Quantitative Finance Qualifying Exam

May 2023

Instructions: (1) You have 4 hours to do this exam. (2) This exam is closed notes and closed books. No electronic devices are permitted. (3) Phones must be turned completely off during the exam. (4) All problems are weighted equally.

Part 1: Do 2 out of problems 1, 2, 3. (AMS511)
Part 2: Do 2 out of problems 4, 5, 6. (AMS512)
Part 3: Do 2 out of problems 7, 8, 9. (AMS513)
Part 4: Do 2 out of problems 10, 11, 12. (AMS517)

Problems to be graded: Please write down which eight problems you want graded here.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (Print clearly):

Student ID:

Signature:

Stony Brook University
Applied Mathematics and Statistics

1. An asset's price S(t) follows the dynamics of the constant coefficient geometric process described by the stochastic differential equation (SDE):

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$$

Carrying costs are continuously charged at the rate c and dividends continuously accrue at the rate b, both proportionally to the current price. Denote the risk-free rate of return by r_f.

Derive the SDE which describes the price dynamics under the risk-neutral measure. Fully explain your logic in deriving the solution.

2. The interest rates in the UK $r_{\mathrm{UK}} = 0.015$ and US $r_{\mathrm{US}} = 0.02$, compounded continuously. The spot price of the UK pound is \$1.20 and the forward price for the UK pound deliverable in six months is \$1.25. Does an arbitrage opportunity exist? Show clearly why one is or is not available.

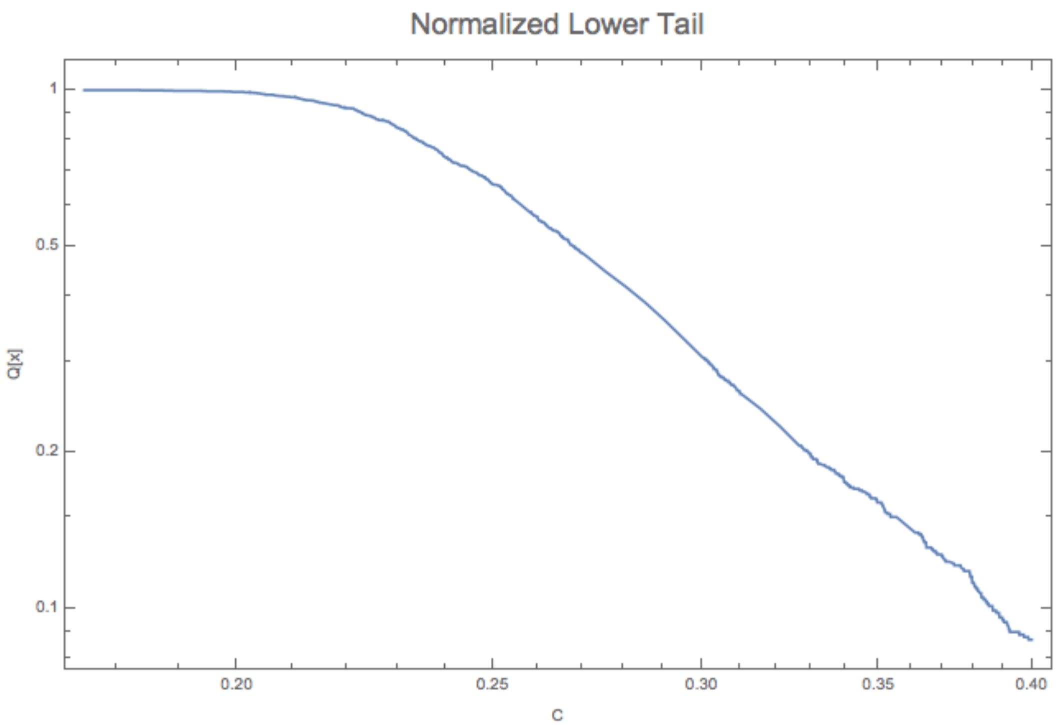
3. Consider a custom European option F on underlying S whose pay-off at expiry T follows

$$F(T) = \min[B, \max[K_1 - S(T), S(T) - K_2, 0]]$$

where B is a cash position and $0 < B < K_1 < K_2$.

