Quantitative Finance Qualifying Exam

May 2021

Instructions: (1) You have 4 hours to do this exam. (2) This exam is closed notes and closed books. No electronic devices are permitted. (3) Phones must be turned completely off during the exam. (4) All problems are weighted equally.

Part 1: Do 2 out of problems 1, 2, 3. (AMS511)
Part 2: Do 2 out of problems 4, 5, 6. (AMS512)
Part 3: Do 2 out of problems 7, 8, 9. (AMS513)
Part 4: Do 2 out of problems 10, 11, 12. (AMS517)

Problems to be graded: Please write down which eight problems you want graded here.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (Print clearly):

Student ID:

Signature:

Stony Brook University
Applied Mathematics and Statistics

Detailed Instructions

- This exam is conducted via Zoom on May 26 from 9 am to 1 pm EDT.
- The entire Zoom meeting and chat messages are being recorded.
- This is a closed book, closed note exam.
- Hand calculators (or other computing devices) may not be used during the exam.
- You should join the Zoom meeting from two devices: Your computer/laptop/tablet (with webcam), and your smartphone (with camera).
- Audio should be muted and video must be kept on during the exam.
- Your computer webcam must fully show your face; your smartphone camera should show your computer monitor, your hands and workspace, with the pages of paper being used for the exam.
- At the very beginning of the exam, during set up, you will be asked to do a brief "environment scan", showing the workspace where your computer is and the desk/table/floor where you will be writing your work.
- You are required to bring enough blank pieces of paper to use for the exam. You will show the blank pages at the beginning, during the "environment scan" on Zoom.
- You are not allowed to use the internet for any searching or communication with others, with the sole exception of communication privately with the proctors via Zoom chat (which is set so that your chats only go privately to hosts, not to others).
- It is recommended that you print the exam and write your answers on it. However, you can write your answers on your blank papers if you do not have a printer with you.
- After you finish the exam, scan your pages, ordered and oriented appropriately, into a single pdf file. Email the pdf file to
  haipeng.xing@stonybrook.edu and cc to **hongshik.ahn@stonybrook.edu**
  no later than 5 minutes after completion of the exam (i.e., **by 1:05 pm EDT**).
- No students are allowed to leave the Zoom meeting until the exam is over.
  - If you finish the exam early, then submit your exam and remain in the Zoom meeting until the conclusion of the exam at 1 pm EDT.
  - After submitting your exam, you can study for another exam or work on anything else, while staying in view in the Zoom meeting.
- If the answers are not submitted by 1:05 pm EDT, the exam will not be graded, and a score of zero will be given.
- If you have a question during the exam, then send a chat message to the host privately.

1. An Itô process $X(t)$ follows the dynamics of the constant coefficient geometricprocess described by the stochastic differential equation (SDE):

$$dX(t) = \mu\, X(t)\, dt + \sigma\, X(t)\, dW(t)$$

Let $Y(y) = ln\, X(t)$. Derive the SDE which describes the dynmaics of $Y(t)$. Show all work in deriving the solution.

2. The interest rates in the UK $r_{UK} = 0.02$ and US $r_{US} = 0.03$, compounded continuously. The spot price of the UK pound is \$1.25 and the forward price for the UK pound deliverable in 6-months is \$1.30. Does an arbitrage opportunity exist? Show clearly why one is or is not available.

3. Solve the following problem in terms of vanilla European puts and calls and, if required, a cash position. An investor owns a stock with current price $S(t)$. The company in question is facing a lawsuit where if it wins, the price will increase dramatically and if it loses the price will decline dramatically. The judge deciding the suit has indicated that she will issue her decision on or before time T. The outcome is uncertain and the investor believes the company is as likely to win as lose the suit. Construct an options portfolio that will have the potential for the investor to profit regardless of the judge's decision and describe the conditions where this strategy will fail to realize a profit.

4. A distribution is said to have a power law tail if its survival function has the form:

$$Prob[R > r] = 1 - F(r) = L(r)r^{-\alpha}, \quad \alpha > 0$$

where $F(r)$ is the cumulative distribution of the return R and $L(r)$ is a slowly varying function such that

$$\lim_{r \to \infty} [L(\lambda\, r) \,/\, L(r)] = 1, \quad \forall\, \lambda > 0$$

- For a return distribution with a power law tail, demonstrate mathematically which moments of R $(E\left[R^i\right], i = \{1, 2, 3, \ldots\})$ exist depending upon the value of the tail exponent.
- Given a sample of data, describe how an appropriately constructed plot of the survival function can be used to identify if a power law tail appears to exist and, if so, how that plot can be used to estimate the tail exponent α.

5. You are given the returns for $N = 30$ assets over $T = 60$ time periods. You wish to examine the sample correlaton matrix.

