About these notes: these notes are based on the US Particle Accelerator School (USPAS) series on plasma accelerators, which were taught by Alec Thomas, Warren Mori, and myself in winter of 2019.

Preamble: relativistic electrodynamics:

A prerequisite for this class is being well-versed in electrodynamics and relativity. In particular, you are expected to know Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

as well as the fundamental relations in special relativity

proper time: $d\tau = \frac{dt}{\gamma}$

$$\left.\begin{aligned} E &= \gamma m c^2 \\ \vec{p} &= \gamma m \vec{v} \end{aligned}\right\} \Rightarrow E^2 = \underbrace{p^2c^2}_{\text{kinetic energy}} + \underbrace{(mc^2)^2}_{\text{rest mass}}$$

$$\gamma = \left(1-\left(\frac{v}{c}\right)^2\right)^{-1/2} = \left(1-\beta^2\right)^{-1/2}$$

Equation of motion:

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v}\times\vec{B} + \vec{F}_{other}$$

this is not emphasized in this class, but includes effects such as the particle's radiation

Potentials ϕ & $\vec{A}$ defined through

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

With a gauge condition. The two common gauges are

Coulomb Gauge: $\vec{\nabla}\cdot\vec{A} = 0$

Lorentz Gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$

<u>Strength parameters for single-particle motion in laser and beam fields</u>

We start with the equation of motion for a particle in the fields of a laser or particle beam, ignoring force terms due to radiation pressure, etc:

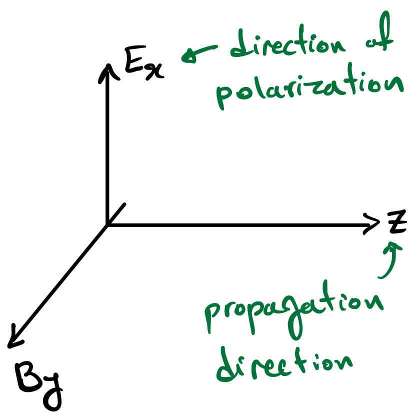
$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B})$$

Consider two <u>idealized</u> situations

1. Linearly polarized laser field of an infinite plane wave
2. An infinite cylinder of charge moving at $v_b \approx c$ at radius r_0 with uniform density of ρ_0

These two cases are selected because they have simple solutions

Case 1: linearly polarized laser field



$$\vec{E} = \hat{x}\, E_0 \sin(k_0 z - \omega_0 t)$$

$$\vec{B} = \hat{y}\, \frac{E_0}{c} \sin(k_0 z - \omega_0 t)$$

$k_0 = \frac{2\pi}{\lambda_0}$: wave number , λ_0: wavelength

$\omega_0 = c k_0$

It is often convinient to express the wave quantities such as fields in terms of the vector potential:

$$\vec{\nabla}\times\vec{A} = \vec{B} \quad \ldots (1)$$

$$-\vec{\nabla}\times\vec{E} = \frac{\partial B}{\partial t} = \frac{\partial}{\partial t}\left[\vec{\nabla}\times\vec{A}\right]$$

$$\Rightarrow \quad \vec{\nabla}\times\left[\vec{E}+\frac{\partial\vec{A}}{\partial t}\right]=0 \Rightarrow \vec{E}=-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\underbrace{\phi}_{\text{constant of integration}}$$

constant of integration since $\vec{\nabla}\times(\vec{\nabla}[\,])=0$

However, for a wave in vacuum, the scalar potential is zero, since there are no source terms ($\rho=0$).

$$\therefore \quad \vec{E}=-\frac{\partial\vec{A}}{\partial t} \Rightarrow \vec{A}=-\int\vec{E}\,dt$$

$$\vec{A}=-\underbrace{\frac{E_0}{\omega_0}}_{\equiv A_0}\cos(k_0 z-\omega_0 t)\,\hat{x}$$

$\equiv A_0$, amplitude of vector potential

One can check that this solution satisfies $\underline{1}$ as well.

We can derive constants of motion here from the equations of motion for a relativistic particle:

$$\frac{d\vec{p}}{dt}=q(\vec{E}+\vec{v}\times\vec{B})$$

$$\frac{dP_z}{dt}=q\,V_x B_y \quad \cdots (4)$$

$$\frac{dP_x}{dt}=q\left[E_x-V_z B_y\right] \quad \cdots (5)$$

$$\frac{d}{dt}\left[\gamma m c^2\right]=q\,V_x E_x \cdots (6) \qquad \text{(Conservation of energy)}$$

Note: $B_y=E_x/c$

Subtract (4) from (6)/c

$$\frac{d}{dt}\left[\gamma m c-P_z\right]=q\,V_x\left[\frac{E_x}{c}-B_y\right]=0$$

$\therefore$ $\boxed{\gamma mc - P_z}$ is a constant of motion

Next, write E_x & B_y in terms of A_x in (5):

$$\frac{d}{dt} P_x = q\left[-\frac{\partial A_x}{\partial t} - V_z \frac{\partial A_x}{\partial z}\right] = -q\left[\frac{\partial A_x}{\partial t} + (\vec{V}\cdot\vec{\nabla}) A_x\right]$$

$$\frac{d}{dt} P_x = -q \frac{dA_x}{dt}$$

$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla}$
convective derivative

$$\therefore \frac{d}{dt}\Big[\underbrace{P_x + qA_x}_{\text{called (transverse) cannonical momentum}}\Big] = 0 \Rightarrow \boxed{P_x + qA_x} \text{ is a constant of motion}$$

To summarize, we have found two constants of motions,

$$\gamma mc - P_z = C_1$$

$$P_x + qA_x = C_2$$

Consider a particle starting from rest "before" laser arrives. For this particle, we have

$$\gamma = 1, \quad P = 0, \quad A = 0 \Rightarrow \begin{cases} C_1 = mc \\ C_2 = 0 \end{cases}$$

$$\Rightarrow \begin{matrix} \gamma mc - P_z = mc \\ P_x + qA_x = 0 \end{matrix} \Rightarrow \begin{cases} \gamma - \frac{P_z}{mc} = 1 & \cdots (7) \\ \frac{P_x}{mc} = -\frac{qA_x}{mc} & \cdots (8) \end{cases}$$

note: every expression in these two equations is unitless: $\frac{P}{mc}$ & $\frac{qA}{mc}$ are called normalized momentum & normalized vector potential, respectively

$$\boxed{\frac{P_x}{mc} = -\frac{qA_x}{mc} = \frac{qE_0}{mc\omega_0}\cos(k_0 z - \omega_0 t)} \quad \cdots \quad (9)$$

$$(7) \Rightarrow \gamma^2 = \left(1 + \frac{P_z}{mc}\right)^2$$

note: $[\gamma mc^2]^2 = P^2c^2 + (mc^2)^2 \Rightarrow \gamma^2 = \frac{P^2}{(mc)^2} + 1$ ← relativistic relation between particle energy & momentum

$$\Rightarrow \cancel{1} + \cancel{\frac{P_z^2}{m^2c^2}} + \frac{P_x^2}{m^2c^2} = \cancel{1} + \cancel{\frac{P_z^2}{m^2c^2}} + \frac{2P_z}{mc}$$

(P_y remains equal to zero, since there are no forces along z)

$$\left.\begin{array}{l} \therefore \dfrac{P_z}{mc} = \dfrac{1}{2}\dfrac{P_x^2}{m^2c^2} \\ (8): \dfrac{P_x}{mc} = -\dfrac{qA_x}{mc} \end{array}\right\} \rightarrow \boxed{\frac{P_z}{mc} = \frac{q^2A_x^2}{2m^2c^2}} \quad \cdots \quad (10)$$

$$\Rightarrow \frac{P_z}{mc} = \frac{1}{2}\frac{q^2}{m^2c^2}\frac{E_0^2}{\omega_0^2}\cos^2(k_0 z - \omega_0 t)$$

$$\boxed{\frac{P_z}{mc} = \frac{1}{4}\left(\frac{qE_0}{mc\omega_0}\right)^2\left(1 + \cos 2(k_0 z - \omega_0 t)\right)} \quad (11)$$

Note that both P_x & P_z are exhibiting oscillation behaviors, where the amplitude of oscillation is directly related to the normalized amplitude of the vector potential.

