Quantitative Finance Qualifying Exam

May 2024

Instructions: (1) You have 4 hours to do this exam. (2) This exam is closed notes and closed books. No electronic devices are permitted. (3) Phones must be turned completely off during the exam. (4) All problems are weighted equally.

Part 1: Do 2 out of problems 1, 2, 3. (AMS511)
Part 2: Do 2 out of problems 4, 5, 6. (AMS512)
Part 3: Do 2 out of problems 7, 8, 9. (AMS513)
Part 4: Do 2 out of problems 10, 11, 12. (AMS517)

Problems to be graded: Please write down which eight problems you want graded here.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (Print clearly):

Student ID:

Signature:

Stony Brook University
Applied Mathematics and Statistics

1. Spot Rate Computation and Applications

   You are given the following information. Assume annual compounding throughout.

   - A 1-year zero-coupon bond with a face value of $10,000 sells at a discount of $9,750.
   - A 2-year bond with a face value of $10,000 and an annual coupon of $350 sells at premium of $10,100.
   - A 3-year bond with a face value of $100,000 and an annual coupon of $4,000 sells at a discount of $98,000.

   Solve for the following:

   a) Spot Rate Curve: Using the market data, above bootstrap the 3-year spot curve.

   b) Bond Valuation: Using the spot rates computed above compute the price of a 3-year bond with a face value of $1,000,000 and annual coupon of $4,000.

   c) Forward Rate: Compute the forward rate $f_{2,3}$.

2. Market Portfolio

Consider the following simple quadratic program representing the portfolios on the Capital Market Line (CML) and its solution to proportionality. The parameters □ and □, are the returns' mean vector and covariance matrix, respectively, and r_f is the risk-free rate. The value of □ ≥ 0 parameterizes the CML:

$$\min\left\{ \tfrac{1}{2}\mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} - \boldsymbol{\lambda}(\boldsymbol{\mu} - r_f)^T \mathbf{x} \right\}$$

Assume the Capital Asset Pricing Model (CAPM), *i.e.*, at time t for an asset i with return $r_i(t)$, market M with return $r_M(t)$ and risk-free rate r_f, and mean-zero, uncorrelated error terms $□_i(t)$, the following expression holds

$$r_i(t) - r_f = \beta_i (r_M(t) - r_f) + \varepsilon_i(t)$$

Let $m_M = E[r_M]$ and $s_M^2 = Var[r_M]$.

a) Express the mean vector □ in terms of the CAPM parameters.□

b) Express the covariance matrix □ in terms of the CAPM parameters.

c) Show that the covariance matrix □ is positive definite.

d) If there are 100 assets, compare the number of parameters of a direct estimate of their mean vector and covariance matrix with that derived from the CAPM.

e) Derive a closed form expression for the optimal Markowitz mean-variance portfolio allocation. You only need to solve to within proportionality.

3. Credit Risk

Consider a 10-year zero coupon with a face value F = \$100,000. The risk-free rate is r_f = 4.2%. The credit spread for the bond is 150 basis points. Based on the intensity model, compute the following at $t = 0$:

a) The default probability of the bond.

b) The value of the bond I there is no possibility of partial recovery on default.

c) The value of the bond if the recovery is 10% of the face amount.

d) The value of the bond if there is no possibility of default.

4. Power Law Tail

A distribution is said to have a power law tail if its survival function has the form :

$$\text{Prob}[R > r] = 1 - F\,(r) = L\,(r)r^{-\alpha}, \alpha > 0$$

where $F(r)$ is the cumulative distribution function of R and $L(r)$ is a slowly varying function such that

$$\lim_{r \to \infty} [\frac{L\,(\lambda\, r)}{L\,(r)}] = 1, \forall \lambda > 0$$

For a return distribution with a power law tail, demonstrate mathematically which moments (*i.e.*, $E[R^{i}]$, i={1, 2, 3, ...}) of R exist depending upon the value of the tail exponent a.