    - Compute the parameter q for the Marchenko-Pastur distribution of eigenvalues for a correlation matrix of uncorrelated assets for a problem of this dimension and sample size.
    - Compute the lower and upper bound for the associated Marchenko-Pastur distribution.
    - You are given the following partial list of sorted eigenvalues of the sample correlation matrix: $\{10.1, 7.4, 6.2, 5.5, 3.1, 1.8, 0.9, 0.8, 0.4, \ldots\}$. Based solely on the distribution and without any adjustment for sample size, which eigenvalues appear to be statistically meaningful.
    - Using this information, denoise the eigenvalues. What are the values of the eigenvalues not found to be statistically meaningful.

6. You are given three stocks, $i = \{1, 2, 3\}$. Each stock's return r_i can be modeled by the Captial Asset Pricing Model (CAPM):

$$r_i - r_f = \beta_i(m - r_f) + \epsilon_i$$

Given $r_f = 0.003$, $\beta = \{0.7, 0.9, 1.2\}$, $\mu_m = 0.065$, $\sigma_m^2 = 0.01$, and $\sigma_e^2 = \{0.008, 0.006, 0.010\}$:

- What is the mean vector and covariance matrix for the three stocks?
- Assuming short positions are permitted and no constraints, derive to proportionality a closed form solution for the mean-variance portfolio optimization problem.
- Assuming unit capital, apply the proportional solution derived to determine the optimal portfolio.

7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a symmetric random walk such that the j-step is defined as

$$Z_j = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

where Z_i and Z_j are independent, for all $i \neq j$. By setting $0 = k_0 < k_1 < \ldots < k_t$, we let

$$M_{k_i} = \sum_{j=1}^{k_i} Z_j, \quad i = 1, 2, \ldots, t$$

where $M_0 = 0$.
Show that the symmetric random walk

(i) has independent increments such that the random variables

$$M_{k_1} - M_{k_0}, M_{k_2} - M_{k_1}, \ldots, M_{k_t} - M_{k_{t-1}}$$

are independent;

(ii) has first two central moments given by $\mathbb{E}(M_{k_{i+1}} - M_{k_i}) = 0$ and $\mathrm{Var}(M_{k_{i+1}} - M_{k_i}) = k_{i+1} - k_i$;

(iii) is a martingale;

(iv) define

$$W_t^{(n)} = \frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} Z_i$$

for a fixed time t, show that

$$\lim_{n \to \infty} W_t^{(n)} = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} Z_i \xrightarrow{D} \mathcal{N}(0, t).$$

(Extension of this result to the functional case is known as a functional central limit theorem, or Donsker's invariance principle. You do not have to prove this more general result. You compute the limit for a fixed t.)

8. (Ornstein-Uhlenbeck Process) Suppose X_t follows the Ornstein-Uhlenbeck process with SDE

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where κ, θ, and σ are constants.

(i) By applying Ito's formula to $Y_t = e^{\kappa t} X_t$ and taking integrals show that for $t < T$

$$X_T = X_t e^{-\kappa(T-t)} + \theta[1 - e^{-\kappa(T-t)}] + \int_t^T \sigma e^{-\kappa(T-s)} \mathrm{d}W_s.$$

(ii) Using the properties of stochastic integrals on the above expression, find the mean and variance of X_T, given $X_t = x$. Deduce that X_T follows a normal distribution.

9. (Asset-or-Nothing Option - Probabilistic Approach) Let $\{W_t : t \geq 0\}$ be a $\mathbb{P}$-standard Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let the stock price S_t follow a GBM with the following SDE

$$dS_t = S_t(\mu - D)dt + S_t \sigma dW_t,$$

where μ is the drift parameter, D is the continuous dividend yield, $\sigma > 0$ is the volatility parameter, and let $r > 0$ denote the risk-free interest rate.

An asset-or-nothing call option is a contract that pays S_T at expiry time T if the spot price $S_T > K$ and nothing if $S_T \leq K$. In contrast, an asset-or-nothing put pays $S_T > 0$ at expiry time T if the spot price $S_T < K$ and nothing if $S_T \geq K$.

(i) By denoting $C_a(S_t, t; K, T)$ and $P_a(S_t, t; K, T)$ as the asset-or-nothing call and put option prices, respectively, at time t, $t < T$ show using the risk-neutral valuation approach that

$$C_a(S_t, t; K, T) = S_t e^{-D(T-t)} \Phi(d_+) \quad \text{and} \quad P_a(S_t, t; K, T) = S_t e^{-D(T-t)} \Phi(-d_+)$$

where

$$d_+ = \frac{\log(S_t/K) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du.$$

(ii) Verify that the put-call parity for an asset-or-nothing option is

$$C_a(S_t, t; K, T) + P_a(S_t, t; K, T) = S_t e^{-D(T-t)}$$

10. (ARMA(1,1) Process) Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary and causal solution of

$$X_t - \phi_1 X_{t-1} = \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad t \in \mathbb{Z}, \tag{1}$$

for some constants $\phi_1 \neq 0$, $\theta_1 \neq 0$ and $(\varepsilon_t)_{t\in\mathbb{Z}} \sim WN(0, \sigma_\varepsilon^2)$, $0 < \sigma_\varepsilon < \infty$.