Def'n

$$a_0 = \frac{qA_0}{mc} = \frac{qE_0}{mc\omega_0} = 0.85 \times 10^{-9}\sqrt{I[\mathrm{W\,cm^{-2}}]}\,\lambda[\mu m]$$

Since $\left(\frac{P}{mc}\right)^2 = \gamma^2 - 1$, $\frac{P}{mc} > 1$ marks the transition to relativistic kinematics. So $a_0 > 1$ is used as a figure of merit for a laser pulse, where a particle's momentum becomes relativistic in the laser field.

e.g. for a $1\mu m$ Laser, $a_0 > 1$ if $I \gtrsim 10^{18}\ W cm^{-2}$

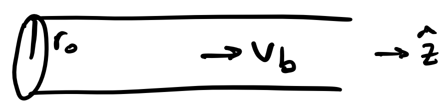
Also, note that whereas P_y averages to zero over a laser cycle, P_x does not. This is the physical origin of a phenomenon called the ponderomotive drift.

Another thing to note is the importance of constants of motion, which greatly simplified the steps needed to get to the solution here and will be used extensively throughout the class. So go ahead and memorize these now!

Case 2: An infinite cylinder of charge moving at $V_b \approx c$ w/ a radius r_0 & uniform density ρ_0

$\vec{J} = \rho V_b \hat{z}$ $\quad r < r_0$

$\vec{J} = 0$ $\quad r > r_0$



Consider a test particle within the beam experiencing the fields

Ampere's Law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

use a circular Amperian loop of radius $r < r_0$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{s}$

$2\pi r B_\phi = \mu_0 (2\pi) V_b \int_0^r \rho_0 r' dr'$ ... (12)

Gauss's Law:

$\vec{\nabla} \cdot \vec{E} = \frac{\rho_0}{\epsilon_0}$

$\vec{E} = E_r \hat{r}$ (cylindrically symmetric case)

$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r E_r) = \frac{\rho_0}{\epsilon_0}$

$\Rightarrow E_r = \frac{1}{r} \int_0^r \frac{\rho}{\epsilon_0} r' dr'$ ... (13)

Def'n $\lambda_0 = \int_0^{r_0} \rho_0 r' dr'$ as the <u>charge density per unit length</u>

for the beam. This is an important parameter as it allows us to convert a multi-dimentional equations into a 1D eqn. Recall from electrostatics:

$$dq = \lambda' dL' \Rightarrow E = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda' dL'}{r'^2}$$

where $\vec{r}' = \vec{r} - \vec{r}'$ is the distance from src to observer

$$\text{Eqn 12 \& 13} \Rightarrow \begin{cases} E_r = \dfrac{\lambda_0}{\epsilon_0}\dfrac{r}{r_0^2} \\ B_\phi = E_r \dfrac{V_b}{c^2} \approx \dfrac{E_r}{c} \quad (\text{since } V_b \approx c) \end{cases}$$

Equation of motion for the test charge is:

$$\frac{dP_r}{dt} = q\left(E_r + \vec{V}\times\vec{B}\big|_{\hat{r}}\right) = q\left(E_r + V_z B_\phi\right)$$

$$= q\frac{\lambda_0}{\epsilon_0}\frac{r}{r_0^2}(1-\beta_z)$$

This eqn describes a simple harmonic motion, provided that q has the opposite charge to ρ. Multiplying this eqn by $P_r = \gamma m \frac{dr}{dt}$, we get

$$\frac{d}{dt}\left(\frac{P_r^2}{2}\right) = \frac{q\lambda_0}{\epsilon_0}\left(\gamma - \frac{P_z}{mc}\right) r\frac{dr}{dt}$$

In the case of a test charge in a laser field, we showed that this was a constant of motion. We will show later that this is generally the case & so we will use $\gamma - \frac{P_z}{mc} = 1$ here.

$$\therefore \frac{d}{dt}\left[\frac{P_r^2}{2} - \frac{q\lambda_0 m}{\epsilon_0 r_0^2}\frac{r^2}{2}\right] = 0$$

$$\therefore \boxed{\frac{P_r^2}{2} - \frac{q\lambda_0 m}{\epsilon_0 r_0^2}\frac{r^2}{2} = \text{constant}}$$

The implied mechanical dynamic is that as the radius of motion increases, the momentum decreases. The maximum value for the oscillation momentum, corresponding to r=r0 is given by

$$@\ r=r_0\ ,\quad P_r=0 \Rightarrow -\frac{q\lambda_0 m}{2\epsilon_0} = \text{constant}$$

$$\max(P_r) = P_{r0} \text{ occurs at } r=0$$

$$\Rightarrow \frac{P_{r0}^2}{2} = \frac{q\lambda_0 m}{2\epsilon_0} \Rightarrow P_{r0} = \left[\frac{q\lambda_0 m}{\epsilon_0}\right]^{1/2}$$

in normalized units $\Rightarrow$ $\boxed{\frac{P_{r0}}{mc} = \left[\frac{q\lambda_0}{\epsilon_0 mc^2}\right]^{1/2}}$

Therefore, we define a strength parameter for the fields of relativistic beams in a similar way as for a laser pulse:

$$\Lambda_0 = \frac{e\lambda_0}{m\epsilon_0 c^2} = \frac{e}{m\epsilon_0 c^2}\int_0^{r_0} \rho r' dr'$$

Summary of Strength Parameters

For Lasers: $a_0 = \frac{eE_0}{mc\omega_0} = 0.86\left[\frac{\lambda_0}{\mu m}\right]\sqrt{\frac{I}{10^{18}\,W cm^{-2}}}$

intensity

Laser angular frequency

Laser peak E field strength

Wavelength

Particle beams: $\Lambda_0 = \frac{e\lambda_0}{m\epsilon_0 c^2} = 90\,\frac{Q[nC]}{\sigma_z(\mu m)}$

charge

rms beam length in z direction

<u>Normalizations:</u>

In this section, we made extensive use of the normalized parameters, where we divided a quantity by a “natural scale”, e.g. p/mc. Normalized quantities are very useful because they allow you to understand the relative strength of a parameter, e.g. a laser pulse with $a_0 \sim 0.01$ creates only a minor perturbation (regardless of what combination of intensity and wavelength create that value of a_0), whereas a laser with $a_0 \sim 10$ creates highly nonlinear phenomena. The normalized parameters are used extensively in description of plasma accelerators, so here we review them for the quantities in our studies.

We have already seen the natural scales for velocity and momentum:

$$p \rightarrow \frac{p}{mc} \qquad\qquad v \rightarrow \frac{v}{c}$$

For time, we need a normalizing frequency:

$$t \rightarrow \omega_N t$$

This also sets the space normalization:

$$\vec{x} \rightarrow \vec{x}\frac{\omega_N}{c}$$

To get the normalizations for fields & charge density, let's normalize the equation of motion:

$$\frac{1}{mc}\frac{d\vec{p}}{dt\cdot\omega_N} = \frac{qE}{mc\omega_N} + \frac{\vec{v}}{c}\times\frac{q\vec{B}}{m\omega_N}$$

The normalized quantities are:

$$\frac{e\vec{E}}{mc\omega_N} \rightarrow \vec{E}$$

$$\frac{e\vec{B}}{m\omega_N} \rightarrow \vec{B}$$

Recall $\omega_c = \frac{e\vec{B}}{m}$ is the cyclotron frequency of e^- in field $\vec{B}$.