Price this option in terms of simpler instruments, *i.e,* cash positions and vanilla European options, as needed.

4. We wish to investigate the lower tail of a return distribution. Let $Q(x)$ = Prob$[X \geq x]$ denote the *survival function* of x. A log-log plot of the survival function for $x \geq 0$ is shown below.



a) Does the distribution of x display at any point evidence that the tail of the distribution follows a power law? Explain what you looked for to determine this.

b) If so, at what point does that behavior emerge? Explain your answer.

c) If there is evidence of a power law in the upper tail, estimate its exponent. Employ a simple visual approximation but explain how you accomplished it. If not, hypothesize a reasonable return distribution.

d) Based on your work above, define to proportionality the PDF and CDF of the upper tail in the power law region.

e) What can you say about the existence of the moments of the distribution based on the work above? Explain you answer.

5. Assume that returns follow a multivariate Normal distribution with mean vector $\boldsymbol{\mu}$, positive-definite covariance matrix $\boldsymbol{\Sigma}$ and risk-free rate r_f. The mean-variance portfolio optimization with unit capital is the quadratic program below. Note that both long and short positions are allowed in this instance.

$$\mathcal{M} = \min_{\mathbf{x}} \left\{ \frac{1}{2} \mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} - \lambda\, (\boldsymbol{\mu}^T - r_f)^{\boldsymbol{T}} \mathbf{x} \mid \mathbf{1}^T \mathbf{x} = 1 \right\}$$

where the risk-reward trade-off is controlled by the parameter $0 \leq \lambda$.

a) Assuming an investor population of mean-variance optimizers, derive an expression for the market (*i.e.*, tangent) portfolio.

b) Given that different investors have different return goals or risk preferences, explain how an investor uses cash and market portfolio to achieve them.

c) Explain why the approach you described in (b) above is superior in mean-variance terms to any other strategy.

6. You are given the returns of N = 120 assets over T = 300 time periods. You wish to examine the sample correlation matrix.

a) Compute the parameter q for the Marchenko-Pastur distribution of eigenvalues for a correlation matrix of uncorrelated assets for an estimation problem of this type.

b) Compute the lower and upper bound for the associated Marchenko-Pastur distribution given q.

c) You are given the partial list of sorted eigenvalues of the sample correlation matrix: {15.2, 8.2, 4.2, 3.1, 2.8, 2.2, 1.8, 1.6, 1.5, 1.4, 1.3, …}. Based solely on the distribution (without any adjustment for sample size), which eigenvalues appear to be statistically meaningful?

d) Briefly explain how you adjust the spectrum based on these results.

7. (Brownian bridge) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{B_t : t \geq 0\}$ be a standard Brownian motion. Suppose X_t follows the Brownian bridge process with SDE

$$dX_t = \frac{y - X_t}{1 - t} dt + dB_t, \quad X_1 = y$$

By applying Itō's formula to $Y_t = (y - X_t)/(1 - t)$ and taking integrals show that under an initial condition $X_0 = x$ and for $0 \leq t < 1$

$$X_t = yt + (1 - t)\left(x + \int_0^t \frac{1}{1 - s} dB_s\right).$$

Using the properties of stochastic integrals on the above expression, find the mean and variance of X_t, given $X_0 = x$ and show that X_t follows a normal distribution.

8. (Forward price dynamics) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{B_t : t \geq 0\}$ be a standard Brownian motion. Suppose the asset price S_t at time t follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

where μ is the drift parameter and σ is the volatility. We define the forward price $F(t, T)$ as an agreed-upon price set at time t to be paid or received at time T, $t \leq T$ and is given by the relationship

$$F(t, T) = \mathbb{E}\left(S_T \mid \mathcal{F}_t\right).$$

(i) By applying Itō's formula to $Y_t = \log X_t$ find a closed form solution for the stock price at time T given S_t, and compute the closed form expression for $F(t, T)$ as a function of stock price S_t.