5. Tangent Portfolio

Assume that returns follow a multivariate Normal distribution with mean vector □□ positive-definite covariance matrix □ and risk-free rate r_f. The optimal mean-variance portfolio $\mathbf{x}$ with unit capital is the quadratic program below. Note that both long and short positions are allowed in this instance.

$$\mathcal{M} = \min_{\mathbf{x}} \left\{ \frac{1}{2}\mathbf{x}^T \boldsymbol{\Sigma}\, \mathbf{x} - \lambda\, (\boldsymbol{\mu}^T - r_f)^{\boldsymbol{T}} \mathbf{x} \mid \mathbf{1}^T \mathbf{x} = 1 \right\}$$

where the risk-reward trade-off is controlled by the parameter 0 ≤ □.

a) Assuming an investor population of mean-variance optimizers, derive an expression for the market (*i.e.*, tangent) portfolio.

b) Given that different investors have different return goals or risk preferences, explain how an investor uses cash and the market portfolio to achieve them.

c) Explain why the approach you described in (b) above is superior in mean-variance terms to any other strategy.

6. Correlation Denoising

You are given the returns of $N = 100$ assets over $T = 360$ time periods. You wish to examine the sample correlation matrix.

a) Compute the parameter q for the Marchenko-Pastur distribution of eigenvalues for a correlation matrix of uncorrelated assets for an estimation problem of this type.

b) Compute the lower and upper bound for the associated Marchenko-Pastur distribution given q.

c) You are given the partial list of sorted eigenvalues of the sample correlation matrix: {15.2, 8.2, 4.2, 3.1, 2.8, 2.2, 1.8, 1.6, 1.5, 1.4, 1.3, …}. Based solely on the distribution (without any adjustment for sample size), which eigenvalues appear to be statistically meaningful?

d) Briefly explain how you adjust the spectrum based on these results.

7. **(American Options)**

(i) Consider a game where a player simultaneously throws one fair coin and one fair six-sided die. After each throw, if the coin shows heads, the player receives an amount of money equal to the number shown on the die. However, if the coin shows tails, the player receives zero dollars. The player is allowed to throw the coin and die up to three times in total. The player can choose to stop the game at time $n = 1, 2, 3$. Compute the fair value of the game at time 0 (before the first throw).

(ii) Consider the prices of two identical American put options, denoted as $P(S_t, t; K, T_1)$ and $P(S_t, t; K, T_2)$, where T_1 and T_2 are their respective expiry times with $T_1 < T_2$. Here, S_t represents the spot price at time t, and K is the common strike price for both options. Construct an arbitrage portfolio that demonstrates that the price of the option with the earlier expiry time is less than or equal to the price of the option with the later expiry time, i.e.,

$$P(S_t, t; K, T_1) \leq P(S_t, t; K, T_2)$$

given that $T_1 < T_2$.

(iii) Consider an economy consisting of a risk-free asset and a stock, whose values at time t are B_t and S_t, respectively. Assume that these values evolve according to the following dynamics:

$$dB_t = rB_t\,dt, \quad dS_t = \mu S_t\,dt + \sigma S_t\,dW_t,$$

where r is the risk-free rate, μ is the stock price growth rate, and $\sigma > 0$ is the stock price volatility. Additionally, $\{W_t : 0 \leq t \leq T\}$ is a $\mathbb{P}$-standard Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{F}_t$, $0 \leq t \leq T$, is the filtration generated by the standard Wiener process. Show that a perpetual American option $V(S_t)$ (which has not yet been exercised) satisfies the following second-order ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2 S_t^2 \frac{d^2V}{dS_t^2} + rS_t\frac{dV}{dS_t} - rV(S_t) = 0.$$

Also, show that the general solution of the above equation is

$$V(S_t) = AS_t^{\alpha+} + BS_t^{\alpha-},$$

where A and B are unknown constants, $\alpha_+ > 0$, and $\alpha_- < 0$. Find α_- and α_+.