Particle number density is commonly used as part of ρ & $\vec{J}$. In the context of plasma, we normalize this variable to the background (or initial) density:

$$\frac{n}{n_0} \rightarrow n$$

What about permittivity/permeability of free space?

$$\frac{c}{\omega_N}\cdot\vec{\nabla}\cdot\frac{e\vec{E}}{mc\omega_N} = \frac{\rho}{\epsilon_0} = \frac{q}{e}\frac{e^2}{m}\frac{n_0}{\epsilon_0}\frac{n}{n_0}\cdot\frac{1}{\omega_N^2}$$

1/spatial units

Since $\omega_p^2 = \frac{e^2 n_0}{m\epsilon_0}$, $\frac{1}{\epsilon_0}$ is normalized as follows:

$$\frac{e^2 n_0}{m\epsilon_0}\cdot\frac{1}{\omega_N^2} = \frac{\omega_p^2}{\omega_N^2}$$

Using Ampere's Law, it can be shown that once normalized, μ_0 will also become

$$\mu_0 \rightarrow \frac{\omega_p^2}{\omega_N^2}$$

For the scalar and vector potential, we have also already seen the normalizations:
From the definition of a_0,

$$\frac{eA}{mc} \rightarrow A$$

$$\frac{e\phi}{mc^2} \rightarrow \phi$$

Using units of φ:

$$\frac{e\psi}{mc^2} \rightarrow \psi$$ → sometimes written as upper-case Ψ

The choice of ω_N in the normalizations above depends on the relevant frequency in the problem of interest. Several possible choices include:

$\omega_N = \omega_p = \left(\frac{e^2 n}{m\epsilon_0}\right)^{1/2}$, plasma frequency. Used most often in plasma problems, such as the plasma waves that are of interest in this class.

$\omega_N = \omega_0$, Laser frequency, natural normalizin factor for problems of Laser interaction with a charged particle

$\omega_N = \omega_p = \left(\frac{e^2 n_b}{m\epsilon_0}\right)^{1/2}$, beam "plasma frequency", natural normalization for problems of beam interaction with a charged particle.

The co-moving coordinates

This is one of the most important concepts and a source of much confusion for those who are just starting out the study in this field. Because this fields includes drivers of physical phenomena that are moving at near the speed of light, many variables depend on the quantity $ct - z$, rather than on 't' or 'z' alone. Therefore, a new coordinate system is developed to work with this explicit dependence:

$$\text{Cartesian: } \begin{matrix} x \\ y \\ z \\ t \end{matrix} \Longrightarrow \text{co-moving frame: } \begin{matrix} x' = x \\ y' = y \\ t' = t \\ \xi = ct - z \end{matrix}$$

Physical interpretation: Those of you who have taken laser physics classes will immediately recognize this variable as the normalized phase of a laser pulse traveling in vacuum, i.e.

$$\vec{E} \propto \cos(\omega t - kz) \Rightarrow \phi(t,z) = \omega t - kz$$

The phase velocity is calculated as the velocity with which a "particle" would need to move to be at a constant phase. Mathematically, this is equivalent to

$$\frac{d\phi}{dt} = 0 \Rightarrow \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dt} = 0$$

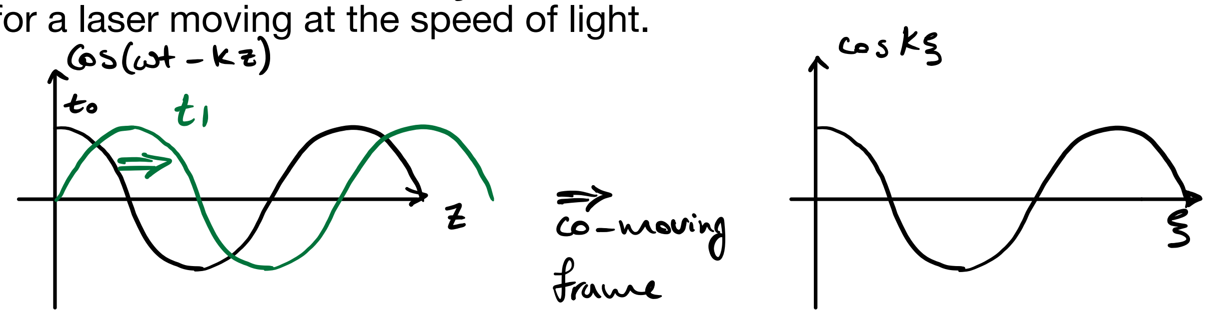
$$\omega - k v_\phi = 0 \Rightarrow v_\phi = \omega / k$$

where the laser is moving at the speed of light: $v_\phi = \frac{\omega}{k} = c$

Therefore,

$$\phi(t,z) = k\left(\frac{\omega}{k} t - z\right) = k(ct - z) = k\xi$$

Physically then, this new variable, ξ, indicates the location of stationary phase for a laser moving at the speed of light.



position of $\phi = \pi$ moves in z coordinate at different times $\Longrightarrow$ position of $\phi = \pi$ is fixed in ξ coordinate for all times

Similarly, a particle moving at the speed of light will maintain its position in ξ while it is moving in the Cartesian coordinates.

Note that while you may see this coordinate transform being referred to as "going to the speed of light frame", there are no Lorentz transforms performed in this operation, and as such, this is **not** a proper change of frame. I prefer to call this "a change of coordinate systems to a co-moving variable", because this operation is simply a relabeling of variable that allows for much more intuitive interpretation of the equations of motion and fields.

Derivatives in this new coordinate system can be obtained with the use of chain rule: Recall,

$$x' = x \qquad t' = t$$
$$y' = y \qquad \xi = ct - z$$

Consider the general case of each cartesian coordinate as a function of new coordinates, i.e. $x = x(x', y', t', \xi)$. Then the chain rule gives

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial \xi}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \quad \text{(trivial solution)}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + c\frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial z} = -\frac{\partial}{\partial \xi}$$

Total time derivative: recall that the total derivative is meaningful when all the variables are a function of time, which is the case when we look at a single particle. In that case,

$$\frac{d}{dt} = \frac{\partial}{\partial t}\cdot\frac{dt}{dt} + \frac{\partial}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial z}\cdot\frac{dz}{dt}$$

$$= \frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underbrace{\vec{v}\cdot\vec{\nabla}}$$

commonly refered to as the convective derivative

Total ξ derivative: Since $\xi = ct - z$, where $z \equiv z(t)$, e.g. where we follow a particle, total ξ derivative for a function of ξ can be expressed as:

$$\frac{d f(\xi)}{dt} = \frac{df}{d\xi}\frac{d\xi}{dt} = (c - v_z)\frac{df}{d\xi}$$

Note that from above definitions,

$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial z} \longrightarrow \frac{\partial}{\partial t'}$$

This means that the wave equation operator becomes

$$\frac{\partial^2}{\partial t^2} - c^2\nabla^2 = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2} - c^2\underbrace{\nabla_\perp^2}_{x \,\&\, y \text{ components}}$$

$$= \frac{\partial^2}{\partial t'^2} + 2c\frac{\partial}{\partial t'}\frac{\partial}{\partial \xi} - c^2\nabla_\perp^2$$

This means that if there are no explicit time variation, i.e. $\partial/\partial t' = 0$,the the wave equation reduces to a 2D Poisson-like solution

$$\text{i.e.}\quad \frac{\partial}{\partial t'} = 0 \Rightarrow \frac{\partial^2}{\partial t^2} - c^2\nabla^2 \longrightarrow -c^2\nabla_\perp^2$$

Another useful situation to consider is when the object is dependent on $v_b t - z$, instead of $ct - z$. This is the scenario for laser drivers for instance, which have a group velocity in plasma slower than the speed of light in vacuum.

In this case, $\frac{\partial}{\partial t}$ is not exactly zero, but the time variation is slow.

$$f(v_b t - z) = f(ct - z + (v_b - c)t)$$
$$= f(\xi + (v_b - c)t')$$

$$\therefore \frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - (c - v_b) = c\frac{\partial}{\partial \xi}\left(1 - \left(1 - \frac{v_b}{c}\right)\right)$$

If we were to cheat & just set $\frac{\partial}{\partial t} \sim c\frac{\partial}{\partial \xi}$, how good an approximation would we be making?

$$\left(1 - \frac{v_b}{c}\right)\frac{(1 + v_b/c)}{(1 + v_b/c)} \approx \frac{1}{2\gamma_b^2}$$

e.g. for the beam driver at SLAC, where $\gamma_b \sim 10^4$, $1 - \frac{v_b}{c} \approx 10^{-8}$!!
So, there would be a correction almost on the order on 1 in a billion! So, these approximations are pretty accurate.

