Applied Mathematics and Statistics

Foundation Qualifying Examination Part B

in Computational Applied Mathematics,

Spring 2025 (January)

(Closed Book Exam)

Instructions: There are 3 problems, and you are required to solve all of them. All problems are weighted equally. Please show detailed work for full credit. Start each answer on a new page. Print your name, and the appropriate question number at the top of every page used to answer any question. Hand in all answer pages.

NAME ____

Student ID ____

Date of Exam: January 23, 2025

Time: 11:15 AM – 13:15 PM

B1. Consider the following initial value problem (IVP) with a small positive parameter ε

$$y'' + (1+\varepsilon)y = 0, \quad x \in (0, \infty),$$

$$y(0) = 1, \quad y'(0) = 0.$$

a) Solve the IVP problem exactly.

b) Obtain a first order perturbative approximation $y(x) = y_0(x) + \varepsilon y_1(x)$ to the IVP.

c) Compare behavior of the perturbative solution at large x with the exact solution. In which x-domain is this approximate solution valid?

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B2. Let $A \in \mathbb{C}^{m\times n}$ be a full-rank matrix with $m \geq n$.

(a) (4 points) Using Householder reflectors, describe an efficient algorithm to compute the **QL factorization** of A, i.e.,

$$A = QL,$$

where $Q \in \mathbb{C}^{m\times m}$ is unitary (i.e., $Q^*Q = I$) and $L \in \mathbb{C}^{m\times n}$ is lower triangular. The algorithm should have the same computational cost as computing the QR factorization of A using Householder reflectors. Either explain how to convert this problem into a QR factorization or describe a direct algorithm to compute the QL factorization. Clearly indicate how to construct the Householder reflectors stably.

(b) (4 points) Explain how to apply the QL factorization from part (a) to solve the least squares problem

$$\min_{x\in\mathbb{C}^n} \|b - Ax\|_2,$$

where $b \in \mathbb{C}^m$ is a given vector. Provide the necessary equations, especially on how to apply the Householder reflectors to the right-hand side vector b, assuming that the matrix Q from part (a) is stored implicitly as Householder reflectors.

(c) (2 points) Discuss the efficiency and numerical stability of computing the QL factorization via Householder reflectors (as in part (a)) and solving the least squares problem (as in part (b)) versus using the QR factorization computed via **modified Gram-Schmidt** orthogonalization.

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B3. Consider the linear system $Ax = b$, where $A \in \mathbb{R}^{m \times m}$ is a large, sparse, nonsingular matrix.

(a) (4 points) Define the n dimensional Krylov subspace $\mathcal{K}_n(A, b)$. Explain the basic properties of this Krylov subspace that makes it useful for solving linear systems, especially for **symmetric** matrices.

(b) (3 points) Assume A is **unsymmetric**, describe the Arnoldi iterations for constructing an orthonormal basis for $\mathcal{K}_k(A, b)$. Provide pseudocode for the algorithm.

(c) (3 points) Discuss the connection between the Arnoldi iterations and the GMRES (Generalized Minimal Residual) method.

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