(i) Assume that $\phi_1 \neq -\theta_1$. Show that a necessary and sufficient condition for the equations (1) to have a stationary and causal solution is $|\phi_1| < 1$.

(ii) Compute the autocorrelation function of $(X_t)_{t\in Z}$ in this case.

(iii) What is the solution of the equations (1) when $\phi_1 = -\theta_1$?

11. (Non)-Subadditivity of Value-at-Risk (VaR))

(i) Consider a portfolio of $d = 100$ defaultable corporate bonds. We assume that defaults of different bonds are independent; the default probability is identical for all bonds and is equal to 2%. The current price of the bonds is 100 USD. If there is no default, a bond pays in $t+1$ (one year from now, say) an amount of 105 USD; otherwise there is no repayment. Hence L_i, the loss of bond i, is equal to 100 USD when the bond defaults and to -5 USD otherwise. Denote by Y_i the default indicator of firm i, i.e. Y_i is equal to one if bond i defaults in $[t, t+1]$ and equal to zero otherwise. We get $L_i = 100Y_i - 5(1-Y_i) = 105Y_i - 5$ USD. Hence the L_i form a sequence of iid rvs with $\mathbb{P}(L_i = -5) = 0.98$ and $\mathbb{P}(L_i = 100) = 0.02$. Your goal is to compare two portfolios in terms of their VaR values, both with current value equal to 10000 USD.

- Portfolio A is fully concentrated and consists of 100 units of bond one.
- Portfolio B is completely diversified: it consists of one unit of each of the bonds.

Economic intuition suggests that portfolio B is less risky than portfolio A and hence should have a lower VaR. Compute VaR at a confidence level of 95% for both portfolios. (You can use the fact that for a binomial distributed random variable $M \sim B(100, 0.02)$, $\mathbb{P}(M \leq 5) \approx 0.984$ and $\mathbb{P}(M \leq 4) \approx 0.949$.) Is the VaR subadditive in this example?

(ii) Suppose $\mathbf{X} = (X_1, \ldots, X_d) \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let $\mathcal{M}$ denote the space of all risks of the form $L = \boldsymbol{\lambda}'\mathbf{X} = \sum_{j=1}^{d} \lambda_j X_j$. Prove that VaR_α is subadditive on $\mathcal{M}$ for all $\alpha \in [1/2, 1)$.

12. (An Interpretation of the Principal Components Transform) Let $\mathbf{X}$ be a random vector with $\mathbb{E}(\mathbf{X}) = 0$ and $cov(\mathbf{X}) = \mathbf{\Sigma}$. Let $\mathbf{Y}$ be the vector given by the principal components transform of $\mathbf{X}$. In other words, $\mathbf{Y} = \mathbf{\Gamma X}$, where $\mathbf{\Sigma} = \mathbf{\Gamma \Lambda \Gamma}'$, the columns of $\mathbf{\Gamma}$ contain the orthonormal eigenvectors of $\mathbf{\Sigma}$ and $\mathbf{\Lambda} = diag(\lambda_1, \ldots, \lambda_d)$, where $\lambda_1 \geq \cdots \geq \lambda_d$ denote the sorted eigenvalues of $\mathbf{\Sigma}$. The jth principal component Y_j is thus $Y_j = \boldsymbol{\gamma}_j' \mathbf{X}$, where j is the jth column of $\mathbf{\Gamma}$ (the eigenvector corresponding to the jth largest eigenvalue of $\mathbf{\Sigma}$), also known as jth vector of loadings or jth principal axis.

   (i) Show that the first principal component Y_1 of $\mathbf{X}$ satisfies

   $$Var(Y_1) = \max_{\mathbf{a} \in \mathbb{R}^d, \|\mathbf{a}\|=1} Var(\mathbf{a}'\mathbf{X}),$$

   that is Y_1 is the standardized linear combination $\mathbf{a}'\mathbf{X}$ of $\mathbf{X}$ with maximal variance among all linear combinations.

   (ii) Show that the second principal component Y_2 of $\mathbf{X}$ satisfies

   $$Var(Y_2) = \max_{\mathbf{a} \in \mathbb{R}^d,\ \|\mathbf{a}\|=1,\ \mathbf{a}'\boldsymbol{\gamma}_1 = 0} Var(\mathbf{a}'\mathbf{X}),$$

   that is Y_2 is the standardized linear combination $\mathbf{a}'\mathbf{X}$ of $\mathbf{X}$ with maximal variance among all linear combinations orthogonal to the first principal axis.