<u>Developing Exact Solutions and Constants of Motion Using the Framework of Co-Moving Variables:</u> We start by reframing the equations of motion in terms of the scalar and vector potentials

Equation of motion: $\frac{d\vec{p}}{dt} = q\vec{E} + \vec{v} \times \vec{B}$ — momentum eqn

$$\therefore \vec{v} \cdot \frac{d\vec{p}}{dt} = q\,\vec{v} \cdot \vec{E}$$

$$\vec{v} \cdot \frac{d\vec{p}}{dt} = \frac{\vec{p}}{\gamma m}\frac{d\vec{p}}{dt} = \frac{1}{2\gamma m}\frac{d}{dt}p^2 = \frac{1}{2\gamma m}\frac{d}{dt}\left[\frac{1}{c^2}\left\{(\gamma m c^2)^2 - (mc^2)^2\right\}\right]$$

$$= \frac{1}{2\gamma m c^2}\frac{d}{dt}(\gamma m c^2)^2 = \frac{d}{dt}(\gamma m c^2)$$

$$\frac{d}{dt}(\gamma m c^2) = q\,\vec{v} \cdot \vec{E}$$ — energy eqn

Substitute $\vec{A}$ & ϕ:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\frac{d\vec{P}}{dt} = -q\frac{\partial \vec{A}}{\partial t} - q\nabla\phi + q\,\vec{v}\times(\vec{\nabla}\times\vec{A})$$

use vector identity: $\vec{a}\times\vec{\nabla}\times\vec{b} = (\vec{\nabla}\vec{b})\cdot\vec{a} - \vec{a}\cdot\vec{\nabla}\vec{b}$

See Appendix for nerding out w/ Einstien notation & for derivation of this identity.

$$\therefore \frac{d\vec{P}}{dt} = \underbrace{-q\frac{\partial \vec{A}}{\partial t} - q\vec{v}\cdot\vec{\nabla}\vec{A}}_{\text{complete derivative of } \vec{A}} - q\nabla\phi + q(\vec{\nabla}\vec{A}\cdot\vec{v})$$

$$\therefore \boxed{\frac{d}{dt}(\vec{P}+q\vec{A}) = -\vec{\nabla}\phi - q(\vec{\nabla}\vec{A})\cdot\vec{v}} \quad \ldots (14)$$

similarly,

$$\boxed{\frac{d}{dt}(\gamma mc^2 + q\phi) = -q\vec{v}\cdot\frac{\partial\vec{A}}{\partial t} + q\frac{\partial\phi}{\partial t}} \quad \ldots (15)$$

This set of equations are incredibly important in this class & you should memorize them!

The top equation is an alternative formulation of the momentum equation. If the right hand side is zero for some reason, then we recover the conservation of canonical momentum in Classical Mechanics. These equations also include the seed of the ponderomotive force, when ‘v’ in the second equation is replaced by a term that has the vector potential.

One of the primary constants of motion in the plasma acceleration field is obtained by combining the ‘z’ component of the first equation with the second equation to get the following scalar equation:

$$\frac{d}{dt}(\gamma mc^2 - P_z c + q\phi - qA_z c) = q\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)\phi - q\vec{v}\cdot\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)\vec{A}$$

If we change the variables to the co-moving coordinates and drop the primes, we get:

$$\frac{d}{dt}(\gamma mc^2 - P_z c + q\phi - qA_z c) = q\frac{\partial\phi}{\partial t} - q\vec{v}\cdot\frac{\partial\vec{A}}{\partial t}$$

But if both $\vec{A}$ & ϕ are functions of ξ only or only weakly depend on t, the right hand side is zero &

$$\boxed{\frac{d}{dt}\left[\gamma mc^2 - P_z c + q(\phi - A_z c)\right] = 0} \quad \cdots (6)$$

This is a very important constant of motion, and it can also be derived from the analysis of the Lagrangian of the system (see Appendix).

One specific case that will be very useful later is the case of a particle that starts from rest in the region where there are no fields. In this case, the constant of motion becomes:

$$\underbrace{\underset{\substack{\uparrow \\ \gamma=1}}{\gamma mc^2} - \underset{\substack{\uparrow \\ P_z=0}}{P_z c}}_{\text{for particle at rest}} + \underbrace{q(\phi - A_z c)}_{\text{These are zero when the fields are zero}} = mc^2$$

So for this particle,

$$\gamma mc^2 - P_z c + q(\underbrace{\phi - A_z c}_{\equiv \varphi}) = mc^2$$

$$\gamma - \frac{P_z}{mc} + \frac{q}{mc^2}\varphi = 1$$

$$\therefore \boxed{\gamma(1-\beta_z) + \frac{q\varphi}{mc^2} = 1}$$

Also note that if $\varphi = 0 \Rightarrow \gamma(1-\beta_z) = 1$

In this case, since $\frac{d}{dt} = (c - V_z)\frac{d}{d\xi}$,

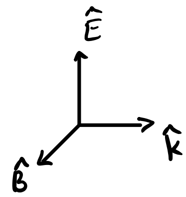
$$\gamma\frac{d}{dt} = \left(\gamma - \frac{P_z}{mc}\right)c\frac{d}{d\xi} = c\frac{d}{d\xi}$$

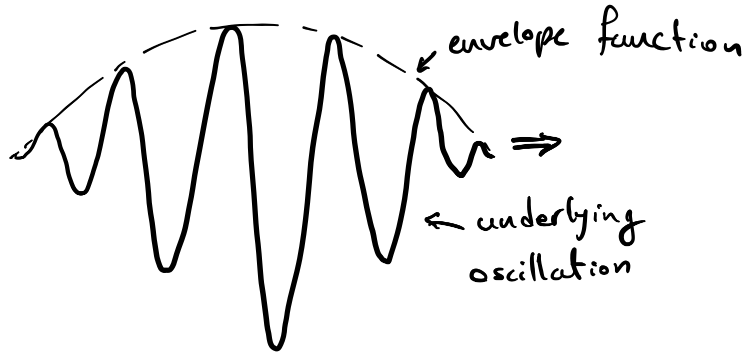
which means that $c\frac{d}{d\xi}$ is equivalent to the particle's proper time in vacuum

<u>Single Particle Equation of Motion in the Fields of a Laser</u>

So now, let's look at the equation motion for an electron in a laser field and then in the field of a beam.

Consider a plane laser pulse (i.e. uniform transverse spatial profile) with a temporal profile moving along the 'z' direction





A plane laser field in vacuum can be described by a transverse only vector potential, which is a function of ξ only.

$$\vec{A} = \vec{A}_\perp(\xi)$$

This motion is more complicated than before because now we are going to have an electron start out in front of the laser where there are no fields. The electron will get accelerated in 'z' also and will spend more time in one part of the laser than another. But while the motion in time is very complicated, the co-moving coordinates allow for a simpler understanding of this motion.

For this laser pulse, $\phi = 0$ & $A_z = 0$ (since $\vec{A} = A_\perp$)

So, $\Psi \equiv \phi - A_z c = 0$

Long before laser arrives, we assume that the electron is at rest & the equation of the constant of motion becomes

$$\gamma(1-\beta_z) = 1 \quad \ldots (17)$$

The transverse component of the momentum equation (Eqn. 14) becomes

$$\frac{d}{dt}\left(\vec{P}_\perp + q\vec{A}_\perp\right) = -q\underbrace{\nabla_\perp\phi}_{0} + q\underbrace{(\vec{\nabla}_\perp\vec{A})}_{0}\cdot\vec{v}$$

0 (no transverse gradient)

i.e. $(\vec{P} + q\vec{A})_\perp$ is a constant of motion

If the particle is initially at rest, then this constant is zero, which means

$$P_\perp = -qA_\perp \Rightarrow \boxed{\frac{P_\perp}{mc} = \frac{q|A|}{mc}} \quad \ldots (18)$$

This is another way to define a_0

using the constant of motion (Eqn. 17):

$$\gamma - \frac{P_z}{mc} = 1 \qquad \text{but also,} \quad \gamma = \left(1 + \frac{P^2}{m^2c^2}\right)^{1/2} = \sqrt{1 + \frac{P_\perp^2}{m^2c^2} + \frac{P_z^2}{m^2c^2}}$$

$$\therefore \gamma^2 = \left(1 + \frac{P_z}{mc}\right)^2 = 1 + \frac{P_\perp^2}{m^2c^2} + \frac{P_z^2}{m^2c^2}$$

$$\cancel{1} + \cancel{\frac{P_z^2}{m^2c^2}} + \frac{2P_z}{mc} = \cancel{1} + \frac{P_\perp^2}{m^2c^2} + \cancel{\frac{P_z^2}{m^2c^2}}$$

$$\therefore \frac{P_z}{mc} = \frac{1}{2}\frac{P_\perp^2}{mc^2} = (\text{from } 18) = \frac{1}{2}\left(\frac{qA}{mc}\right)^2 \quad \dots (19)$$

From Eqn. 19, we see that when we are in a relativistic regime (i.e. $a_0 > 1$), the longitudinal momentum actually becomes larger than the transverse. Since the force of the magnetic field depends on the velocity, the importance of the magnetic field is dependent on this parameter as well.