8. **(Properties of Brownian Motion)** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{W_t : t \geq 0\}$ be a standard Brownian motion.

   (i) Show that
   $$Cov(W_s, W_t) = \min(t, s),$$
   and that
   $$Corr(W_s, W_t) = \sqrt{\frac{\min(t, s)}{\max(t, s)}}.$$

   (ii) Show that the following process defined via time inversion
   $$B_s = \begin{cases} 0 & \text{if } s = 0 \\ sW_{\frac{1}{s}} & \text{if } s \neq 0 \end{cases}$$
   is also a Brownian motion.

   (iii) Demonstrate that the pair of random variables $(W_t, \int_0^t W_s \, ds)$ has the following covariance matrix:
   $$\mathbf{\Sigma} = \begin{pmatrix} t & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & \frac{1}{3}t^3 \end{pmatrix}.$$
   Furthermore, calculate the corresponding correlation coefficient.

9. **Girsanov Theorem & Discounted Portfolio)** Let $\{W_t : t \geq 0\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and suppose θ_t is an adapted process for $0 \leq t \leq T$. Consider the stochastic process

$$Z_t = e^{-\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds}.$$

(i) If it holds that

$$\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T \theta_t^2 \, dt}\right] < \infty,$$

demonstrate that Z_t is a positive $\mathbb{P}$-martingale for $0 \leq t \leq T$.

(ii) By changing the measure $\mathbb{P}$ to a measure $\mathbb{Q}$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t$, show that

$$\widetilde{W}_t = W_t + \int_0^t \theta_u \, du$$

is a $\mathbb{Q}$-standard Wiener process.

(iii) Consider an economy comprising a risk-free asset and a stock, whose values at time t are denoted by B_t and S_t, respectively. Assume that their dynamics are governed by:

$$dB_t = rB_t \, dt, \quad dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

where r is the risk-free rate, μ is the stock price growth rate, and $\sigma > 0$ denotes the stock price volatility. Additionally, $\{W_t : 0 \leq t \leq T\}$ is a $\mathbb{P}$-standard Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathcal{F}_t$, $0 \leq t \leq T$, as the filtration generated by the standard Wiener process.

At time t, consider a trader with a portfolio valued at Π_t, holding Φ_t shares of stock and Ψ_t units invested in the risk-free asset. Define the discounted portfolio value as

$$Y_t = e^{-\int_0^t r_u \, du}\Pi_t.$$

Employing Girsanov's theorem, show that by changing the measure $\mathbb{P}$ to an equivalent risk-neutral measure $\mathbb{Q}$, the discounted portfolio Y_t becomes a $\mathbb{Q}$-martingale.

10. Consider the Clayton copula with $\theta = -1/2$, the copula can be written as $C(u_1, u_2) = \max\{(\sqrt{u_1} + \sqrt{u_2} - 1)^2, 0\}$. Show that its Kendall's tau correlation is $-1/3$.

11. **Properties of Gaussian copula**

Consider the follwoing bivariate Gaussian copula with correlation coefficient ρ

$$C(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(- \frac{s^2+t^2-2\rho st}{2(1-\rho^2)} \right) dsdt. \tag{1}$$

Show that

$$C(u,v) = \int_0^u \int_0^v \frac{\partial^2 C(u,v)}{\partial u \partial v} dudv.$$

12. **VaR and ES of ARMA-GARCH models**

Suppose returns y_t of an asset follow the AR(2)-GARCH(1,1) model ($a \in (0,1), a \neq 0, \alpha > 0, \beta > 0, \alpha + \beta < 1$)

$$y_t = a^2 y_{t-2} + z_t, \qquad z_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \omega + \alpha z_{t-1}^2 + \beta \sigma_{t-1}^2,$$

where ϵ_t are independent and identically distributed standard normal random variables with mean 0 and variance σ^2. Compute 95% 1-day and 5-day VaR and ES for the long position of the asset?