Applied Mathematics and Statistics

Foundation Qualifying Examination Part B

in Computational Applied Mathematics,

Spring 2023 (January)

(Closed Book Exam)

Instructions: There are 3 problems, and you are required to solve all of them. All problems are weighted equally. Please show detailed work for full credit. Start each answer on a new page. Print your name, and the appropriate question number at the top of every page used to answer any question. Hand in all answer pages.

NAME ____

Student ID ____

Date of Exam: January 19, 2023

Time: 11:15 AM – 13:15 PM

B1.

a) Use the power series method to find the general solution of the following homogeneous equation

$$(x^2 - 1)y'' - 6xy' + 12y = 0.$$

b) Find the general solution of the non-homogeneous equation

$$(x^2 - 1)y'' - 6xy' + 12y = 6x.$$

c) Find the solution of (b) satisfying the initial condition

$$y(0) = 1, \quad y'(0) = 10.$$

d) Does the equation in (b) have a unique solution if the boundary conditions (instead of the initial condition) are given as $y(0) = 1$, $y(1) = 1$? If yes, solve the boundary value problem.

This page is intentionally blank. Continue your answer on this page.

B2. Suppose $A \in \mathbb{R}^{m \times n}$ has full rank, where $m \geq n$. Let α be any positive real number.

a) (4 points) Show that $\begin{bmatrix} \alpha I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ has a solution x that minimizes $\|Ax - b\|$.

b) (4 points) Show that the largest singular value of the matrix $B = \begin{bmatrix} \alpha I & A \\ A^T & 0 \end{bmatrix}$ is $\|A\| + \alpha$ and the smallest singular value of B is the same as that of A.

c) (2 points) What is the 2-norm condition number of B in part (b) in terms of the singular values of A?

This page is intentionally blank. Continue your answer on this page.

B3. Given $A \in \mathbb{C}^{n\times n}$, suppose $A^* = \omega A$, where $\omega \in \mathbb{C}$ is a complex sign, i.e., $|\omega| = 1$. For example, if $\omega = 1$, then A is Hermitian; if $\omega = -1$, then A is skew Hermitian.

a) (4 points) Show that $A + \alpha I$ is normal for any $\alpha \in \mathbb{C}$.

b) (3 points) Show that A has a full set of orthonormal eigenvectors.

c) (3 points) Show that in the reduction to Hessenberg form, $A = QHQ^*$, H must be tridiagonal.

This page is intentionally blank. Continue your answer on this page.