The results obtained so far don't depend on the longitudinal shape of the magnetic field. Let's constrain the problem slightly by considering a specific form of laser field:

$$A_\perp(\xi) = \hat{x}\, A_0(\xi)\, \sin k_0 \xi \quad \dots (20)$$

polarization vector (Linear polarization) — envelope of the laser — fast oscillations

We assume that the envelope varies slowly on the scale of the fast laser oscillations, i.e. the envelope contains many oscillations:

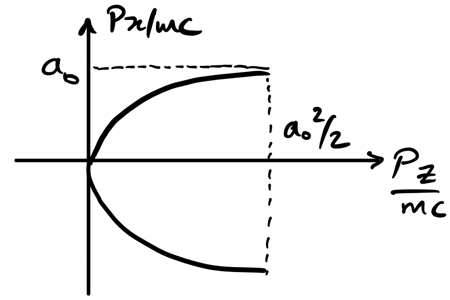
$$\frac{1}{k_0}\frac{\partial A_0}{\partial \xi} \ll A_0$$

The way we have the phase of the fast oscillations, we can also consider the envelope also starting at $\xi = 0$, i.e. $\xi = 0$ is when the laser pulse just reaches the electrons (this simplies the mathematics of the solution). In this case $P_x(\xi = 0) = 0$

$$18 \,\&\, 20 \Rightarrow \frac{P_x}{mc} = a_0 \sin k_0 \xi \quad \dots (21) \qquad \left(\text{Recall},\ a_0 = \frac{qA_0}{mc}\right)$$

$$19 \ \& \ 20 \Rightarrow \quad \frac{P_z}{mc} = \frac{a_0^2 \sin^2 k_0 \xi}{2} = \frac{a_0^2}{4}\left(1 - \cos 2k_0\xi\right) \quad \cdots (22)$$

Notice that these equations describe a parabola in phase space. Also note that P_z is a positive quantity.



Next, the position of the particle can be found through integrating the momentum equations:

$$\frac{d\vec{r}}{dt} = \frac{\vec{P}}{\gamma m} \quad \cdots (23)$$

This is a quite a nonlinear equation, since we showed above that $\vec{P}$ goes as "a_0" & "a_0^2" & it can be shown that $\gamma \sim 1 + a_0^2$. However, we can make use of the co-moving coordinates to greatly simplify these equations:

Since $\vec{A} = \vec{A}_\perp(\xi)$ & $\phi = 0 \Rightarrow \psi = 0$ and as show above, using $\gamma(1-\beta_z) = 1$ as the constant of motion,

$$\gamma \frac{d}{dt} = c \frac{d}{d\xi}$$

$$(23) \Rightarrow \gamma \frac{d\vec{r}}{dt} = \frac{\vec{P}}{m}$$

$$\begin{aligned} (21) &\Rightarrow \left\{ \frac{dx}{d\xi} = a_0(\xi) \sin k_0 \xi \right. \\ (22) &\Rightarrow \left\{ \frac{dz}{d\xi} = \frac{a_0^2}{4}(\xi)\left(1 - \cos 2k_0\xi\right) \right. \end{aligned}$$

If a0 was constant, it would be trivial to integrate, but even if it is not, one can integrate by parts to get

$$x(\xi) = -\frac{a_0(\xi')}{k_0}\cos(k_0\xi')\Big|_0^{\xi} + \frac{1}{k_0}\int_0^{\xi} \frac{da_0}{d\xi'} \cos k\xi' \, d\xi'$$

Assuming all orders of the derivatives for a_0 to be zero at $\xi = 0$ (i.e. the pulse has not arrived yet) allows us to develop an infinite series solution for $x(\xi)$; i.e. the second integration by parts would give:

$$x(\xi) = -\frac{a_0(\xi)}{K_0}\cos(K_0\xi) + \frac{da_0}{d\xi\, K_0^2}\sin(K_0\xi) - \frac{1}{K_0^2}\int_0^{\xi}\frac{d^2a_0}{d\xi'^2}\cos(K_0\xi')\,d\xi'$$

$$\vdots$$

$$x(\xi) = -\sum_{n=0}^{\infty}\frac{d^n a_0(\xi)}{d\xi^n}\left(\frac{1}{K_0}\right)^{n+1}\cos\left(K_0\xi + \frac{n\pi}{2}\right)$$

Note that the ratio of the magnitude for any two succeeding terms are $x_{n+1}/x_n \sim \frac{1}{K_0}\frac{da_0}{d\xi} \ll 1$ based on our assumption of the slowly varying envelope (i.e. many cycles in a pulse)

So we retain the leading order term to get

$$x(\xi) \approx -\frac{a_0(\xi)}{K_0}\cos(K_0\xi) \quad \ldots \quad (24)$$

This is the same solution as we would get if a0 was constant, except that now the solution is modulated by the slowly varying amplitude of the laser pulse.

Likewise, for $z(\xi)$ we get

$$z(\xi) = \int d\xi\, \frac{a_0^2}{4}(\xi)\,[\underbrace{1}_{\text{slow drift term}} - \underbrace{\cos(2K_0\xi)}_{\text{fast oscillations, averages to zero over oscillation period}}]$$

We can treat the fast oscillation term in the same way as we did for "x", keeping the lowest order of fast oscillations modulated by the laser envelope, but this cannot be done for the slow drift term, and that one just has to get integrated. Let's look at the results for several canonical cases:

(1) for an infinite plane wave: $\frac{a_0^2}{4} = \text{constant}$

$$\therefore\ z = \frac{a_0^2}{4}(ct - z) - \{\text{oscillating term}\}$$

$$\Rightarrow z\left(\frac{a_0^2}{4}+1\right) = \frac{a_0^2}{4}ct - \{\text{oscillating term}\}$$

$$\Rightarrow z = \frac{a_0^2}{4\left(1+\frac{a_0^2}{4}\right)}ct - \{\text{oscillating term}\}$$

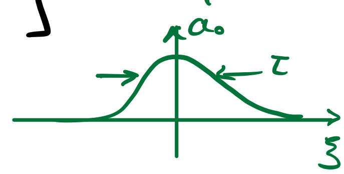
Implies a drift velocity

$$v_d = \frac{dz}{dt} = \frac{a_0^2}{4(1+a_0^2/4)}c$$

Note that at low a_0, $a_0 \to 0$, $v_d \to 0$, meaning that if the relativistic effect is small, the particle oscillates back & forth & comes to rest when the laser is gone. On the other hand at relativistic intensities, $a_0 \gtrsim 1$, the electrons will have a drift velocity along the direction of the laser propagation even after the laser is gone & in the limiting case, $a_0 \to \infty$, $v_d \to c$. So for intense lasers we expect to snow plow electrons forward at near the speed of light.

② For a Gaussian envelope

$$\frac{a_0^2(\xi)}{4} = \frac{a_{00}^2}{4}\exp\left[-\frac{(\xi-\xi_0)^2}{\tau^2}\right]$$ Gaussian envelope



Solve for z_{drift} :

$$z_{drift} = \frac{a_{00}^2}{8}\sqrt{\pi}\,\tau\left[\mathrm{erf}\left(\frac{\xi-\xi_0}{\tau}\right) + \mathrm{erf}\left(\frac{\xi_0}{\tau}\right)\right]$$

error function

for $\tau \to \infty$ (plane wave approximation)

$$\Rightarrow \mathrm{erf}\left(\frac{\xi-\xi_0}{\tau}\right) = \frac{2}{\sqrt{\pi}}\frac{\xi-\xi_0}{\tau}$$ (first term in the Taylor expansion)

$$\Rightarrow z_{drift} = \frac{a_{00}^2}{4}\xi$$ ⟸ plane wave result as expected.

