\omega(z)\Phi^*_{\boldsymbol{\lambda}} :, \quad \Phi^*_{\boldsymbol{\lambda}}\varphi^+_\omega(z) =: \varphi^+_\omega(z)\Phi^*_{\boldsymbol{\lambda}} :,\\
&\varphi^-_\omega(z)\Phi^*_{\boldsymbol{\lambda}} =: \varphi^-_\omega(z)\Phi^*_{\boldsymbol{\lambda}} :, \quad \Phi^*_{\boldsymbol{\lambda}}\varphi^-_\omega(z) = f^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1}\frac{f^{[\boldsymbol{\lambda}]}_{\omega+2\nu_1+2\nu_2}(q_3^{-2}z)}{\mathring{f}^{[\boldsymbol{\lambda}]}_{\omega+2\nu_1+2\nu_2}(q_3^{-2}z)}\Psi^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z) : \varphi^-_\omega(z)\Phi^*_{\boldsymbol{\lambda}} :,
\end{aligned}\tag{K.2.4}$$

and the zero-modes part,

$$\begin{aligned}
&P_\omega(z)t_{\boldsymbol{\lambda}} = \mathring{f}^{[\boldsymbol{\lambda}]}_\omega(z) : P_\omega(z)t_{\boldsymbol{\lambda}} :, \quad [Q_\omega(z), t_{\boldsymbol{\lambda}}] = 0,\\
&P_\omega(z)t^*_{\boldsymbol{\lambda}} = \mathring{f}^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1} : P_\omega(z)t^*_{\boldsymbol{\lambda}} :, \quad t^*_{\boldsymbol{\lambda}}Q_\omega(z) = \mathring{f}^{[\boldsymbol{\lambda}]}_{\bar\omega}(q_3^{-1}z)^{-1} : t^*_{\boldsymbol{\lambda}}Q_\omega(z) : .
\end{aligned}\tag{K.2.5}$$

From these relations, we to deduce the normal ordering relations for the currents $x^\pm_\omega$ and $\psi^\pm_\omega$. Then, the relation K.2.1 and K.2.2 for the Cartan currents $\psi^\pm_\omega$ follow directly, provided that the weights and levels satisfy the relation 6.4.2. This condition is related to the difference between the functions $f_\omega^{[\boldsymbol{\lambda}]}$ and $\mathring{f}_\omega^{[\boldsymbol{\lambda}]}$,

$$\frac{u'_\omega z^{-n'_\omega}}{u_\omega z^{-n_\omega}} = \frac{\mathring{f}_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)}{f_{\bar{\omega}}^{[\boldsymbol{\lambda}]}(q_3^{-1}z)}. \tag{K.2.6}$$

The relations K.2.1 and K.2.2 involving the currents $x_\omega^\pm$ are harder to prove. This is done by decomposition of the functions $\mathcal{Y}_\omega^{[\boldsymbol{\lambda}]}(z)$ as sum over poles. We refer the reader to [51] for a more detailed explanation.

Appendix L

Spinor conventions

The spinor indices in ψ_α and $\tilde{\psi}^{\dot{\alpha}}$ are raised and lowered by

$$\begin{aligned} &\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \tilde{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\psi}^{\dot{\beta}} \\ &\psi_\beta = -\psi^\alpha\epsilon_{\alpha\beta}, \quad \tilde{\psi}^{\dot{\beta}} = -\tilde{\psi}_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}, \end{aligned} \tag{L.0.1}$$

where $\epsilon^{12} = -\epsilon_{12} = \epsilon^{\dot{1}\dot{2}} = -\epsilon_{\dot{1}\dot{2}} = 1$. We use the convention for the spinor index contraction

$$\psi\chi = \psi^\alpha\chi_\alpha, \quad \tilde{\psi}\tilde{\chi} = \tilde{\psi}_{\dot{\alpha}}\tilde{\chi}^{\dot{\alpha}}. \tag{L.0.2}$$

The symplectic-Majorana spinors ψ_A and $\tilde{\psi}_A$ are defined by

$$(\psi_{\alpha A})^\dagger = \epsilon^{AB}\epsilon^{\alpha\beta}\psi_{\beta B}, \quad (\tilde{\psi}_{\dot{\alpha}A})^\dagger = \epsilon^{AB}\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\psi}_{\dot{\beta}B}, \tag{L.0.3}$$

where the $SU(2)_R$ indices are raised and lowered as $X^A = \epsilon^{AB}X_B$ and $X_A = \epsilon_{AB}X^B$ with $\epsilon^{12} = -\epsilon_{12} = 1$.

The σ-matrices are defined by

$$\sigma^a_{\alpha\dot{\alpha}} = (i\vec{\tau}, \mathbb{1})_{\alpha\dot{\alpha}}, \quad \tilde{\sigma}^{a\dot{\alpha}\alpha} = (-i\vec{\tau}, \mathbb{1})^{\dot{\alpha}\alpha}, \tag{L.0.4}$$

where $\vec{\tau}$ are the Pauli matrices.

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