(ii) Using Itō's lemma on the closed form expression from (i) show that the forward curve satisfies

$$\frac{dF(t, T)}{F(t, T)} = \sigma dB_t$$

9. (Derivation of Dupire local volatility formula) Let $\{B_t : t \geq 0\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose the asset price S_t has the following dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dB_t,$$

where μ are constants and the volatility σ_t is a continuous process. In addition, let r be the risk-free interest rate from a money-market account. Consider a European call option $C\left(S_t, t; K, T\right)$ written at time t on S_t with strike price K and expiry time $T (T > t)$. The Black-Scholes formula for the price of this option is

$$C\left(S_t, t; K, T\right) = S_t \Phi\left(d_+\right) - K e^{-r(T-t)} \Phi\left(d_-\right)$$

where $\Phi(\cdot)$ is the cdf of a standard normal and

$$d_\pm = \frac{\log\left(S_t/K\right) + \left(r \pm \frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}} \quad \text{with} \quad \bar{\sigma}^2 = \frac{1}{T-t}\int_t^T \sigma_u^2 du$$

(i) Show the following identities

$$C\left(S_t, t; K, T\right) = S_t \frac{\partial C}{\partial S_t} + K \frac{\partial C}{\partial K} \quad \text{and} \quad S_t^2 \frac{\partial^2 C}{\partial S_t^2} = K^2 \frac{\partial^2 C}{\partial K^2}.$$

(ii) Using equalities above and the local-volatility PDE (which is a direct generalization of Black-Scholes PDE)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2} + r S_t \frac{\partial C}{\partial S_t} - rC = 0,$$

derive Dupire local volatility formula

$$\sigma_t^2 = \frac{\frac{\partial C}{\partial t} + rK\frac{\partial C}{\partial K}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}.$$

10. (Attainable correlations of lognormal random variables) Let $Z \sim N(0,1)$ and consider two random variables $X_1 = e^Z$ and $X_2 = e^{\sigma Z}$. Compute the correlation of X_1 and X_2 and show that the minimum and maximum of the correlations are

$$\rho_{\min} = \frac{e^{-\sigma} - 1}{\sqrt{(e-1)(e^{-\sigma}-1)}}, \qquad \rho_{\max} = \frac{e^{\sigma} - 1}{\sqrt{(e-1)(e^{-\sigma}-1)}}.$$

11. Consider the GARCH(1,1) process

$$u_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2,$$

in which $\omega, \alpha, \beta > 0$, $\alpha + \beta < 1$, and ϵ_t are i.i.d. random variables with mean 0 and variance 1. Suppose that the fourth moment of the process exist and denote by κ_u and κ_ϵ the kurtosis of u_t and ϵ_t, respectively. Show that

$$\kappa_u = \frac{(1 - (\alpha + \beta)^2)\kappa_\epsilon}{1 - (\alpha + \beta)^2 - (\kappa_\epsilon - 1)\alpha^2}.$$

12. (Maxima of exponential and Pareto distributions) Suppose that $x_1, \ldots, x_n$ are i.i.d. samples drawn from the distribution $F(x)$. Let $x_{(n)} = \{x_1, \ldots, x_n\}$ be the maxima of the sample. Choosing normalizing sequence c_n and d_n, we then have $P((x_{(n)} - d_n)/c_n \le x) = F^n(c_n x + d_n)$.

    (1) If the distribution $F(x)$ is an exponential distribution function, i.e., $F(x) = 1 - \exp(-\beta x)$ for $\beta > 0$ and $x \ge 0$. Choosing normalizing sequence $c_n = 1/\beta$ and $d_n = (\log n)/\beta$, show that

    $$\lim_{n\to\infty} P((x_{(n)} - d_n)/c_n \le x) = \exp(-e^{-x}), \quad x \in \mathbb{R}.$$

    (2) If the distribution $F(x)$ is a Pareto distribution function, i.e., $F(x) = 1 - (\kappa/(\kappa + x))^{\alpha}$ for $\alpha > 0$, $\kappa > 0$ and $x \ge 0$. Choosing normalizing sequence $c_n = \kappa n^{1/\alpha}/\alpha$ and $d_n = \kappa n^{1/\alpha} - \kappa$, show that

    $$\lim_{n\to\infty} P((x_{(n)} - d_n)/c_n \le x) = \exp\big(-(1 + (x/\alpha))^{-\alpha}\big), \quad 1 + (x/\alpha) > 0.$$