These two cases are illustrated in the figures below. The Gaussian vector potential is implemented for both set. The momentum equations are implemented exactly as above and the ‘z’ and ‘x’ positions are calculated by numerically integrating the momentum along ξ in each respective direction.

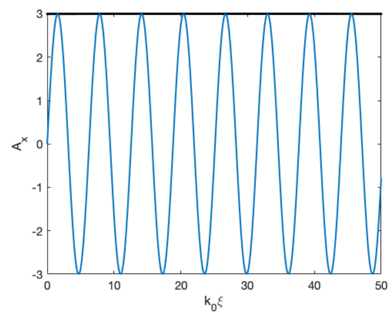
Case 1. Plane Wave

The plane wave is achieved by choosing the variables such that $\tau \gg k_0 \xi$, i.e. the pulse contains many fast oscillations, i.e.

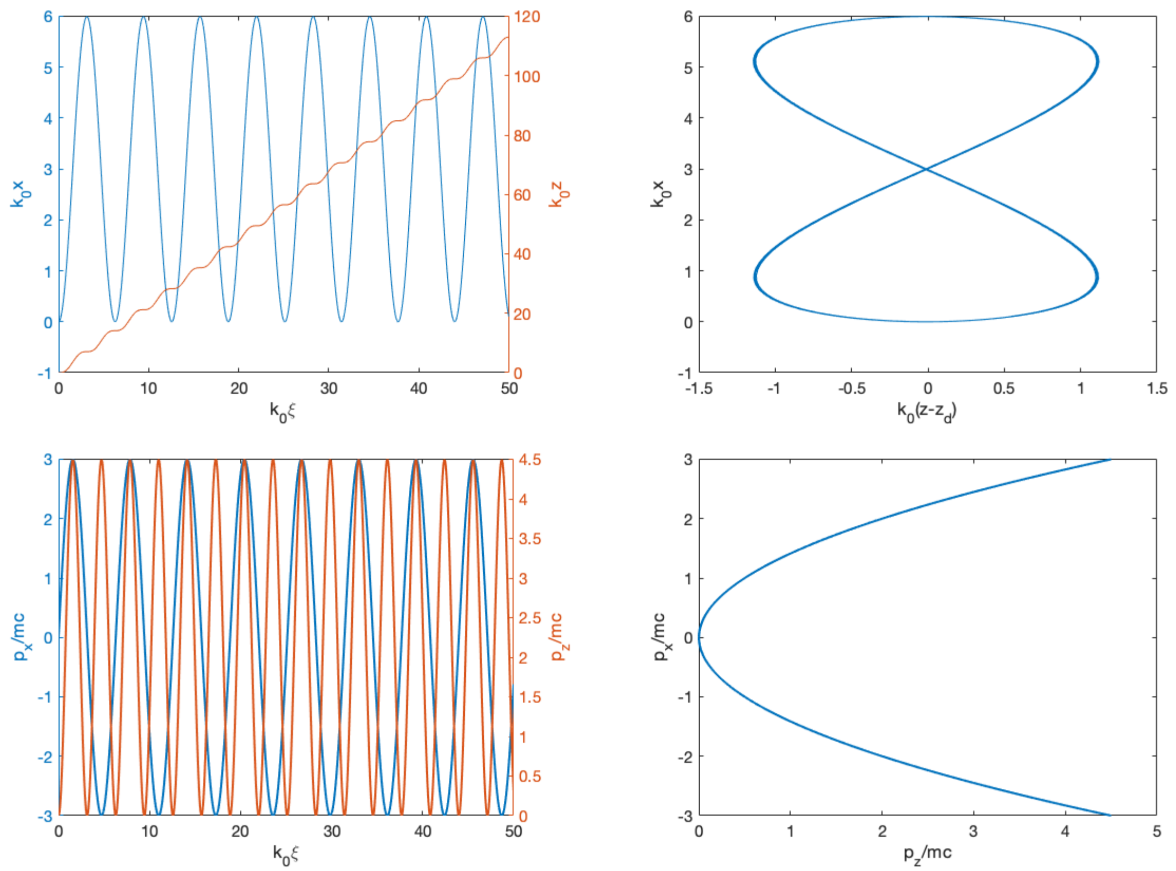
$$a_0(\xi) = a_{00} \exp\left[-\frac{(\xi-\xi_0)^2}{2\tau^2}\right]$$

$a_{00} = 3$, $\sqrt{2}\tau = 10$, $k_0 = 100$, $\xi \in [0, 0.5]$ $\xi_0 = 0.25$

k_0 & ξ are arbitrary, but the range of ξ is chosen to span several oscillation wavelengths



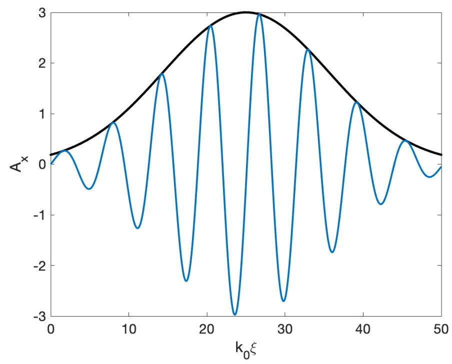
The top row shows the position and the bottom row shows the momentum curves. Note these important features:



- The position drift in ‘z’ is linear, indicating a constant drift velocity as expected
- The “figure 8” curve that emerges when ‘x’ is plotted against ‘z’ (with the drift taken out) is a very famous curve and can be used to understand the radiation due to this motion. It also shows that ‘z’ oscillates at twice the frequency of ‘x’
- The momentum curves plotted individually as a function of phase show sinusoidal oscillation with ‘z’ oscillating with twice the frequency of ‘x’
- Plotted against each other, the momentum curves trace a parabola as expected, with the maximum of p_x being equal to a_0 and the maximum of p_z being equal to $a_0^2/2$ as expected.

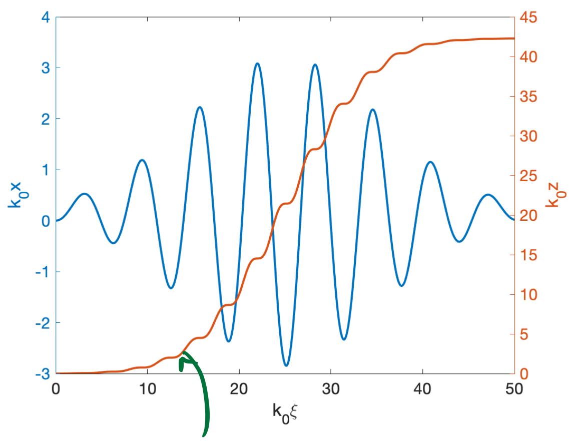
<u>Case 2. Gaussian Wave</u>

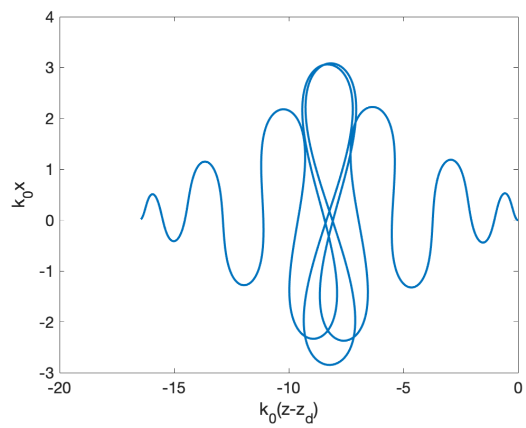
Now we are going to give the wave an envelope by the proper choice of τ with respect to $k_0\xi$: $\sqrt{2}\,\tau = 0.15$



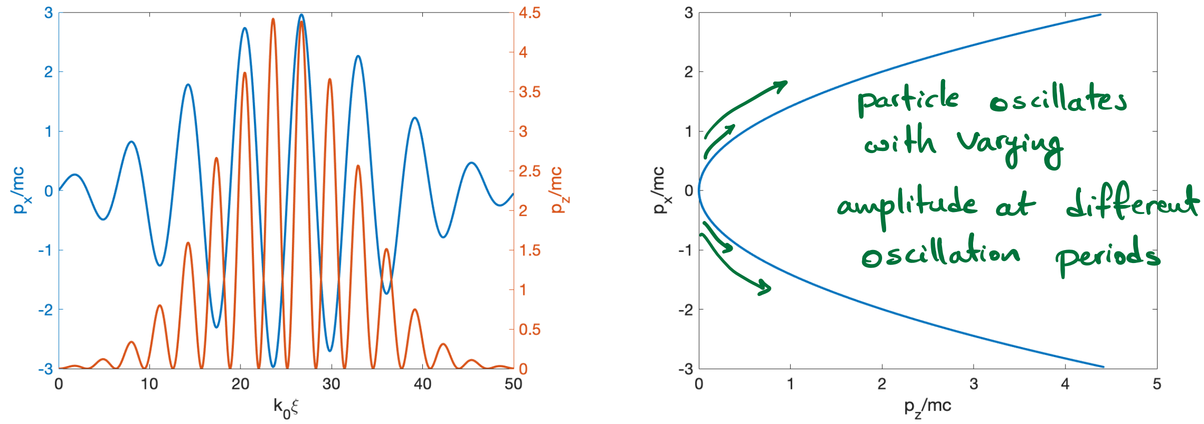
$K_0 = 100$ & $\xi \in [0, 0.5]$ as before

The black curve shows the slowly varying envelope (a_0) & the blue curve shows the vector potential, including its fast oscillations: $A = a_0 \sin(K_0 \xi)$





The drift now traces a nonlinear curve rather than a Line with a constant slope



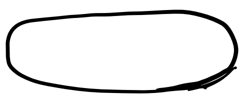
Again, the top row shows the position and the bottom row show the momentum curves. Note the following important features of these curves

- The envelope now covers only a few periods.
- The ‘x’ position just oscillates about a centroid as it follows the laser pulse. The ‘z’ curve shows a drift again, but now the drift is not linear. So if we took the derivative of the drift to get the velocity, unlike the plane-wave case where we would get a constant velocity, now we would get a varying velocity, which implies an acceleration. This implies a time-averaged force, which is called a ponderomotive force, which we will get back to later.
- The momentum curves follow the vector potential, so one can see the amplitude of oscillation of momentum curves following the envelope of the vector potential.
- The longitudinal period oscillates at the second harmonic and is only positive. So while might think of the electron moving back and forth in the oscillating field of the laser, in the longitudinal direction, the electrons are only pushed forward.
- Looking at the momentum phase space (the right curve) one can see however that the relation between the momentum curves stays the same as before even though now the amplitude of the momenta are changing. However because the amplitude of momenta are changing, the peaks are not reached during every oscillation. Nevertheless the particle in phase space still traces the parabolic curve.

Note that so far, we have assumed a transversely uniform laser pulse. In reality, laser pulses have finite width, which complicated the problem, but for now, we are going to leave this topic and look at a particle beam.

Single Particle Equation of Motion in the Fields of a Particle Beam Driver

Assume we have a cylindrically symmetric beam traveling with negligible focusing effects (e.g. because of a long focal length optic)



$\vec{P} = \gamma m v_b \hat{z}$

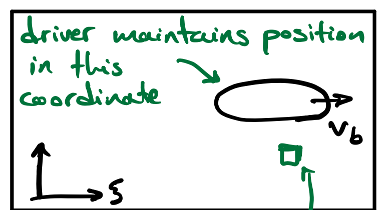
Assume $p \gg mc \Rightarrow \gamma \gg 1$

e.g. SLAC beam at 20 GeV $\rightarrow \gamma \sim 4\times10^4$

& v_b differs from c by $< 10^{-8}$!!

For a driver that is moving at a velocity close to the speed of light, we can often enormously simplify the problems by introducing the quasistatic approximation:

$$\frac{\partial}{\partial t} \ll \frac{\partial}{\partial \xi}$$



$\partial/\partial t$ means observing quantities at a point in space

Physically, this means that temporal variation at some particular location (ξ, x, y) is much smaller than the spatial variation at a particular time. Why is that? It is because the driver is at the same location in this coordinate & as long as its velocity remains close enough to the speed of light, its charge density, current density, & other parameters that depend on them will change very slowly in time. The spatial variation however is on the scale of the beam size/c, which occurs much more quickly.

The error in this approximation is related to the difference between v & c. Consider this relation from above: $\frac{d}{dt} = (c - v_z)\frac{d}{d\xi}$

For a particle moving at the speed of light, $v_z = c$, $\frac{d}{dt} = 0$, meaning that the particle stays where it is in the co-moving coordinate.

A particle moving near (but not at) the speed of Light will slip back in ξ. The time variation in some quantity at position (ξ, x, y) will be related to this slippage. This slippage is on the order of $(c - v_z)$ meaning the error $\frac{\partial}{\partial t} \ll \frac{\partial}{\partial \xi}$ is also on the same order.

With this quasistatic approximation, and assuming that the fields associated with accelerating the beam have propagated far away, we calculate the fields starting from the Maxwell's equations in the Cartesian coordinates in terms of potentials and in the Lorentz Gauge:

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t} \quad \ldots (25)$$

Ampere's Law
$$\frac{\partial^2 \vec{A}}{\partial t^2} - c^2 \nabla^2 \vec{A} = \frac{\vec{J}}{\epsilon_0} \quad \ldots (26)$$

Gauss's Law
$$\frac{\partial \phi}{\partial t^2} - c^2 \nabla^2 \phi = \frac{\rho c^2}{\epsilon_0} \quad \ldots (27)$$

Wave equations with source terms

We saw earlier that in the case where explicit time variation can be ignored (i.e. $\partial/\partial t = 0$, here due to our quasistatic approx.) the wave operator in the co-moving variable becomes: $-c^2 \nabla_\perp^2$

$$(26) \Rightarrow \quad -c^2 \nabla_\perp^2 \vec{A} = \frac{\vec{J}}{\epsilon_0} \quad \ldots (28)$$

$$(27) \Rightarrow \quad -c^2 \nabla_\perp^2 \phi = \frac{\rho c^2}{\epsilon_0} \quad \ldots (29)$$

Note that since the beam moving in the $\hat{z}$ direction with velocity near c,

$$\vec{J} \simeq \rho v_b \hat{z} \approx \rho c \hat{z} \quad \ldots (30)$$

$$(28), (30) \Rightarrow \quad \nabla_\perp^2 A_z = -\frac{\rho}{c \epsilon_0}$$

$$\nabla_\perp^2 \phi = -\frac{\rho}{\epsilon_0}$$

$$\overline{\nabla_\perp^2 (\phi - A_z c) = 0} \quad \ldots (31)$$

Also, from (28), (30) $\Rightarrow$ $\nabla_\perp^2 A_\perp = 0$ ... (32)

Boundary condition $A_\perp \to 0$ at ∞ (open boundary)

Since $A_\perp = 0$ satisfies both equation (32) & the boundary condition, the uniqueness theorem dictates that $A_\perp = 0$ is the only solution $\Rightarrow$ $\boxed{A_\perp = 0}$ ... (33)

$$(25), (33) \Rightarrow \partial_z A_z + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \;\cdots\; (34)$$

In the co-moving coordinate,

$$\left.\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial t'} + c\frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial z} &= -\frac{\partial}{\partial \xi} \end{aligned}\right\} \Rightarrow (34) \Rightarrow -\frac{\partial A_z}{\partial \xi} + \frac{1}{c^2}\Big(\underbrace{\frac{\partial \phi}{\partial t'}}_{} + c\frac{\partial \phi}{\partial \xi}\Big) = 0$$

$\partial/\partial t = 0$ in the co-moving coordinate

$$\Rightarrow \quad \frac{\partial}{\partial \xi}(\phi - A_z c) = 0 \;\cdots\; (35)$$

From 31 & 35 & considering the open boundary condition,

$$\boxed{\begin{aligned} &\phi = A_z c \;\cdots\; (36) \\ (33) \Rightarrow\; &A_\perp = 0 \end{aligned}}$$

The two equations above are very important as they completely define the vector potential and its relation to the scalar potential. Scalar potential in turn can be solved from equation 29, which is a 2D Poisson's equation. Moreover,

$$\psi = \phi - A_z = 0 \Rightarrow \gamma(1-\beta_z) = 1 \;\ldots\; (37)$$

i.e. The constant of motion equation becomes the same as the Laser case (eqn. 17) & we can take the same approach to solving the equations of motion as in the laser field.

Assuming cylindrical symmetry, we can solve for the fields using the Maxwell's equations:

$$\left.\begin{aligned}\vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}\right\} \text{ In Cartesian coordinates}$$

In co-moving coordinates,

$$E_r = -\frac{\partial \phi}{\partial r} \qquad (\text{since } A_\perp = 0) \quad \ldots (38)$$

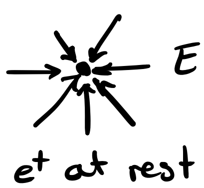
$$E_z = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t}$$

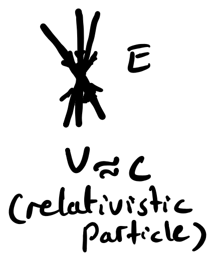
$$= \frac{\partial \phi}{\partial \xi} - \cancel{\frac{\partial A_z}{\partial t'}} - c\frac{\partial A_z}{\partial \xi}$$

° quasi-static approximation

$$\Rightarrow E_z \cong \frac{\partial}{\partial \xi}(\phi - A_z c) \cong 0 \qquad \ldots (39)$$

Physically, this is understood as the field lines bunching transversely



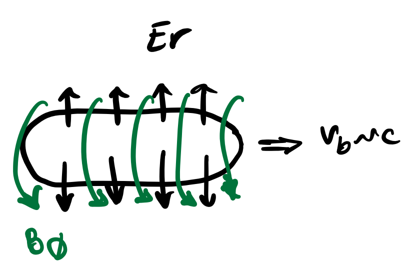


E_z becomes almost zero (or more presicely, neglyable compared to E_r)

$$\left.\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A} \\ A_\perp &= 0\end{aligned}\right\} \Rightarrow B_\phi = -\frac{\partial A_z}{\partial r} = -\frac{1}{c}\frac{\partial \phi}{\partial r}$$

$$\Rightarrow \boxed{B_\phi = \frac{E_r}{c}} \quad \ldots (40)$$

for a positron beam



For an electron beam, the directions are reversed.

So far, we haven't said anything about the profile of the beam. The only stipulation has been that the beam is highly relativistic and cylindrically symmetric. Now we look at the motion of a particle in the field of the beam:

From constant of motion, Eqn. 37: $\gamma - \frac{p_z}{mc} = 1$ for a particle initially at rest.

Also,

$$\gamma^2 = 1 + \left(\frac{p_z}{mc}\right)^2 + \left(\frac{p_\perp}{mc}\right)^2$$

$$\therefore \boxed{\frac{p_z}{mc} = \frac{1}{2}\frac{p_\perp^2}{m^2c^2}} \quad \ldots \quad (41)$$

This is similar to the laser

To solve for $p_\perp$, we look at the radial eqn of motion:

$$\frac{dp_r}{dt} = qE_r + q(\vec{v}\times\vec{B})_r$$

note: $B = B_\phi \hat{\phi} = \frac{E_r}{c}\phi$

$$\frac{dp_r}{dt} = -q\left(1 - \frac{v_z}{c}\right)\frac{\partial \phi}{\partial r} \quad \ldots \quad (42)$$

$$\Rightarrow p_r \frac{dp_r}{dt} = -q\underbrace{\gamma m\left(1 - \frac{v_z}{c}\right)}_{=1}v_r\frac{\partial \phi}{\partial r}$$

constant of motion

$$\frac{d}{dt}\left(\frac{p_r^2}{2}\right) = -qm\frac{dr}{dt}\frac{\partial \phi}{\partial r}$$

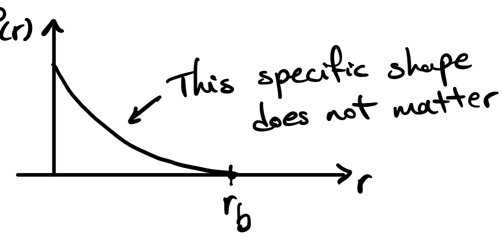
chain rule

$$\frac{d}{dt}\left(\frac{p_r^2}{2}\right) = -qm\frac{d}{dt}\left[\int \frac{\partial \phi}{\partial r}dr\right]$$

partial derivative, so the expression in square brackets is <u>not</u> simply ϕ

$$\Rightarrow \boxed{p_r = \sqrt{-2qm\int \frac{\partial \phi}{\partial r}dr}} \quad \ldots \quad (43)$$

In general, this is fairly complicated, but as we see later, we are interested in a tightly focused beam, and we want to look at particles outside of this beam. So choose a beam with compact support, i.e. a beam with nonzero charge density only up to radius r_b, e.g.



ϕ satisfies eqn 29 in the co-moving coordinates:

$$\text{(29)} \Rightarrow \quad -\nabla_\perp^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi}{\partial r}\right] = \frac{\rho}{\epsilon_0} \quad \leftarrow \frac{\partial}{\partial \phi} = 0 \text{ due to cylindrical symmetry}$$

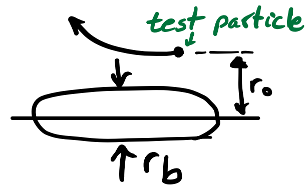
for $r > r_b$ (outside of beam radius),

$$\frac{\partial \phi}{\partial r} = \frac{1}{\epsilon_0 r}\underbrace{\int_0^{r_b} \rho(r)\, r\, dr}_{\text{defined as } \lambda(\xi)\text{, charge density per unit length (see Eqn. 13)}}$$

$$\frac{\partial \phi}{\partial r} = \frac{\lambda(\xi)}{\epsilon_0 r} \quad \ldots \text{(44)}$$

Now, we can substitute this expression in equation 43 to find an expression for the momentum of a particle.

Start w/ an e^- at radius r_0:



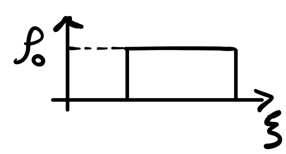
$$\int \frac{\partial \phi}{\partial r} dr = \int_{r_0}^{r_0+\Delta r} \frac{\lambda(\xi)}{\epsilon_0 r} dr = \lambda_0 \int_{r_0}^{r_0+\Delta r} \frac{\lambda(\xi)}{\lambda_0 r} dr$$

here, we make use of $\lambda_0 = \max(\lambda(\xi))$, where it is understood that the beam has a profile in ξ (e.g. a Gaussian) & we normalize $\lambda(\xi)$ to its maximum value.

$$\text{So,} \quad P_r = \sqrt{2\Lambda_0 \int_{r_0}^{r_0+\Delta r} \frac{\lambda}{\lambda_0 r} dr} \quad \ldots \text{(45)}$$

For a top-hat beam,

$$\rho(\xi) = \rho_0 \qquad \xi_1 < \xi < \xi_2$$



$$\text{So,} \quad \frac{\lambda}{\lambda_0} = 1 \rightarrow P_r = \sqrt{2\Lambda_0 \underbrace{\ln\left(1+\frac{\Delta r}{r_0}\right)}_{\text{usually order } 2}}$$

So, the parameter Λ_0 once again plays the role of a strength parameter. In SLAC experiments in 1998,

$\sigma_z \sim 600\,\mu m$, $Q \sim 1nC$

$$\Lambda = \frac{90\,nC}{\sigma_z\,[\mu m]} < 1 \quad \text{in } 1998$$

in 2003, $\sigma \sim 10\,\mu m \Rightarrow \frac{P_r}{mc} \sim 4$